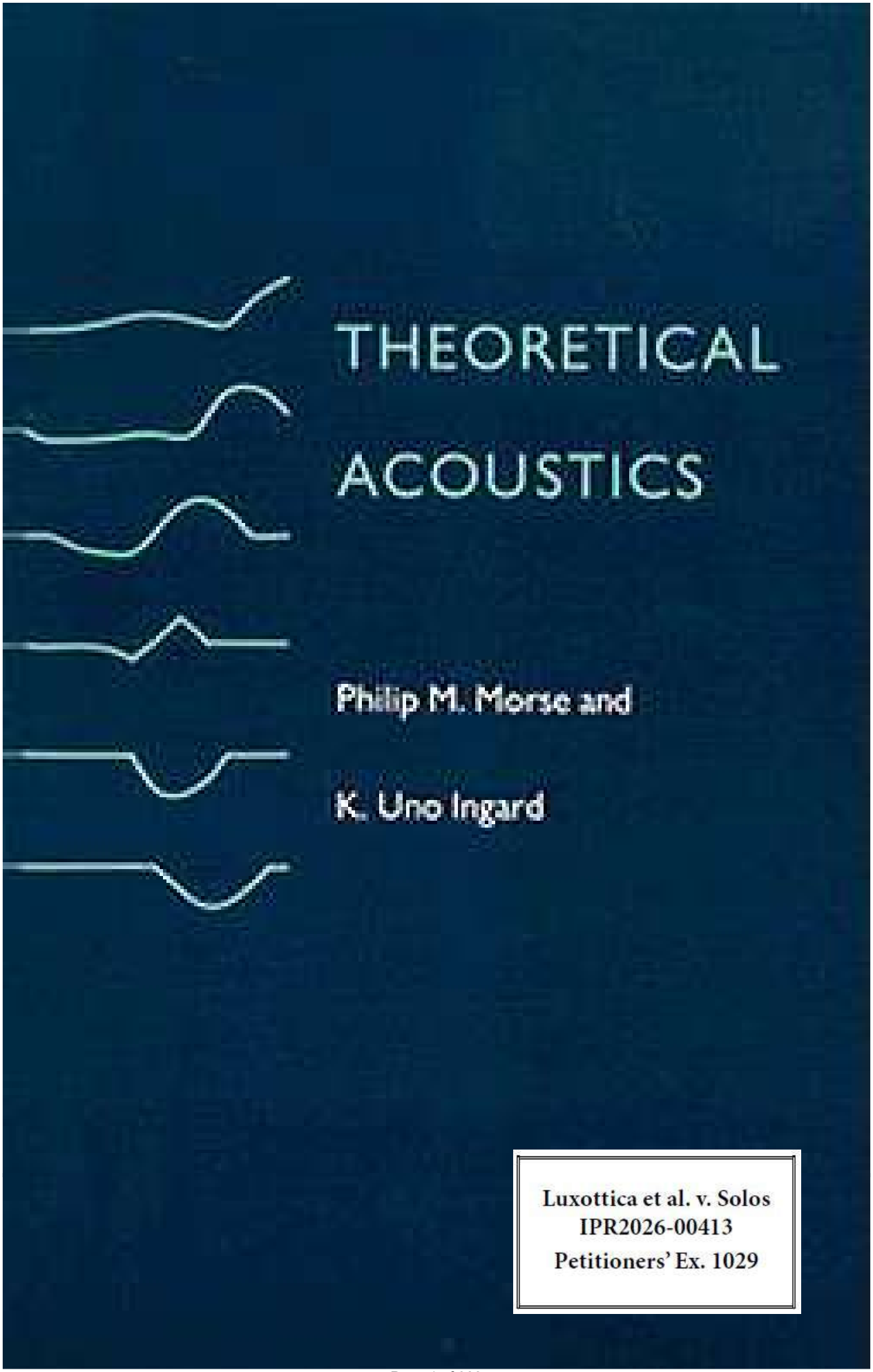


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THEORETICAL ACOUSTICS

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PREFACE

In the thirty years since the textbook "Vibration and Sound" first appeared, the science of acoustics has expanded in many directions. Products of the jet age have added economic incentive to the solution of problems related to the generation and transmission of noise; the behavior of underwater sound now is of commercial as well as of scientific and military importance; new mathematical techniques for the solution of problems involving coupled acoustic systems have been developed and the correspondingly great improvement in acoustical measurement techniques has made the theoretical solutions of more than academic interest. Acoustical measurements are being increasingly used in exploring the properties of matter; the interaction between sound fields and electromagnetic waves is an important part of plasma physics; and magneto-hydrodynamic wave motion is a phenomenon of growing importance in the sciences of meteorology and of astrophysics. The phenomena of acoustics have taken on new importance and significance, both scientifically and technologically.

