

Geometric Design of Linkages

Second Edition



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Geometric Design of Linkages

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For Kendall and Wyatt

Preface

2nd Edition

This second edition of *Geometric Design of Linkages* revises and updates our formulation of the kinematic theory of linkages. Four new chapters have been added that present the analysis and synthesis of multiloop planar and spherical linkages, and the synthesis theory for spatial serial chains.

An introduction to linkage graphs and linkage enumeration has been added to Chapter 1 to provide background for the new chapters on the synthesis of multiloop linkages, and the use of the Dixon determinant to analyze planar multiloop linkages has been added to Chapter two.

Chapters three, four, and five are the same as before with corrections of minor errors. Chapter six is new and presents a methodology for the synthesis of planar six-bar and eight-bar linkages by constraining three-degree-of-freedom 3R open chains or 6R closed chains, respectively. Examples are provided that demonstrate the technique.

Chapter seven is the same as Chapter six in the previous edition but now includes a section on the analysis of multiloop spherical linkages. Chapters eight and nine are the same as Chapters seven and eight in the previous edition, again with corrections of minor errors,. The new Chapter 10 parallels the new Chapter six and presents a way to design spherical six-bar and eight-bar linkages by constraining 3R open and 6R closed spherical chains.

Chapters 11, 12, and 13 are the same as Chapters 9, 10, and 11 in the previous edition. Chapters 14 and 15 on the synthesis of spatial serial chains are new and were the result of research with Haijun Su and Alba Perez-Gracia. Chapter 14 formulates and solves using numerical homotopy the synthesis equations for five-degree-of-freedom serial chains that can position a wrist center on specific algebraic surfaces, termed reachable surfaces. Chapter 15 introduces the Clifford algebra of dual quaternions and its use in formulating the synthesis equations for general spatial serial chains. Chapter 16 is the same as Chapter 12 in the previous edition.

I continue to benefit from the contributions of teachers, colleagues, and students toward a geometric synthesis theory for linkage systems, recently from Jeff Ge, Mohan Bodduluri, John Dooley, Pierre Larochelle, Andrew Murray, Fangli Hao, Curtis Collins, Alba Perez-Gracia, Haijun Su, Nina Robson, Duanling Li, and my coauthor Gim Song Soh. In addition, I am grateful for the continued inspiration of Qizheng Liao and Bernard Roth.

Finally, I gratefully acknowledge the support of the Engineering Design Program of the National Science Foundation that has made this book possible.

Irvine, CA,

J. Michael McCarthy
September 2010

First Edition

This book is an introduction to the mathematical theory of design for articulated devices that rely on simple mechanical constraints to provide a complex workspace for a workpiece or end-effector. Devices ranging from windshield wipers to robot manipulators and mechanical hands are examples of these systems each of which has a skeleton of links connected by joints called a linkage. The function or task for the device is defined as a set of positions to be reached by the end-effector. The goal is to determine the dimensions of all of the devices that can achieve a specific task. Formulated in this way the design problem is purely geometric in character.

This text blends two approaches to this design problem in order to develop the intuition needed to move from planar to spatial linkage design. One approach considers the geometric configurations of points and lines generated as a moving body is displaced through a finite set of positions. This is the foundation for graphical methods for planar linkage synthesis and can be generalized to spherical and spatial linkage design. A separate approach focuses directly on solving the nonlinear constraint equations that characterize a mechanical connection. This provides convenient equations for planar and spherical linkage design, and is crucial to addressing the geometric challenge of spatial linkage design.

This unified formulation requires a range of mathematical tools. The basic language is vector algebra and matrix theory, which should be familiar to junior and senior university students. However, something among the techniques ranging from graphical constructions, spherical trigonometry, complex vectors, and quaternions to line geometry and dual vector algebra is certain to be unfamiliar. For this reason, the presentation is designed to introduce these techniques, and additional background is provided in appendices.

The first chapter presents an overview of the articulated systems that we will be considering in this book. The generic mobility of a linkage is defined, and we separate them into the primary classes of planar, spherical, and spatial chains.

The second chapter presents the analysis of planar chains and details their movement and classification. Chapter three develops the graphical design theory for pla-

nar linkages and introduces many of the geometric principles that appear in the remainder of the book. In particular, geometric derivations of the pole triangle and the center-point theorem anticipate analytical results for the spherical and spatial cases.

Chapter four presents the theory of planar displacements, and Chapter five presents the algebraic design theory. The bilinear structure of the design equations provides a solution strategy that emphasizes the geometry underlying linear algebra. The five-position solution includes an elimination step that is probably new to most students, though it is understood and well received in the classroom.

Chapters six and seven introduce the properties of spherical linkages and detail the geometric theory of spatial rotations (now Chapters seven and eight, 2nd edition). Chapter eight presents the design theory for these linkages (now Chapter nine), which is analogous to the planar theory. This material exercises the student's use of vector methods to represent geometry in three dimensions. Perpendicular bisectors in the planar design theory become perpendicular bisecting planes that intersect to define axes. The analogue provides students with a geometric perspective of the linear equations that they are solving.

