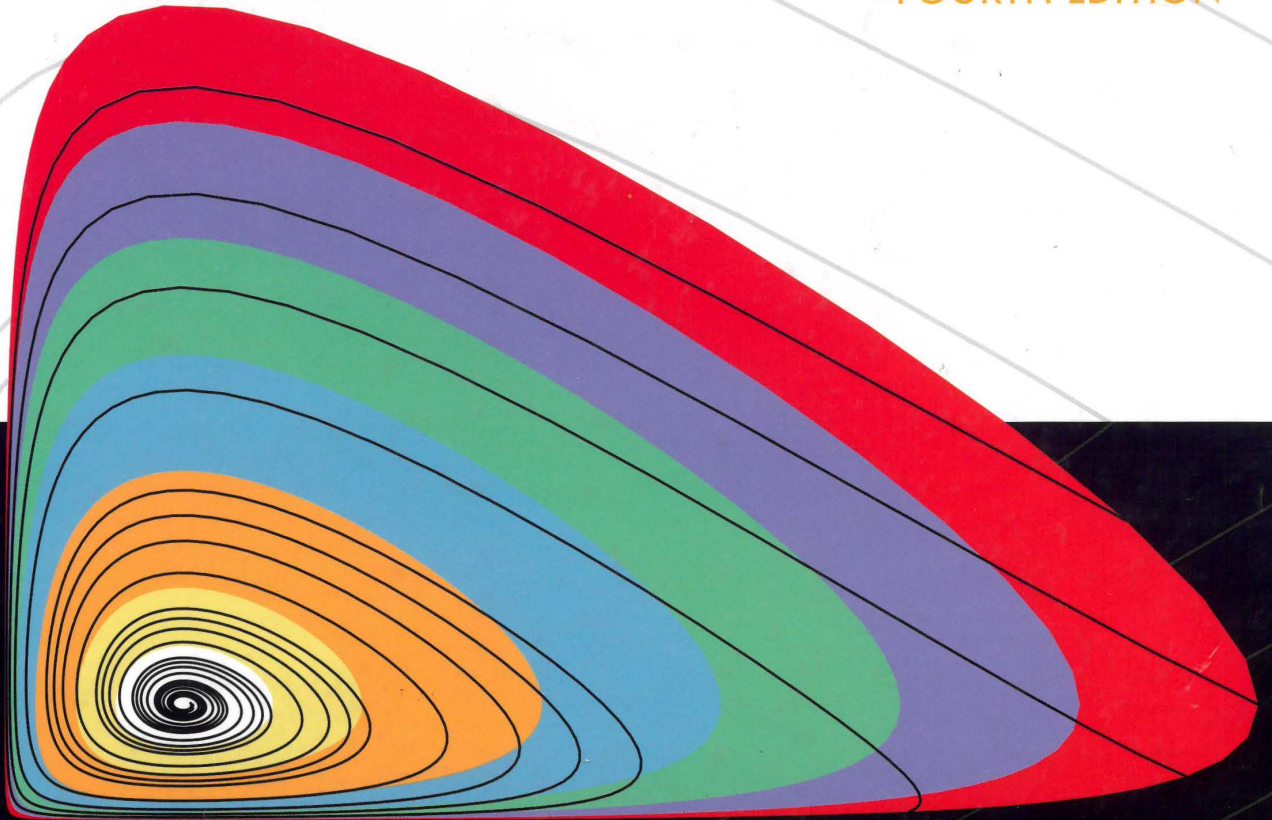


ELEMENTARY DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE PROBLEMS

FOURTH EDITION



C. HENRY EDWARDS
DAVID E. PENNEY



ELEMENTARY DIFFERENTIAL EQUATIONS with Boundary Value Problems

Fourth Edition

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PRENTICE HALL, Upper Saddle River, NJ 07458

Library of Congress Cataloging-in-Publication Data

Edwards, C. H. (Charles Henry).

Elementary differential equations with boundary
value problems / C. Henry Edwards, David E. Penney. -- 4th ed.
p. cm.

Includes bibliographical references and index.

ISBN 0-13-011301-8

1. Differential equations. 2. Boundary value problems.

I. Penney, David E. II. Title.

QA371.E33 2000

515'.35--dc21

99-43376

CIP

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Upper Saddle River, New Jersey 07458

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Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN 0-13-011301-8

Prentice-Hall International (UK) Limited, *London*

Prentice-Hall of Australia Pty. Limited, *Sydney*

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Editora Prentice-Hall do Brasil, Ltda., *Rio de Janeiro*

49. Solve the initial value problem

$$y^{(4)} = y^{(3)} + y'' + y' + 2y;$$

$$y(0) = y'(0) = y''(0) = 0, \quad y^{(3)}(0) = 30.$$

50. The differential equation

$$y'' + (\operatorname{sgn} x)y = 0 \quad (25)$$

has the discontinuous coefficient function

$$\operatorname{sgn} x = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Show that Eq. (25) nevertheless has two linearly independent solutions $y_1(x)$ and $y_2(x)$ defined for all x such that

- Each satisfies Eq. (25) at each point $x \neq 0$;
- Each has a continuous derivative at $x = 0$;
- $y_1(0) = y_2'(0) = 1$ and $y_2(0) = y_1'(0) = 0$.

(Suggestion: Each $y_i(x)$ will be defined by one formula for $x < 0$ and by another for $x \geq 0$.)