With these developments in mind, the decisions involved in a revision of "Vibration and Sound" were difficult indeed. The second edition of "Vibration and Sound" is still useful as a fairly elementary text; simple addition of much new material, necessarily more advanced, would destroy its compactness and probably would reduce its utility to beginning students. What has been decided is to write a new book, with a new title. Some of the material in the earlier book has been retained, as introduction to the more advanced portions and to keep the book as self-contained as possible. The result is a graduate, rather than an undergraduate text, and one which we hope will be useful to physical scientists and engineers, who wish to learn about new developments in acoustical theory. As with "Vibration and Sound," each chapter begins with a discussion of the basic physical concepts and the more elementary theory; the later sections can be returned to later for detailed study or for reference.

The choice of new material has had to be limited, of course, by the constraints of size and unity. Experimental equipment and techniques are not discussed; other books cover this field. Specialized aspects of ultrasonics, of underwater sound, and of wave motion in elastic solids are not treated. Coverage is restricted primarily to the subjects of the generation, propagation, absorption, reflection, and scattering of compressional waves in fluid media, in the distortion of such waves by viscous and thermal effects

as well as by solid boundaries, and in their coupling through the induced vibration of walls and transmission panels. New material has been included on the acoustics of moving media, on plasma acoustics, on nonlinear effects, and on the interaction between light and sound. Much of this material has not previously been presented in unified form.

Theoretical techniques, developed for quantum mechanics, field theory, and communications theory, to mention a few, have been adapted to solve these acoustical problems. Autocorrelation functions are introduced to deal with the radiation from randomly vibrating surfaces and with the scattering of sound from turbulence, for example. The Green's function, which enables differential equation and boundary conditions to be combined into an integral equation for the wave motion, is introduced early and is used extensively throughout the rest of the book. Variational techniques are discussed and are used to obtain improved approximations in many complex problems. In all of these subjects we have tried to explain and motivate the theoretical methodology, as well as to use it to elucidate the physics. Problems, illustrating other applications, are given at the end of each chapter. While questions of rigor could not be deeply explored, it is hoped the book will be its own introduction to the more advanced mathematical techniques which have been used. Anyone familiar with calculus and vector analysis should be able to understand the mathematical techniques used here; if he is willing also to spend some time and mental effort, he should be able to master them.

The harmonic oscillator and coupled oscillators are reviewed primarily as a simple means of introducing the concepts of resonance, impedance, transient and steady-state motion, and the techniques of the Fourier transform. Transverse waves on a string are examined in some detail, again as a means of introducing the concepts of wave motion, of the flow of energy and momentum, and as a simple model with which to demonstrate the use of Green's functions and variational methods. There is a short chapter on the vibratory properties of those solid bodies, bars, membranes, and thin plates, which are of importance in later discussions.

The rest of the book is concerned with acoustic wave motion in fluid media. The equations of mass, momentum, and energy flow are discussed in some detail and the interrelations between pressure, density, velocity, temperature, and entropy are examined. The range of validity of the linear wave equation, the origin and consequences of the additional terms present in the Stokes-Navier equation, and the significance of the elements of the stress-energy tensor are developed in a unified manner. Based on these equations, the chapters on radiation, scattering, transmission, and acoustical coupling follow. The final chapters are concerned with more recent ramifications of the theory and with an introductory discussion of nonlinear oscillations and the nonlinear behavior of high intensity sound.

The authors are indebted to Professors Herman Feshbach and Wallace Dean for many suggestions regarding the organization and contents of the textual material. The authors must bear the responsibility for any lapses in clarity and the errors that may remain. We would welcome notice from our readers of such errors they may find, so later printings can be corrected.

Philip M. Morse
K. Uno Ingard

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THEORETICAL ACOUSTICS

CHAPTER I

INTRODUCTORY

1.1 DEFINITIONS AND METHODS

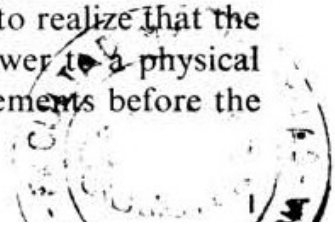
The subject of this book is the application of the methods of classical dynamics to the description of acoustical phenomena. Its object is twofold: to present the salient facts regarding acoustical phenomena, in both descriptive and mathematical form, to students and specialists in the field; and to provide an extended series of illustrations of the way mathematical physics can aid in the understanding of a small facet of nature.