Chapter nine introduces the analysis of spatial linkages including open chains that are closely related to robot manipulators (now Chapter 11). The complexity of spatial linkages requires the introduction of new techniques. However, we maintain a point of view that emphasizes the similarity to the planar and spherical theories. For example, the constraint equations of planar and spherical linkages are shown to be special cases of those for spatial linkages.

Chapter 10 develops the geometry of spatial displacements (now Chapter 12). Here, we find that the screw triangle and the center-axis theorem must be formulated using lines rather than points. Dual vector algebra is introduced to provide vector operations for calculations with the Plücker coordinates of lines. The result is that geometric calculations with line coordinates are identical to the more familiar vector calculations with point coordinates.

Chapter 11 presents the design theory for spatial chains, and Chapter 12 introduces the geometry of linear combinations of lines that arise in the construction of spatial linkage systems (now Chapters 13 and 16). While the design techniques for planar linkages are well developed, there is room for much more work in the design and use of spatial linkages.

I am pleased to express my gratitude for the contribution of many teachers and colleagues whose work over the years has developed and clarified linkage design theory. This book includes results of the insight, commitment, and hard work by Jeff Ge, Mohan Bodduluri, John Dooley, Pierre Laroche, Andrew Murray, Fangli Hao, Curtis Collins, Alan Ruth, Shawn Ahlers, and Alba Perez. I have also benefited from the insight of Qizheng Liao and the inspiration of Bernard Roth. Finally, the support by the Division of Design, Manufacturing, and Industrial Innovation of the National Science Foundation that has made this book possible is gratefully acknowledged.

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Chapter 1

Introduction

A mechanical system, or machine, generally consists of a power source and a mechanism for the controlled use of this power. The power may originate as the flow of water or the expansion of steam that drives a turbine and rotates an input shaft to the mechanism. It may be that instead the turbine rotates a generator and the resulting electricity is used to actuate a distant electric motor connected to the mechanism input. Another power source is the expansion of pressurized fluid or burning air–fuel mixture against a piston in order to drive its linear movement inside a cylinder. The purpose of the mechanism is to transform this input power into a useful application of forces combined with a desired movement. For this reason, machines are often defined abstractly as devices that transform energy from one form, such as heat or chemical energy, into another form, usually work.

In this book our focus is on devices that transform an input rotary or linear motion into more general movement. We assume that a power source is available that can provide the force and torque needed to drive the system. The primary concern is determining the mechanical constraints that provide a desired movement. This is known as the *kinematic synthesis*, or *geometric design*, of a mechanism.

A mechanism is often described as assembled from gears, cams, and linkages, though it usually contains other specialized components, such as springs, ratchets, brakes, and clutches, as well. Of these it is the linkage that provides versatile movement. Gears and cams generally rotate or translate, though with important torque and timing properties. And the other components are used to apply forces. Therefore, because our goal is to obtain a desired movement, we focus on the design of linkages.

1.1 Linkages

A *linkage* is a collection of interconnected components, individually called *links*. The physical connection between two links is called a *joint*. This definition is general enough to encompass gears and cams where the joint is formed by direct contact

between two gear teeth or between a cam and follower. However, we limit our attention to joints that do not include the type of rolling and sliding contact that is found in gears and cams. In fact, the linkages that we study can be viewed as constructed from two elementary joints, the rotary hinge, called a *revolute* joint (denoted by R), and the linear slider, or *prismatic* joint (denoted by P). These joints allow *one-degree-of-freedom* movement between the two links that they connect. The configuration variable for a hinge is the angle measured around its axis between the two bodies, and for a slider it is the distance measured along the linear slide of the joint.

Other joints are available to form linkages, such as the universal joint, the ball-in-socket, and a circular cylinder on a rod. In these cases, it is possible to identify an equivalent assembly of hinges and sliders that provide the same geometric constraint. The universal joint, or T joint, can be viewed as constructed from two revolute joints that are at right angles to each other. This joint allows a two-degree-of-freedom movement between the two links that it connects. The ball-in-socket joint, or S joint, is formed from three revolute joints with concurrent axes. This joint allows three-degree-of-freedom rotational movement and is often found in robot wrists with each joint actuated. Finally, the cylindric joint, or C joint, is constructed by mounting a hinge on a slider so that the axis of the hinge is parallel to the direction of the slider. This joint allows two-degree-of-freedom movement.

These joints constrain the trajectories of points in one link to lie on simple geometric objects in the other link, assuming that the links do not bend or distort as they move. For example, the revolute joint constrains points in one body to follow circular trajectories relative to the other link, and a slider generates linear trajectories. The T and S joints both constrain these trajectories to lie on a sphere, while the cylindric joint forces them to lie on a cylinder. This feature leads to algebraic equations that characterize the mechanical constraint imposed by a linkage on the movement of a floating link, or workpiece. The design problem consists of solving these equations for a specified set of task positions for the workpiece.

1.2 Mobility

An important characteristic of an assembly of bodies forming a linkage is the *generic mobility* of the system. This is, generally speaking, the number of independent parameters such as joint angles and slide distances that are needed to specify the configuration of the linkage ignoring any deformation of the links. It is also known as the dimension of the *configuration space* of the system. The generic mobility of a linkage is the sum of the unconstrained degree of freedom for the links in the system less the constraints imposed by the joints.