2.4 MECHANICAL VIBRATIONS

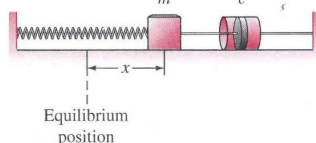


FIGURE 2.4.1. A mass-spring-dashpot system.

The motion of a mass attached to a spring serves as a relatively simple example of the vibrations that occur in more complex mechanical systems. For many such systems, the analysis of these vibrations is a problem in the solution of linear differential equations with constant coefficients.

We consider a body of mass m attached to one end of an ordinary spring that resists compression as well as stretching; the other end of the spring is attached to a fixed wall, as shown in Fig. 2.4.1. Assume that the body rests on a frictionless horizontal plane, so that it can move only back and forth as the spring compresses and stretches. Denote by x the distance of the body from its **equilibrium position**—its position when the spring is unstretched. We take $x > 0$ when the spring is stretched, and thus $x < 0$ when it is compressed.

According to Hooke's law, the restorative force F_S that the spring exerts on the mass is proportional to the distance x that the spring has been stretched or compressed. Because this is the same as the displacement x of the mass m from its equilibrium position, it follows that

$$F_S = -kx. \quad (1)$$

The positive constant of proportionality k is called the **spring constant**. Note that F_S and x have opposite signs: $F_S < 0$ when $x > 0$, $F_S > 0$ when $x < 0$.

Figure 2.4.1 shows the mass attached to a dashpot—a device, like a shock absorber, that provides a force directed opposite to the instantaneous direction of motion of the mass m . We assume the dashpot is so designed that this force F_R is proportional to the velocity $v = dx/dt$ of the mass; that is,

$$F_R = -cv = -c \frac{dx}{dt}. \quad (2)$$

The positive constant c is the **damping constant** of the dashpot. More generally, we may regard Eq. (2) as specifying frictional forces in our system (including air resistance to the motion of m).

If, in addition to the forces F_S and F_R , the mass is subjected to a given **external force** $F_E = F(t)$, then the total force acting on the mass is $F = F_S + F_R + F_E$. Using Newton's law

$$F = ma = m \frac{d^2x}{dt^2} = mx'',$$

we obtain the second-order linear differential equation

$$mx'' + cx' + kx = F(t) \quad (3)$$

that governs the motion of the mass.

If there is no dashpot (and we ignore all frictional forces), then we set $c = 0$ in Eq. (3) and call the motion **undamped**; it is **damped** motion if $c > 0$. If there is no external force we replace $F(t)$ with 0 in Eq. (3). We refer to the motion as **free** in this case and **forced** in the case $F(t) \neq 0$. Thus the homogeneous equation

$$mx'' + cx' + kx = 0 \quad (4)$$

describes free motion of a mass on a spring with dashpot but with no external forces applied. We will defer discussion of forced motion until Section 2.6.

For an alternative example, we might attach the mass to the lower end of a spring that is suspended vertically from a fixed support, as in Fig. 2.4.2. In this case the weight $W = mg$ of the mass would stretch the spring a distance s_0 determined by Eq. (1) with $F_s = -W$ and $x = s_0$. That is, $mg = ks_0$, so that $s_0 = mg/k$. This gives the **static** equilibrium position of the mass. If y denotes the displacement of the mass in motion, measured downward from its static equilibrium position, then we ask you to show in Problem 9 that y satisfies Eq. (3); specifically, that

$$my'' + cy' + ky = F(t) \quad (5)$$

if we include damping and external forces.

The Simple Pendulum

The importance of the differential equation that appears in Eqs. (3) and (5) stems from the fact that it describes the motion of many other simple mechanical systems. For example, a **simple pendulum** consists of a mass m swinging back and forth on the end of a string (or better, a *massless rod*) of length L , as shown in Fig. 2.4.3. We may specify the position of the mass at time t by giving the counterclockwise angle $\theta = \theta(t)$ that the string or rod makes with the vertical at time t . To analyze the motion of the mass m , we will apply the law of the conservation of mechanical energy, according to which the sum of the kinetic energy and the potential energy of m remains constant.

The distance along the circular arc from 0 to m is $s = L\theta$, so the velocity of the mass is $v = ds/dt = L(d\theta/dt)$, and therefore its kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m \left(\frac{ds}{dt} \right)^2 = \frac{1}{2}mL^2 \left(\frac{d\theta}{dt} \right)^2.$$

We next choose as reference point the lowest point O reached by the mass (see Fig. 2.4.3). Then its potential energy V is the product of its weight mg and its vertical height $h = L(1 - \cos \theta)$ above O , so

$$V = mgL(1 - \cos \theta).$$

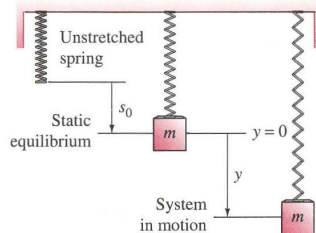


FIGURE 2.4.2. A mass suspended vertically from a spring.

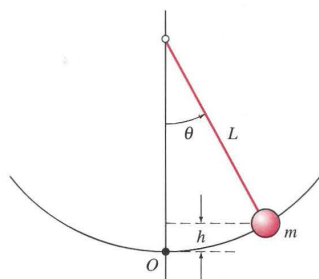


FIGURE 2.4.3. The simple pendulum.