In a linkage assembled from n bodies, one link is designated as the fixed frame, or ground, against which the movement of the remaining links is measured. This link has no freedom of movement. Thus, the unconstrained degree of freedom for a linkage constructed from n links is $(n - 1)K$, where K is the number of parameters required to specify the position of a single link. In three-dimensional space $K = 6$

because three orientation parameters and three coordinates for a reference point are required to locate this body relative to the ground link. A body that can only rotate in space has $K = 3$; and one constrained to planar movement also has $K = 3$.

Revolute and prismatic joints, and joints constructed from them, reduce the dimension of the configuration space of a system by introducing constraint equations among the configuration parameters. These joints are said to impose *holonomic constraints* on the system. In contrast, nonholonomic constraints, such as the contact of a rolling wheel or knife edge on a surface, do not reduce the dimension of the configuration space of the system. They restrict the instantaneous movement from one configuration to the next. The number of constraint equations imposed by a joint is $u = K - f$, where f is known as the freedom allowed by the joint. Thus, the number of constraint equations imposed by j joints is $\sum_{i=1}^j (K - f_i)$, where f_i is the freedom of the i th joint.

The generic mobility of a linkage is the difference between the unconstrained freedom of the links and the number of constraints imposed by the joints, that is,

$$F = (n - 1)K - \sum_{i=1}^j (K - f_i) = K(n - j - 1) + \sum_{i=1}^j f_i. \quad (1.1)$$

This mobility formula places a lower bound on the degree of freedom of a linkage. The mobility may actually be greater due to special dimensions and internal symmetries in the linkage. An interesting example is the spatial 4R closed chain known as Bennett's linkage, which moves with one degree of freedom though this formula predicts that it is a structure with mobility $F = -2$. Such linkages are termed over-constrained (Waldron [141]).

If a linkage consists of a series of links separated by individual joints forming a *serial open chain*, then there is always one more link than the number of joints. This means that $n = j + 1$ and the mobility formula simplifies to

$$F = \sum_{i=1}^j f_i. \quad (1.2)$$

Thus, the mobility of a serial open chain is simply the sum of the freedom at each joint.

We now consider separately the cases of planar, spherical, and spatial linkages.

1.2.1 Planar Linkages

A planar linkage has the property that all of its links move in parallel planes. Most linkages found in practice are, in fact, planar linkages. A body moving in the plane is located by the x and y components of a reference point and a rotation angle ϕ measured relative to ground. Thus, its unconstrained degree of freedom is $K = 3$.

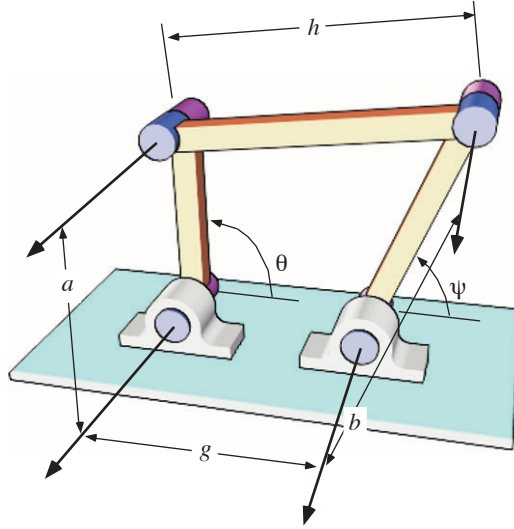


Fig. 1.1 The planar 4R linkage.

The only joints compatible with this movement are revolute joints with axes that are perpendicular to the plane, prismatic joints that move along lines parallel to it, and the direct contact joints of gears and cams that have lines of action parallel to the plane. Of the joints that we are considering, only the R and P joints can be used for planar linkages. Both of these joints have freedom $f = 1$. Thus, the mobility formula for planar linkages is

$$F = 3n - 2j - 3. \quad (1.3)$$

If the links are arranged to form a *single-loop closed chain*, then $j = n$ and the mobility formula becomes

$$F = n - 3. \quad (1.4)$$

It is easy to see that the four-bar linkage, which is a closed chain formed by four links and four joints, has mobility $F = 4 - 3 = 1$. [Figure 1.1](#) is an example of a planar 4R closed chain. Another example is the RRRP slider-crank linkage.

It is interesting to consider how many joints j are needed to constrain a planar assembly of n links to a generic mobility of one. Set $F = 1$ in (1.3) and solve for j to obtain

$$j = \frac{3}{2}n - 2. \quad (1.5)$$

Clearly, n must be even, and we find, for example, that a one-degree-of-freedom six-bar linkage must have $j = 7$ revolute joints. [Figure 1.2](#) shows the two classes of single-degree-of-freedom six-bar linkages.

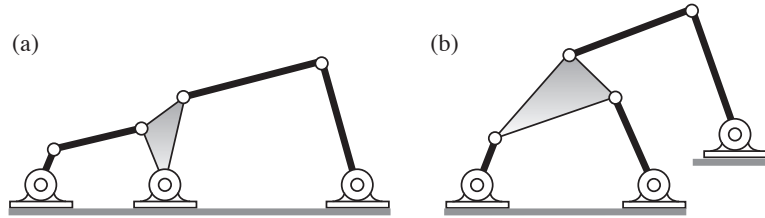


Fig. 1.2 Examples of the two classes of six bar linkages: (a) the Watt six-bar linkage; and (b) the Stephenson six-bar linkage.

1.2.2 Linkage Graphs

A linkage consists of links connected by joints, which can be represented as a graph the links as vertices and the joints as edges. A graph with n vertices and j edges is called an (n, j) graph, and following Tsai [138] we call the associated linkage an (n, j) linkage, usually termed an n -bar linkage.

Figure 1.3 shows a serial chain formed by four bodies including the ground frame which are connected by three revolute joints. Also shown is its linkage graph consisting of four vertices connected by three edges. Two 3R serial chains connected to the same workpiece form a 6R closed chain, Figure 1.4.

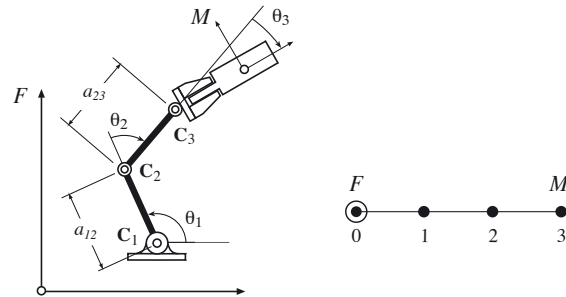


Fig. 1.3 A planar 3R serial chain formed from four links and three revolute joints, and its linkage graph.

The graph of a simple closed loop linkage has the same number of vertices as joints, and it divides the plane into inner and outer regions, or faces, F such that

$$F = j - n + 2. \quad (1.6)$$

This is Euler's equation, which relates the number of faces, edges, and vertices of a polyhedron. This equation applies to linkage graphs that are obtained by the projection of a polyhedron onto a plane.

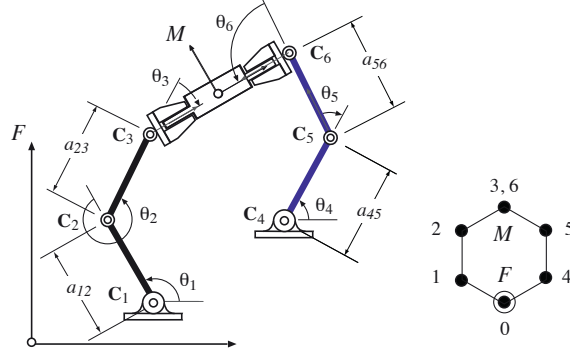


Fig. 1.4 A planar closed chain formed by two 3R serial chains connected to the same workpiece, and its linkage graph.

The faces of a linkage graph are the loops of the linkage. However, the outside face is usually not counted, so the graph is considered to consist of $L = F - 1$ loops. Thus, we obtain the relationship

$$L = j - n + 1. \quad (1.7)$$

We find that the four-, six-, eight- and ten-bar linkages given by $(4, 4)$, $(6, 7)$, $(8, 10)$ and $(10, 13)$ graphs have consecutively one, two, three, and four loops.

The enumeration of the graphs associated with a given set of links and joints is a useful design tool known as *type synthesis*. Tsai [138] presents atlases for the graphs associated with a variety of (n, j) linkages. Table 1.1 lists the number of distinct graphs associated with each of four different linkage types.

Table 1.1 Number of distinct linkage graphs for four-, six-, eight-, and ten-bar linkages.

(n, j)	Loops (L)	Graphs
$(4, 4)$	1	1
$(6, 7)$	2	2
$(8, 10)$	3	16
$(10, 13)$	4	230

A linkage graph can be viewed as a specialized application of constraint graphs used in geometric modeling, Bouma et al. [6]. A constraint graph maps geometric objects to vertices and geometric relationships to edges to provide a versatile tool for characterizing mechanical assemblies. For a four-bar linkage, the links and joints are represented as lines and points and form the vertices of the constraint graph. The distances between points, angles between lines, and the incidence of a point on a line form the edges of the constraint graph, Figure 1.5.

A subgraph of a constraint graph is called a *cluster* if the coordinates of all of its vertices can be computed once one pair of vertices is determined. For example, in Figure 1.5 the vertices $\mathbf{P}_O$, L_{OC} , and $\mathbf{P}_C$ form the cluster C_{OC} . The reduced constraint graph that identifies clusters as vertices and shared geometric elements as edges becomes the linkage graph. This relationship between the constraint graph and the linkage graph provides a way to add dimensional information to the linkage graph, Li et al. [65].