The fact that the sum of T and V is a constant C therefore gives

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1 - \cos\theta) = C.$$

We differentiate both sides of this identity with respect to t to obtain

$$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) + mgL(\sin\theta)\frac{d\theta}{dt} = 0,$$

so

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \quad (6)$$

after removal of the common factor $mL^2(d\theta/dt)$.

Now recall that $\sin\theta \approx \theta$ when θ is small; in fact, $\sin\theta$ and θ agree to two decimal places when $|\theta|$ is at most $\pi/12$ (that is, 15°). In a typical pendulum clock, for example, θ would never exceed 15° . It therefore seems reasonable to simplify our mathematical model of the simple pendulum by replacing $\sin\theta$ with θ in Eq. (6). If we also insert a term $c\theta'$ to account for the frictional resistance of the surrounding medium, the result is an equation in the form of Eq. (4):

$$\theta'' + c\theta' + k\theta = 0, \quad (7)$$

where $k = g/L$. Note that this equation is independent of the mass m on the end of the rod. We might, however, expect the effects of the discrepancy between θ and $\sin\theta$ to accumulate over a period of time, so that Eq. (7) will probably not describe accurately the actual motion of the pendulum over a long period of time.

In the remainder of this section, we first analyze free undamped motion and then free damped motion.

Free Undamped Motion

If we have only a mass on a spring, with neither damping nor external force, then Eq. (3) takes the simpler form

$$mx'' + kx = 0. \quad (8)$$

It is convenient to define

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad (9)$$

and rewrite Eq. (8) as

$$x'' + \omega_0^2 x = 0. \quad (8')$$

The general solution of Eq. (8') is

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t. \quad (10)$$

To analyze the motion described by this solution, we choose constants C and α so that

$$C = \sqrt{A^2 + B^2}, \quad \cos \alpha = \frac{A}{C}, \quad \text{and} \quad \sin \alpha = \frac{B}{C}, \quad (11)$$

as indicated in Fig. 2.4.4. Note that, although $\tan \alpha = B/A$, the angle α is *not* given by the principal branch of the inverse tangent function (which gives values only in the interval $-\pi/2 < x < \pi/2$). Instead, α is the angle between 0 and 2π whose cosine and sine have the signs given in (11), where either A or B or both may be negative. Thus

$$\alpha = \begin{cases} \tan^{-1}(B/A) & \text{if } A > 0, B > 0 \text{ (first quadrant),} \\ \pi + \tan^{-1}(B/A) & \text{if } A < 0 \text{ (second or third quadrant),} \\ 2\pi + \tan^{-1}(B/A) & \text{if } A > 0, B < 0 \text{ (fourth quadrant),} \end{cases}$$

where $\tan^{-1}(B/A)$ is the angle in $(-\pi/2, \pi/2)$ given by a calculator or computer.

In any event, from (10) and (11) we get

$$x(t) = C \left(\frac{A}{C} \cos \omega_0 t + \frac{B}{C} \sin \omega_0 t \right) = C(\cos \alpha \cos \omega_0 t + \sin \alpha \sin \omega_0 t).$$

With the aid of the cosine addition formula, we find that

$$x(t) = C \cos(\omega_0 t - \alpha). \quad (12)$$

Thus the mass oscillates to and fro about its equilibrium position with

Amplitude	C ,
Circular frequency	ω_0 , and
Phase angle	α .

Such motion is called **simple harmonic motion**. A typical graph of $x(t)$ is shown in Fig. 2.4.5. If time t is measured in seconds, the circular frequency ω_0 has dimensions of radians per second (rad/s). The **period** of the motion is the time required for the system to complete one full oscillation, so is given by

$$T = \frac{2\pi}{\omega_0} \quad (13)$$

seconds; its **frequency** is

$$\nu = \frac{1}{T} = \frac{\omega_0}{2\pi} \quad (14)$$

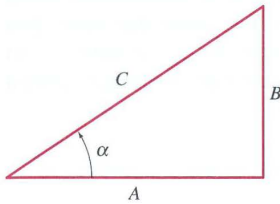


FIGURE 2.4.4. The angle α .

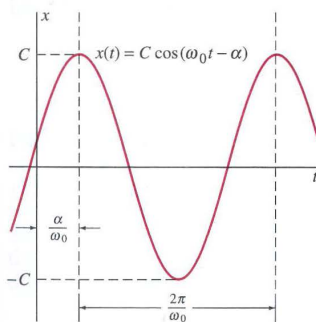


FIGURE 2.4.5. Simple harmonic motion.

in hertz (Hz), which measures the number of complete cycles per second. Note that frequency is measured in cycles per second, whereas circular frequency has the dimensions of radians per second.

If the initial position $x(0) = x_0$ and initial velocity $x'(0) = v_0$ of the mass are given, we first determine the values of the coefficients A and B in Eq. (10), then find the amplitude C and phase angle α by carrying out the transformation of $x(t)$ to the form in Eq. (12), as indicated previously.

EXAMPLE 1 A body with mass $m = \frac{1}{2}$ kilogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force of 100 newtons (N). It is set in motion with initial position $x_0 = 0.5$ (m) and initial velocity $v_0 = -10$ (m/s). (Note that these data indicate that the body is displaced to the right and moving to the left at time $t = 0$.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and phase angle of its motion.