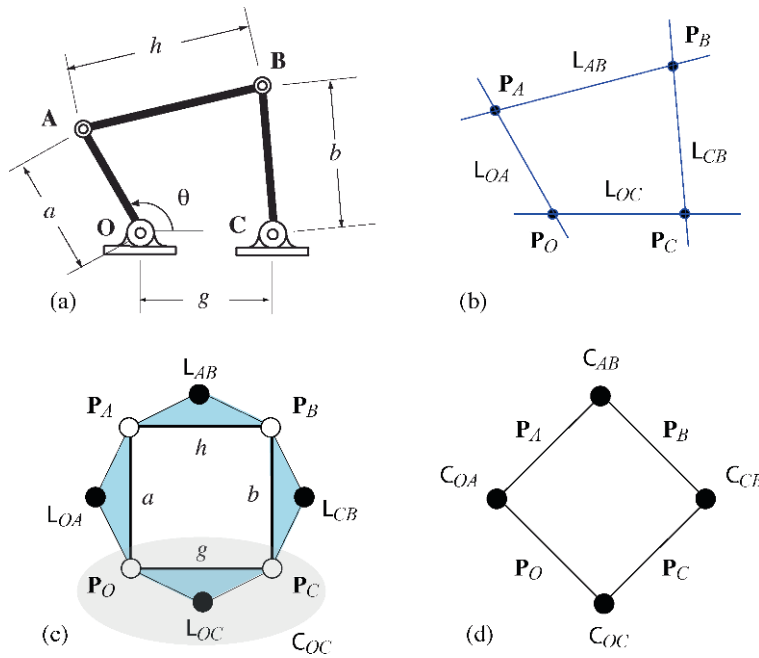


Fig. 1.5 A planar four-bar linkage (a) defined in terms of lines and points (b) is represented by the constraint graph (c). The constraint graph separates into clusters linked by points to become the linkage graph.

1.2.3 Spherical Linkages

Each of the links of a spherical linkage is constrained to rotate about the same fixed point in space, which means that the trajectories of points in each link lie on concentric spheres. The orientation of a link can be defined by three rotation angles often termed *roll*, *pitch*, and *yaw*. For convenience, we visualize yaw as a longitude angle on a globe, and pitch as latitude. Then the roll angle is the rotation about an

axis through a given longitude and latitude. These three parameters are sufficient to define the orientation of a rigid body in space and, as in the plane, $K = 3$.

Only revolute joints assembled so their axes intersect the same fixed point are compatible with the geometric constraint of spatial rotation. Because this joint has freedom $f = 1$, the mobility formula for spherical linkages becomes

$$F = 3n - 2j - 3. \quad (1.8)$$

The spherical four-bar linkage has four revolute joints with axes that radiate from the fixed center point, see Figure 1.6. It has mobility $F = 1$.

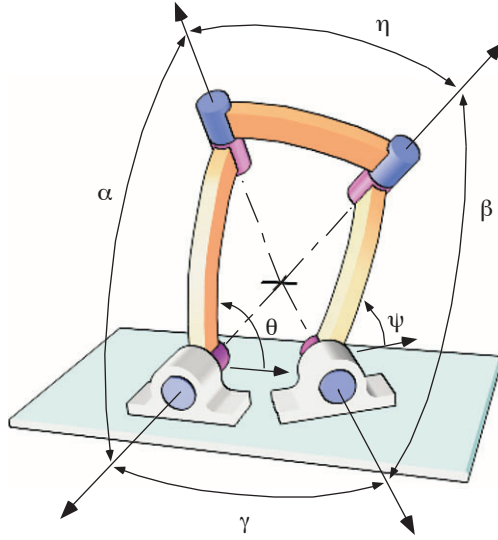


Fig. 1.6 The spherical 4R linkage.

1.2.4 Spatial Linkages

A spatial linkage is characterized by the feature that at least one link in the system moves between two general positions in space. Six parameters are needed to specify the location of this link, three for the coordinates of a reference point and three for the orientation of the body about this point. Therefore, $K = 6$ and we have

$$F = 6(n - j - 1) + \left(\sum_{i=1}^j f_i \right). \quad (1.9)$$

For the case of a single-loop closed chain, where $j = n$, the mobility formula becomes

$$F = \left(\sum_{i=1}^j f_i \right) - 6. \quad (1.10)$$

Thus, a spatial single-loop closed chain must have seven one-degree-of-freedom joints in order to have mobility $F = 1$. Notice that of these seven at most three can be prismatic joints, because these joints constrain only the translational freedom of the body. The spatial four-bar linkage constructed with four cylindric joints, denoted 4C, has mobility $F = 8 - 6 = 2$.

1.2.5 Platform Linkages

An important class of linkages consists of multiple serial chains that have their end-links connected to form a single floating link, or platform. Planar and spherical 4R linkages can each be viewed as platform linkages constructed from a pair of RR open chains. Similarly, the spatial 4C closed chain is obtained by rigidly connecting the end-links of two CC open chains. The 5TS platform linkage consists of a single floating link supported by five separate TS chains (Figure 1.7). The mobility of these systems is easily determined.

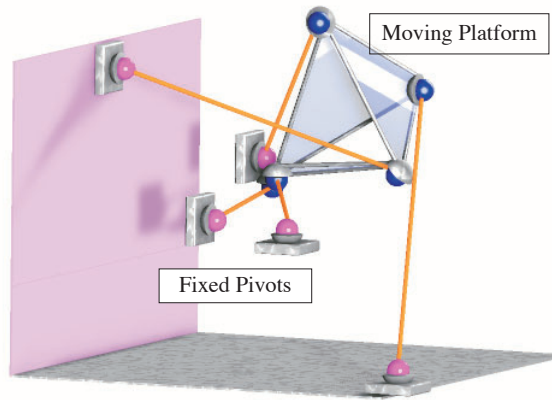


Fig. 1.7 The spatial 5TS platform linkage.

Consider a platform supported by c serial open chains connecting it to ground. Divide the platform among the c chains and let the mobility of the i th chain including its piece of the platform be F_i . Rejoining the end-links of these chains to form a single platform is the same as removing $c - 1$ bodies from the system. Thus, the mobility of a platform manipulator can be calculated to be

$$F = \sum_{i=1}^c F_i - K(c-1) = K - \sum_{i=1}^c (K - F_i). \quad (1.11)$$

A convenient interpretation of this equation is obtained by introducing the *degree of constraint* $U_i = K - F_i$ imposed by the i th open chain. If the mobility of a chain is greater than the unconstrained degree of freedom K , that is, $F_i > K$, then the chain does constrain the platform and $U_i = 0$. Let the degree of constraint of the platform itself be $U = K - F$. Then (1.11) can be written as

$$U = \sum_{i=1}^c U_i. \quad (1.12)$$

The degree of constraint imposed on the platform is the sum of the degrees of constraint of its supporting chains. For example, the TS open chain has five degrees of freedom, therefore it has one degree of constraint. Thus, the 5TS platform has five degrees of constraint, or one degree of freedom.

The formula (1.12) defines the maximum constraint that the supporting chains can impose. Special dimensions and symmetries can reduce this degree of constraint increasing the freedom of movement of the system.

1.3 Workspace

The *workspace* of a robot arm is the set of positions, consisting of both a reference point and the orientation about this point, that are reachable by its end effector. A robot is designed so its end effector has unconstrained freedom of movement within its workspace. However, this workspace does have boundaries, defined in part by the extreme reach allowed by the chain. The shape and size of the workspace for a robot is a primary consideration in its design.

The workspace of a linkage is defined by identifying a specific link as the workpiece. Then the workspace is the set of positions that this workpiece can reach. For serial open chains and platform linkages the dimension of the workspace is exactly the generic mobility F of the system, when $F \leq K$. If the mobility F of the linkage is greater than the unconstrained freedom K , then the system is said to have *redundant* degrees of freedom.

1.4 Linkage Design

Linkage design is often divided into three categories of tasks, called motion generation, function generation, and point-path generation. Our approach, presented in later chapters, is based on techniques developed for motion generation. In this case, it is assumed that the designer has identified positions that represent the desired

movement of a workpiece. This can be viewed as specifying positions that are to lie in the workspace of the linkage. Thus, a discrete representation of the workspace is known, but not the design parameters of the linkage. The constraint equations of the chain evaluated at each of the task positions provide design equations that are solved to determine the linkage.

This approach can also be applied to the design of linkages for function generation. In this case the goal is to coordinate an output crank rotation or slide with a specific input crank rotation or slide. Specialized design procedures have been developed for function generation for various linkage systems. However, we can transform this into a motion generation problem by holding the input link fixed and allowing the ground link to move. By doing this we obtain a set of task positions for the output link relative to the input link, which are defined by the desired coordinated set of input and output joint parameters. We use these positions to design another open chain that connects the input and output links. The result is a closed chain with the desired coordination between the input and output links.

The last category, point-path generation, is a classical problem in linkage design where the primary concern has been the generation of straight-line paths. It is possible to transform a path generation design problem into a motion generation problem by simply adding specifications for the orientation of a reference frame at each location of the moving point. However, this reduces the number of points that can be used to approximate a desired path. In particular, we will see that a planar 4R linkage can be designed to reach as many as five arbitrary task positions. In contrast, it is known that, by ignoring the orientation of the coupler, a four-bar linkage can be designed so the trajectory of point on the coupler passes through as many as nine specified points; see Roth and Freudenstein [108] and Wampler et al. [146]. Unfortunately, this is a difficult design problem even in the plane, and the theory for more general linkage systems has not been developed.

1.4.1 Approximate Straight-Line Mechanisms

One of the interesting problems in classical mechanism theory was the design of a linkage to generate straight-line motion. We take a moment here to present three important planar 4R linkages distinguished by the near straight-line movement of a point on the floating link. Let vertices of the quadrilateral formed by the linkage be labeled $OABC$ in a clockwise manner, such that O and C are the fixed pivots and A and B are the moving pivots; and let the link lengths be labeled $a = |OA|$, $b = |CB|$, $g = |OC|$, and $h = |AB|$.

1. *Watt's linkage*: Let the tracing point P be a distance x along the coupler AB of length h measured from A . If the link lengths OA and BC satisfy the ratio $x/(h-x) = a/b$ and the fixed link has length $g^2 = h^2 + (a+b)^2$, then the point P follows a near straight-line movement over part of its path. See [Figure 1.8](#).
2. *Robert's linkage*: Let the link lengths satisfy the relations $a = b$ and $g = 2h$, and locate the tracing point P to form an isosceles triangle of side $x = a$ with the two

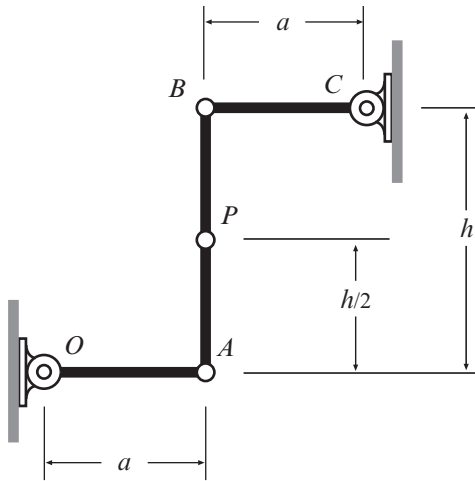


Fig. 1.8 Watt's straight-line linkage: $x = h/2$, $b = a$.

moving pivots. The result is that P moves on a nearly straight line along the part of its path between the two fixed pivots. The height d can be varied to modify this path. See [Figure 1.9](#).