Solution The spring constant is $k = (100 \text{ N})/(2 \text{ m}) = 50 \text{ (N/m)}$, so Eq. (8) yields $\frac{1}{2}x'' + 50x = 0$; that is,

$$x'' + 100x = 0.$$

Consequently, the circular frequency will be $\omega_0 = 10 \text{ (rad/s)}$. So the body will oscillate with

$$\text{frequency: } \frac{10}{2\pi} \approx 1.59 \text{ Hz}$$

and

$$\text{period: } \frac{2\pi}{10} \approx 0.63 \text{ s.}$$

We now impose the initial conditions $x(0) = 0.5$ and $x'(0) = -10$ on the general solution $x(t) = A \cos 10t + B \sin 10t$, and it follows that $A = 0.5$ and $B = -1$. So the position function of the body is

$$x(t) = \frac{1}{2} \cos 10t - \sin 10t.$$

Hence its amplitude of motion is

$$C = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{1}{2}\sqrt{5} \approx 1.12 \text{ (m)}.$$

To find the phase angle, we write

$$x(t) = \frac{\sqrt{5}}{2} \left(\frac{1}{\sqrt{5}} \cos 10t - \frac{2}{\sqrt{5}} \sin 10t \right) = \frac{5}{2} \cos(10t - \alpha).$$

Thus we require

$$\cos \alpha = \frac{1}{\sqrt{5}} > 0 \quad \text{and} \quad \sin \alpha = -\frac{2}{\sqrt{5}} < 0.$$

Hence α is the fourth-quadrant angle

$$\alpha = 2\pi - \tan^{-1} \left(\frac{\frac{2}{3}\sqrt{5}}{\frac{1}{3}\sqrt{5}} \right) \approx 5.1760 \text{ (rad)}.$$

In the form in which the amplitude and phase angle are made explicit, the position function is

$$x(t) \approx \frac{\sqrt{5}}{2} \cos(10t - 5.1760).$$

Free Damped Motion

With damping but no external force, the differential equation we have been studying takes the form $mx'' + cx' + kx = 0$; alternatively,

$$x'' + 2px' + \omega_0^2 x = 0, \quad (15)$$

where $\omega_0 = \sqrt{k/m}$ is the corresponding *undamped* circular frequency and

$$p = \frac{c}{2m} > 0. \quad (16)$$

The characteristic equation $r^2 + 2pr + \omega_0^2 = 0$ of Eq. (15) has roots

$$r_1, r_2 = -p \pm (p^2 - \omega_0^2)^{1/2} \quad (17)$$

that depend on the sign of

$$p^2 - \omega_0^2 = \frac{c^2}{4m^2} - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}.$$

The **critical damping** c_{cr} is given by $c_{cr} = \sqrt{4km}$, and we distinguish three cases, according as $c > c_{cr}$, $c = c_{cr}$, or $c < c_{cr}$.

OVERDAMPED CASE: $c > c_{cr}$ ($c^2 > 4km$). Because c is relatively large in this case, we are dealing with a strong resistance in comparison with a relatively weak spring or a small mass. Then (17) gives distinct real roots r_1 and r_2 , both of which are negative. The position function has the form

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \quad (18)$$

It is easy to see that $x(t) \rightarrow 0$ as $t \rightarrow +\infty$ and that the body settles to its equilibrium position without any oscillations (Problem 27). Figure 2.4.6 shows some typical graphs of the position function for the overdamped case; we chose x_0 a fixed positive number and illustrated the effects of changing the initial velocity v_0 . In every case the would-be oscillations are damped out.

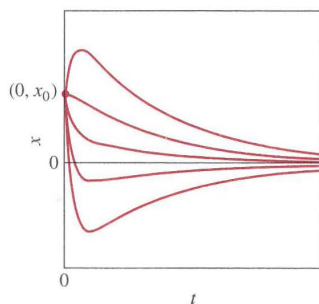


FIGURE 2.4.6. Overdamped motion:
 $x(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t)$
 with $r_1 < 0$ and $r_2 < 0$.
 Solution curves are graphed with the same initial position x_0 and different initial velocities.

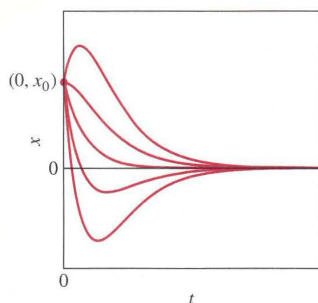


FIGURE 2.4.7. Critically damped motion:

$x(t) = (c_1 + c_2 t)e^{-pt}$ with $p > 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

CRITICALLY DAMPED CASE: $c = c_{cr}$ ($c^2 = 4km$). In this case, (17) gives equal roots $r_1 = r_2 = -p$ of the characteristic equation, so the general solution is

$$x(t) = e^{-pt}(c_1 + c_2 t). \quad (19)$$

Because $e^{-pt} > 0$ and $c_1 + c_2 t$ has at most one positive zero, the body passes through its equilibrium position at most once, and it is clear that $x(t) \rightarrow 0$ as $t \rightarrow +\infty$. Some graphs of the motion in the critically damped case appear in Fig. 2.4.7, and they resemble those of the overdamped case (Fig. 2.4.6). In the critically damped case, the resistance of the dashpot is just large enough to damp out any oscillations, but even a slight decrease in resistance will bring us to the remaining case, the one that shows the most dramatic behavior.