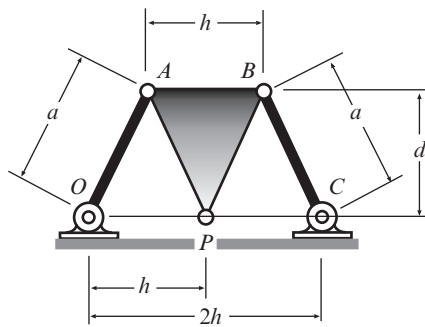


Fig. 1.9 Robert's straight-line linkage.

3. *Chebyshev's linkage*: Chebyshev's linkage has dimensions $g = 2h$ and $a = b = 2.5h$. In this case locate the point P midway between the two moving pivots to trace a near straight-line movement. See [Figure 1.10](#).

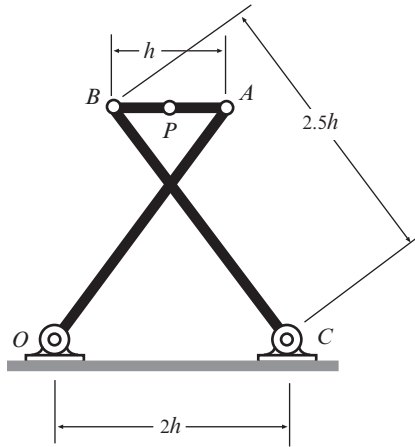


Fig. 1.10 Chebyshev's straight-line linkage: $a = 2.5h$, $b = 2.5h$, $g = 2h$.

1.5 Summary

This chapter has introduced the linkage systems that are the focus of the mathematical theory that follows. Linkages are classified by the type and number of joints that form the assembly. Joints that can be constructed from hinges and sliders to form planar, spherical, and spatial linkages provide a remarkably wide range of devices. The mobility formula defines the generic degree of freedom of a linkage, which is the dimension of its workspace. The size and shape of this workspace characterizes the performance of the device. The goal of the design theory presented in the following chapters is to formulate and solve the constraint equations that ensure that the workspace for a given open chain includes a discrete task space prescribed by the designer.

1.6 References

Dimarogonas [24] and Hartenberg and Denavit [46] present historical accounts of the origins of the theory of machines and mechanisms. The concept of a kinematic chain was introduced by Reuleaux [101] who also identified the central role played by the joints in classifying devices. A systematic notation to define kinematic chains constructed from elementary joints was introduced by Denavit and Hartenberg [22]. Freudenstein and Dobryankyj [36] and Crossley [17] applied graph theory to the problem of identifying all the linkage assemblies available for a certain number of links and type of joint. Crossley [18] described spatial linkages and the larger variety of joints available for these systems. Harrisberger [44] formulated a classification theory for spatial linkages and with Soni [45] enumerated the spatial four-bar link-

ages. See Kota [57] for more information about systematic enumeration of linkage systems.

Tsai[138] shows how graph theory is used in the design of linkages. Dai[19, 20] uses linkage graphs to describe the operation of metamorphic linkages. Li[65] shows that linkage graphs are related to constraint graphs in geometric modeling. See Hoffman[6] for a description of constraint graphs and geometric solvers.

Exercises

1. Determine the degree of freedom of the system formed by two planar 3R manipulators that hold the same workpiece.
2. Determine the degree of freedom of the system formed by a spatial 6R robot that (i) rotates a hinged lever; (ii) rubs a block against a planar surface.
3. Consider a linkage with c independent closed loops, show that $c = j + 1 - n$, and the mobility formula can be written as $F = \sum f_i - 6c$.
4. Show that while the planar and spherical 4R linkages have mobility $F = 1$, the spatial 4R Bennett linkage should not move at all.
5. The mobility formula assumes that a joint is a connection between two links. However, often three or more links are shown connected at one joint. Show that if p links appear connected at one joint, that the mobility formula requires the connection be counted as $p - 1$ joints.
6. Show that if a truss is a linkage with mobility $F = 0$, then its joint forces can be determined by elementary static analysis.
7. Let the links of a mechanism be the vertices of a graph, and its joints the edges. Show that there are only two nonisomorphic graphs of planar six-bar linkages with $F = 1$, the Watt and Stephenson chains.
8. Determine the number of revolute joints that a planar eight-bar linkage must so $F = 1$. Show that there are 16 nonisomorphic graphs for these linkages.
9. Enumerate the $F = 1$ spatial four-bar linkages that can be assembled using R, P, T, S, and C joints.