UNDERDAMPED CASE: $c < c_{cr}$ ($c^2 < 4km$). The characteristic equation now has two complex conjugate roots $-p \pm i\sqrt{\omega_0^2 - p^2}$, and the general solution is

$$x(t) = e^{-pt}(A \cos \omega_1 t + B \sin \omega_1 t), \quad (20)$$

where

$$\omega_1 = \sqrt{\omega_0^2 - p^2} = \frac{\sqrt{4km - c^2}}{2m}. \quad (21)$$

Using the cosine addition formula as in the derivation of Eq. (12), we may rewrite Eq. (20) as

$$x(t) = Ce^{-pt} \left(\frac{A}{C} \cos \omega_1 t + \frac{B}{C} \sin \omega_1 t \right),$$

so

$$x(t) = Ce^{-pt} \cos(\omega_1 t - \alpha) \quad (22)$$

where

$$C = \sqrt{A^2 + B^2}, \quad \cos \alpha = \frac{A}{C}, \quad \text{and} \quad \sin \alpha = \frac{B}{C}.$$

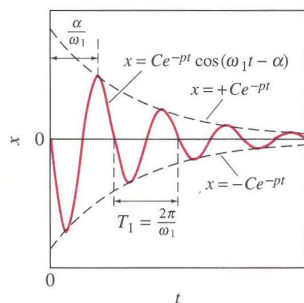


FIGURE 2.4.8. Underdamped oscillations:
 $x(t) = Ce^{-pt} \cos(\omega_1 t - \alpha)$.

The solution in (22) represents exponentially damped oscillations of the body around its equilibrium position. The graph of $x(t)$ lies between the curves $x = -Ce^{-pt}$ and $x = Ce^{-pt}$ and touches them when $\omega_1 t - \alpha$ is an integral multiple of π . The motion is not actually periodic, but it is nevertheless useful to call ω_1 its **circular frequency**, $T_1 = 2\pi/\omega_1$ its **pseudoperiod** of oscillation, and Ce^{-pt} its **time-varying amplitude**. Most of these quantities are shown in the typical graph of underdamped motion shown in Fig. 2.4.8. Note from Eq. (21) that in this case ω_1 is less than the undamped circular frequency ω_0 , so T_1 is larger than the period T of oscillation of the same mass without damping on the same spring. Thus the action of the dashpot has at least three effects:

1. It exponentially damps the oscillations, in accord with the time-varying amplitude.

2. It slows the motion; that is, the dashpot decreases the frequency of the motion.
3. It delays the motion—this is the effect of the phase angle in Eq. (22).

EXAMPLE 2 The mass and spring of Example 1 are now attached also to a dashpot that provides 6 N of resistance for each meter per second of velocity. The mass is set in motion with the same initial position $x(0) = 0.5$ (m) and the same initial velocity $x'(0) = -10$ (m/s). Find the position function of the mass, its new frequency and pseudoperiod, its phase angle, and the amplitudes of its first four (local) maxima and minima.

Solution Rather than memorizing the various formulas given in the preceding discussion, it is better practice in a particular case to set up the differential equation and then solve it directly. Recall that $m = \frac{1}{2}$ and $k = 50$; we are now given $c = 6$ in mks units. Hence Eq. (3) is $\frac{1}{2}x'' + 6x' + 50x = 0$; that is,

$$x'' + 12x' + 100 = 0.$$

The roots of the characteristic equation $r^2 + 12r + 100 = 0$ are

$$r_1, r_2 = \frac{-12 \pm \sqrt{144 - 400}}{2} = -6 \pm 8i,$$

so the general solution is

$$x(t) = e^{-6t}(A \cos 8t + B \sin 8t). \quad (23)$$

The new circular frequency is $\omega_1 = 8$ (rad/s) and the pseudoperiod and new frequency are

$$T_1 = \frac{2\pi}{8} \approx 0.79 \text{ (s)}$$

and

$$\frac{1}{T_1} = \frac{8}{2\pi} \approx 1.27 \text{ (Hz)}$$

(in contrast with 0.63 s and 1.59 Hz, respectively, in the undamped case).

From Eq. (23) we compute

$$x'(t) = e^{-6t}(-8A \sin 8t + 8B \cos 8t) - 6e^{-6t}(A \cos 8t + B \sin 8t).$$

The initial conditions therefore produce the equations

$$x(0) = A = \frac{1}{2} \quad \text{and} \quad x'(0) = -6A + 8B = -10,$$

so $A = \frac{1}{2}$ and $B = -\frac{7}{8}$. Thus

$$x(t) = e^{-6t} \left(\frac{1}{2} \cos 8t - \frac{7}{8} \sin 8t \right),$$

and so with

$$C = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{7}{8}\right)^2} = \frac{1}{8}\sqrt{65}$$

we have

$$x(t) = \frac{\sqrt{65}}{8} e^{-6t} \left(\frac{4}{\sqrt{65}} \cos 8t - \frac{7}{\sqrt{65}} \sin 8t \right).$$

We require

$$\cos \alpha = \frac{4}{\sqrt{65}} > 0 \quad \text{and} \quad \sin \alpha = -\frac{7}{\sqrt{65}} < 0,$$

so α is the fourth-quadrant angle

$$\alpha = 2\pi - \tan^{-1} \left(\frac{7}{4} \right) \approx 5.2315 \text{ (rad)}.$$