Chapter 2

Analysis of Planar Linkages

In this chapter we consider assemblies of links that move in parallel planes. Any one of these planes can be used to examine the movement since the trajectories of points in any link can be projected onto this plane without changing their properties. Our focus is on linkages constructed from revolute joints with axes perpendicular to this plane and prismatic joints that move along lines parallel to it. We examine the RR, PR, and RP open chains and the closed chains constructed from them, as well as the 3R and RPR planar robots. We determine the configuration of the linkage as a function of the independent joint parameters and the physical dimensions of the links.

2.1 Coordinate Planar Displacements

A revolute joint in a planar linkage allows rotation about a point, and a prismatic joint allows translation along a line. These movements are represented by transformations of point coordinates in the plane.

Consider the rotation of a link about the revolute joint $\mathbf{O}$ located at the origin of the fixed coordinate frame F . Let $\mathbf{x} = (x, y)^T$ be the coordinates of a point measured in the frame M of the link. If the moving frame has its origin also located at $\mathbf{O}$, and the angle between the x -axes of these two frames is θ , then the coordinates $\mathbf{X} = (X, Y)^T$ of this point in F are given by the matrix equation

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \quad (2.1)$$

We introduce the extra column in this matrix to accommodate translations typical of prismatic joints as part of the matrix operation.

In particular, consider a prismatic joint that has the x -axis of F as its line of action. Let the distance between the origins of M and F along this line be s . Then

we have

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \quad (2.2)$$

A translation along a prismatic joint parallel to the y -axis is defined in the same way as

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \quad (2.3)$$

The matrices in these equations define the three *coordinate displacements* of planar movement. Planar displacements are constructed from these three basic transformations.

We now introduce the notation $[Z(\theta)]$, $[X(s)]$, and $[Y(s)]$ for these coordinate displacements, so we have

$$\mathbf{X} = [Z(\theta)]\mathbf{x}, \quad \mathbf{X} = [X(s)]\mathbf{x}, \quad \text{and} \quad \mathbf{X} = [Y(s)]\mathbf{x}, \quad (2.4)$$

respectively. Notice that we do not distinguish symbolically between the coordinates $\mathbf{X}$ that are two-dimensional and those that have 1 as a third component. Some authors refer to the former as vectors and the latter as affine points. We do not need this general distinction, and therefore will take the time to make the difference clear when needed in the context of our calculations.

2.1.1 The PR Open Chain

The benefit of this matrix formulation can be seen in considering the movement of a PR open chain. This chain consists of a link that slides along the linear guide of a P-joint relative to the ground. An end-link is attached to the slider by a revolute joint, [Figure 2.1](#). We now determine the movement of a coordinate frame M attached to the end-link relative to a fixed frame F .

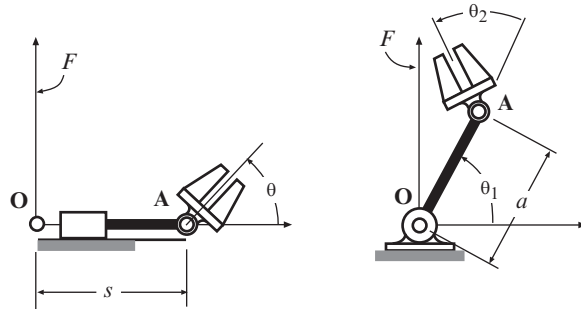


Fig. 2.1 The PR and RR open chain robots.

First, locate F so that its x -axis is parallel to the slide of the P-joint and denote its origin by $\mathbf{O}$. Locate M in the end-link so that its origin is centered on the revolute joint, which we denote by $\mathbf{A}$, and with its x -axis aligned initially with the x -axis of F .

The configuration of the PR chain is defined by the slide s from $\mathbf{O}$ to $\mathbf{A}$, and the rotation angle θ about $\mathbf{O}$ measured from the x -axis of F to the x -axis of M . The transformation of coordinates from M to F is given by the matrix product

$$\mathbf{X} = [X(s)][Z(\theta)]\mathbf{x}, \quad (2.5)$$

or

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & s \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \quad (2.6)$$

The set of planar displacements $[D]$, given by

$$[D] = [X(s)][Z(\theta)], \quad (2.7)$$

is the workspace of the PR open chain. This matrix equation defines the *kinematics equations* for the chain.

An important question in the analysis of an open chain is what parameter values s and θ are needed to reach a given displacement $[D]$ in the workspace of the chain. Assuming the elements of the matrix

$$[D] = \begin{bmatrix} a_{11} & a_{21} & p_x \\ a_{21} & a_{22} & p_y \\ 0 & 0 & 1 \end{bmatrix} \quad (2.8)$$

are known, equation (2.7) can be solved to determine these parameters. Notice that $p_y = 0$ is required for the displacement $[D]$ to be in the workspace reachable by the PR chain. It is now easy to see that s and θ can be determined from the elements of $[D]$ by the formulas

$$s = p_x \quad \text{and} \quad \theta = \arctan \frac{a_{21}}{a_{11}}. \quad (2.9)$$

Its important to note here that the arctan function must keep track of the signs of both a_{21} and a_{11} so the correct value for θ is obtained in the range 0 to 2π .

The arctan function in calculators often incorporates the assumption that the denominator of the fraction a_{21}/a_{11} is positive. If this denominator is actually negative, as occurs when θ is in the second and third quadrants, then π must be added to the angle returned by the calculator in order to obtain the correct result.