Finally,

$$x(t) \approx \frac{\sqrt{65}}{8} e^{-6t} \cos(8t - 5.2315). \quad (24)$$

The local maxima and minima of $x(t)$ occur when

$$0 = x'(t) \approx \frac{\sqrt{65}}{8} [-6e^{-6t} \cos(8t - 5.2315) - 8e^{-6t} \sin(8t - 5.2315)],$$

and thus when

$$\tan(8t - 5.2315) \approx -0.75.$$

Because $\tan^{-1}(-0.75) \approx -0.6435$, we want to find the first four positive values of t such that $8t - 5.2315$ is the sum of -0.6435 and an integral multiple of π . These values of t and the corresponding values of x computed with the aid of (24) are as follows:

t (s)	0.1808	0.5735	0.9662	1.3589
x (m)	-0.2725	0.0258	-0.0024	0.0002

We see that the oscillations are damped out very rapidly, with their amplitude decreasing by a factor of about 10 every half-cycle. See Problems 30 and 31 for a more general discussion of this phenomenon. ■

2.4 Problems

1. Determine the period and frequency of the simple harmonic motion of a 4-kg mass on the end of a spring with spring constant 16 N/m.
2. Determine the period and frequency of the simple harmonic motion of a body of mass 0.75 kg on the end of a spring with spring constant 48 N/m.
3. A mass of 3 kg is attached to the end of a spring that is stretched 20 cm by a force of 15 N. It is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -10$ m/s. Find the amplitude, period, and frequency of the resulting

motion.

4. A body with mass 250 g is attached to the end of a spring that is stretched 25 cm by a force of 9 N. At time $t = 0$ the body is pulled 1 m to the right, stretching the spring, and set in motion with an initial velocity of 5 m/s to the left. (a) Find $x(t)$ in the form $C \cos(\omega_0 t + \alpha)$. (b) Find the amplitude and period of motion of the body.

In Problems 5 through 8, assume that the differential equation of a simple pendulum of length L is $L\theta'' + g\theta = 0$, where $g = GM/R^2$ is the gravitational acceleration at the location of the pendulum (at distance R from the center of the earth; M denotes the mass of the earth).

5. Two pendulums are of lengths L_1 and L_2 and—when located at the respective distances R_1 and R_2 from the center of the earth—have periods p_1 and p_2 . Show that

$$\frac{p_1}{p_2} = \frac{R_1 \sqrt{L_1}}{R_2 \sqrt{L_2}}.$$

6. A certain pendulum keeps perfect time in Paris, where the radius of the earth is $R = 3956$ (mi). But this clock loses 2 min 40 s per day at a location on the equator. Use the result of Problem 5 to find the amount of the equatorial bulge of the earth.
7. A pendulum of length 100.10 in., located at a point at sea level where the radius of the earth is $R = 3960$ (mi), has the same period as does a pendulum of length 100.00 in. atop a nearby mountain. Use the result of Problem 5 to find the height of the mountain.
8. Most grandfather clocks have pendulums with adjustable lengths. One such clock loses 10 min per day when the length of its pendulum is 30 in. With what length pendulum will this clock keep perfect time?
9. Derive Eq. (5) describing the motion of a mass attached to the bottom of a vertically suspended spring. (Suggestion: First denote by $x(t)$ the displacement of the mass below the unstretched position of the spring; set up the differential equation for x . Then substitute $y = x - x_0$ in this differential equation.)
10. Consider a floating cylindrical buoy with radius r , height h , and uniform density $\rho \leq 0.5$ (recall that the density of water is 1 g/cm^3). The buoy is initially suspended at rest with its bottom at the top surface of the water and is released at time $t = 0$. Thereafter it is acted on by two forces: a downward gravitational force equal to its weight $mg = \rho\pi r^2 h g$ and an upward force of buoyancy equal to the weight $\pi r^2 x g$ of water displaced, where $x = x(t)$ is the depth of the bottom of the buoy beneath the surface at time t (Fig. 2.4.9). Conclude that the buoy undergoes simple harmonic motion around its equilibrium position

$x_e = \rho h$ with period $p = 2\pi\sqrt{\rho h/g}$. Compute p and the amplitude of the motion if $\rho = 0.5 \text{ g/cm}^3$, $h = 200 \text{ cm}$, and $g = 980 \text{ cm/s}^2$.

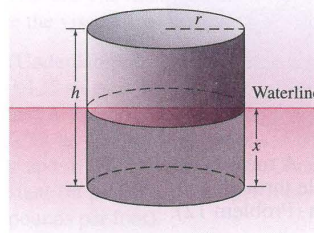


FIGURE 2.4.9. The buoy of Problem 10.

11. A cylindrical buoy weighing 100 lb (thus of mass $m = 3.125$ slugs in ft-lb-s (fps) units) floats in water with its axis vertical (as in Problem 10). When depressed slightly and released, it oscillates up and down four times every 10 s. Assume that friction is negligible. Find the radius of the buoy.
12. Assume that the earth is a solid sphere of uniform density, with mass M and radius $R = 3960$ (mi). For a particle of mass m within the earth at distance r from the center of the earth, the gravitational force attracting m toward the center is $F_r = -GM_r m/r^2$, where M_r is the mass of the part of the earth within a sphere of radius r . (a) Show that $F_r = -GMmr/R^3$. (b) Now suppose that a small hole is drilled straight through the center of the earth, thus connecting two antipodal points on its surface. Let a particle of mass m be dropped at time $t = 0$ into this hole with initial speed zero, and let $r(t)$ be its distance from the center of the earth at time t (Fig. 2.4.10). Conclude from Newton's second law and part (a) that $r''(t) = -k^2 r(t)$, where $k^2 = GM/R^3 = g/R$. (c) Take $g = 32.2 \text{ ft/s}^2$, and conclude from part (b) that the particle undergoes simple harmonic motion back and forth between the ends of the hole, with a period of about 84 min. (d) Look up (or derive) the period of a satellite that just skims the surface of the earth; compare with the result in part (c). How do you explain the coincidence. Or is it a coincidence? (e) With what speed (in miles per hour) does the particle pass through the center of the earth? (f) Look up (or derive) the orbital velocity of a satellite that just skims the surface of the earth; compare with the result in part (e). How do you explain the coincidence? Or is it a coincidence?

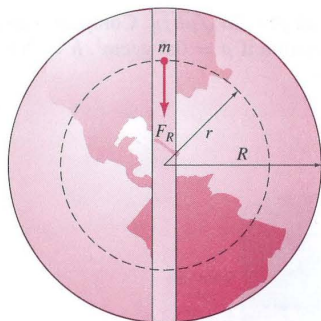


FIGURE 2.4.10. A mass m falling down a hole through the center of the earth (Problem 12).

13. Suppose that the mass in a mass-spring-dashpot system with $m = 10$, $c = 9$, and $k = 2$ is set in motion with $x(0) = 0$ and $x'(0) = 5$. (a) Find the position function $x(t)$ and show that its graph looks as indicated in Fig. 2.4.11. (b) Find how far the mass moves to the right before starting back toward the origin.
14. Suppose that the mass in a mass-spring-dashpot system with $m = 25$, $c = 10$, and $k = 226$ is set in motion with $x(0) = 20$ and $x'(0) = 41$. (a) Find the position function $x(t)$ and show that its graph looks as indicated in Fig. 2.4.12. (b) Find the pseudoperiod of the oscillations and the equations of the “envelope curves” that are dashed in the figure.

The remaining problems in this section deal with free damped motion. In Problems 15 through 21, a mass m is attached to both a spring (with given spring constant k) and a dashpot (with given damping constant c). The mass is set in motion with initial position x_0 and initial velocity v_0 . Find the position function $x(t)$ and determine whether the motion

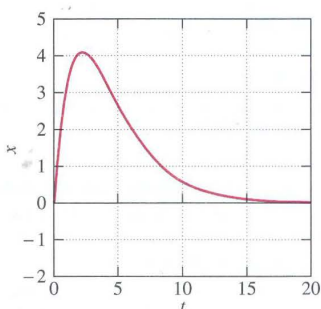


FIGURE 2.4.11. The position function $x(t)$ of Problem 13.

is overdamped, critically damped, or underdamped. If it is underdamped, write $x(t)$ in the form $Ce^{-pt} \cos(\omega_1 t - \alpha)$.

15. $m = \frac{1}{2}$, $c = 3$, $k = 4$; $x_0 = 2$, $v_0 = 0$
16. $m = 3$, $c = 30$, $k = 63$; $x_0 = 2$, $v_0 = 2$
17. $m = 1$, $c = 8$, $k = 16$; $x_0 = 5$, $v_0 = -10$
18. $m = 2$, $c = 12$, $k = 50$; $x_0 = 0$, $v_0 = -8$
19. $m = 4$, $c = 20$, $k = 169$; $x_0 = 4$, $v_0 = 16$
20. $m = 2$, $c = 16$, $k = 40$; $x_0 = 5$, $v_0 = 4$
21. $m = 1$, $c = 10$, $k = 125$; $x_0 = 6$, $v_0 = 50$
22. A 12-lb weight (mass $m = 0.375$ slugs in fps units) is attached both to a vertically suspended spring that it stretches 6 in. and to a dashpot that provides 3 lb of resistance for every foot per second of velocity. (a) If the weight is pulled down 1 ft below its static equilibrium position and then released from rest at time $t = 0$, find its position function $x(t)$. (b) Find the frequency, time-varying amplitude, and phase angle of the motion.
23. This problem deals with a highly simplified model of a car of weight 3200 lb (mass $m = 100$ slugs in fps units). Assume that the suspension system acts like a single spring and its shock absorbers like a single dashpot, so that its vertical vibrations satisfy Eq. (4) with appropriate values of the coefficients. (a) Find the stiffness coefficient k of the spring if the car undergoes free vibrations at 80 cycles per minute (cycles/min) when its shock absorbers are disconnected. (b) With the shock absorbers connected the car is set into vibration by driving it over a bump, and the resulting damped vibrations have a frequency of 78 cycles/min. After how long will the time-varying amplitude be 1% of its initial value?

Problems 24 through 34 deal with a mass-spring-dashpot system having position function $x(t)$ satisfying Eq. (4). We write $x_0 = x(0)$ and $v_0 = x'(0)$ and recall that

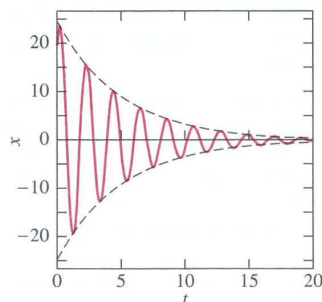


FIGURE 2.4.12. The position function $x(t)$ of Problem 14.

$p = c/(2m)$, $\omega_0^2 = k/m$, and $\omega_1^2 = \omega_0^2 - p^2$. The system is critically damped, overdamped, or underdamped, as specified in each problem.

24. (Critically damped) Show in this case that

$$x(t) = (x_0 + v_0 t + p x_0 t) e^{-pt}.$$

25. (Critically damped) Deduce from Problem 24 that the mass passes through $x = 0$ at some instant $t > 0$ if and only if x_0 and $v_0 + p x_0$ have opposite signs.
26. (Critically damped) Deduce from Problem 24 that $x(t)$ has a local maximum or minimum at some instant $t > 0$ if and only if v_0 and $v_0 + p x_0$ have the same sign.
27. (Overdamped) Show in this case that

$$x(t) = \frac{1}{2\gamma} [(v_0 - r_2 x_0) e^{r_1 t} - (v_0 - r_1 x_0) e^{r_2 t}],$$

where $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$ and $\gamma = (r_1 - r_2)/2 > 0$.

28. (Overdamped) If $x_0 = 0$, deduce from Problem 27 that

$$x(t) = \frac{v_0}{\gamma} e^{-pt} \sinh \gamma t.$$

29. (Overdamped) Prove that in this case the mass can pass through its equilibrium position $x = 0$ at most once.
30. (Underdamped) Show that in this case

$$x(t) = e^{-pt} \left(x_0 \cos \omega_1 t + \frac{v_0 + p x_0}{\omega_1} \sin \omega_1 t \right).$$

31. (Underdamped) If the damping constant c is small in comparison with $\sqrt{8mk}$, apply the binomial series to show that

$$\omega_1 \approx \omega_0 \left(1 - \frac{c^2}{8mk} \right).$$

32. (Underdamped) Show that the local maximum and minima of

$$x(t) = C e^{-pt} \cos(\omega_1 t - \alpha)$$

occur where

$$\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1}.$$

Conclude that $t_2 - t_1 = 2\pi/\omega_1$ if two consecutive maxima occur at times t_1 and t_2 .

33. (Underdamped) Let x_1 and x_2 be two consecutive local maximum values of $x(t)$. Deduce from the result of Problem 32 that

$$\ln \frac{x_1}{x_2} = \frac{2\pi p}{\omega_1}.$$

The constant $\Delta = 2\pi p/\omega_1$ is called the **logarithmic decrement** of the oscillation. Note also that $c = m\omega_1 \Delta/\pi$ because $p = c/(2m)$.

Note: The result of Problem 33 provides an accurate method for measuring the *viscosity* of a fluid, which is an important parameter in fluid dynamics, but which is not easy to measure directly. According to Stokes' drag law, a spherical body of

radius a moving at a (relatively slow) speed through a fluid of viscosity μ experiences a resistive force $F_R = 6\pi\mu av$. Thus if a spherical mass on a spring is immersed in the fluid and set in motion, this drag resistance damps its oscillations with damping constant $c = 6\pi a\mu$. The frequency ω_1 and logarithmic decrement Δ of the oscillations can be measured by direct observation. The final formula in Problem 33 then gives c and hence the viscosity of the fluid.

34. (Underdamped) A body weighing 100 lb (mass $m = 3.125$ slugs in fps units) is oscillating attached to a spring and a dashpot. Its first two maximum displacements of 6.73 in. and 1.46 in. are observed to occur at times 0.34 s and 1.17 s, respectively. Compute the damping constant (in pound-seconds per foot) and spring constant (in pounds per foot).

Differential Equations and Determinism

Given a mass m , a dashpot constant c , and a spring constant k , Theorem 2 of Section 2.1 implies that the equation

$$mx'' + cx' + kx = 0 \quad (25)$$

has a *unique* solution for $t \geq 0$ satisfying given initial conditions $x(0) = x_0$, $x'(0) = v_0$. Thus the future motion of an ideal mass-spring-dashpot system is completely determined by the differential equation and the initial conditions. Of course in a real physical system it is impossible to measure the parameters m , c , and k *precisely*. Problems 35 through 38 explore the resulting uncertainty in predicting the future behavior of a physical system.

35. Suppose that $m = 1$, $c = 2$, and $k = 1$ in Eq. (25). Show that the solution with $x(0) = 0$ and $x'(0) = 1$ is

$$x_1(t) = te^{-t}.$$

36. Suppose that $m = 1$ and $c = 2$ but $k = 1 - 10^{-2n}$. Show that the solution of Eq. (25) with $x(0) = 0$ and $x'(0) = 1$ is

$$x_2(t) = 10^n e^{-t} \sinh 10^{-n} t.$$

37. Suppose that $m = 1$ and $c = 2$ but that $k = 1 + 10^{-2n}$. Show that the solution of Eq. (25) with $x(0) = 0$ and $x'(0) = 1$ is

$$x_3(t) = 10^n e^{-t} \sin 10^{-n} t.$$

38. Whereas the graphs of $x_1(t)$ and $x_2(t)$ resemble those shown in Figs. 2.4.6 and 2.4.7, the graph of $x_3(t)$ exhibits damped oscillations like those illustrated in Fig. 2.4.8, but with a very long pseudoperiod. Nevertheless, show that for each fixed $t > 0$ it is true that

$$\lim_{n \rightarrow \infty} x_2(t) = \lim_{n \rightarrow \infty} x_3(t) = x_1(t).$$

Conclude that *on a given finite time interval* the three solutions are in "practical" agreement if n is sufficiently large.



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