The discussion of any problem in science or engineering has two aspects: the physical side, the statement of the facts of the case in everyday language and of the results in a manner that can be checked by experiment; and the mathematical side, the working out of the intermediate steps by means of the symbolized logic of calculus. These two aspects are equally important and are used side by side in every problem, one checking the other.

The solution of the problems that we shall meet in this book will, in general, involve three steps: the posing of the problem, the intermediate symbolic calculations, and the statement of the answer. The stating of the problem to be solved is not always the easiest part of an investigation. One must decide which properties of the system to be studied are important and which can be neglected, what facts must be given in a quantitative manner and what others need only a qualitative statement. When all these decisions are made for problems of the sort discussed in this book, we can write down a statement somewhat as follows: Such and such a system of bodies is acted on by such and such a set of forces.

We next translate this statement in words into a set of equations and solve the equations (if we can).

The mathematical solution must then be translated back into the physical statement of the answer: If we do such and such to the system in question, it will behave in such and such a manner. It is important to realize that the mathematical solution of a set of equations is not the answer to a physical problem; we must translate the solution into physical statements before the problem is finished.



Units

The physical concept that force causes a change in the motion of a body has its mathematical counterpart in the equation

$$F = \frac{d}{dt}(mv) \tag{1.1.1}$$

In order to link the two aspects of this fact, we must define the physical quantities concerned in a quantitative manner; we must tell how each physical quantity is to be measured and what standard units of measure are to be used. The fundamental quantities, distance, mass, and time, can be measured in any arbitrary units, but for convenience we use those arbitrary units which most of the scientific world is using: the meter, the kilogram, and the second (occasionally, the centimeter, gram, and second). The units of the few other quantities needed, electrical, thermal, etc., will be given when we encounter them.

The units of measure of the other mechanical quantities are defined in terms of these fundamental ones. The equation $F = d(mv)/dt$ is not only the mathematical statement of a physical law, it is also the definition of the unit of measure of a force. It states that the amount of force, measured in *newtons*, equals the rate of change of momentum, in kilogram meters per second per second. If force were measured in other units than newtons, this equation would not be true; an extra numerical factor would have to be placed on one side or the other of the equality sign.

Energy

Another physical concept which we shall often use is that of work, or energy. The wound-up clock spring can exert a force on a gear train for an indefinite length of time if the gears do not move. It is only by motion that the energy inherent in the spring can be expended. The work done by a force on a body equals the distance through which the body is moved by the force times the component of the force in the direction of the motion; and if the force is in newtons and the distance is in meters, the work is given in *joules*. The mathematical statement of this is

$$W = \int \mathbf{F} \cdot \mathbf{ds} \tag{1.1.2}$$

where both the force **F** and the element of distance traveled **ds** are vectors and their scalar product is integrated.

If the force is used to increase the velocity of the body on which it acts, the work that it does is stored up in energy of motion of the body and can be given up later when the body slows down again. This energy of motion is called *kinetic energy*, and when measured in joules, it is equal to $mv^2/2$.

If the force is used to overcome the forces inherent in the system—the "springiness" of a spring, the weight of a body, the pull between two unlike charges, etc.—the work can be done without increasing the body's velocity. If we call the inherent force **F** (a vector), the work done opposing it is $W = -\int \mathbf{F} \cdot \mathbf{ds}$. In general, this amount depends not only on the initial and final position of the body, but also on the particular way in which the body travels between these positions. For instance, because of friction, it takes more work to run an elevator from the basement to the third floor, back to the first, and then to the fourth than it does to run it directly to the fourth floor.

In certain ideal cases where we can neglect friction, the integral W depends only on the position of the end points of the motion, and in this case it is called the *potential energy* V of the body at its final position with respect to its starting point. If the body moves only in one dimension, its position being given by the coordinate x, then

$$V = -\int F\,dx \qquad \text{or} \qquad F = -\frac{dV}{dx} \tag{1.1.3}$$

In such a case we can utilize Newton's equation $F = m(dv/dt)$ to obtain a relation between the body's position and its velocity.

$$V = -\int F\,dx = -m\int \frac{dv}{dt}\,dx = -m\int v\,dv = -\tfrac{1}{2}mv^2 + \text{const}$$

or

$$\tfrac{1}{2}mv^2 + V = \text{const} \tag{1.1.4}$$

This is a mathematical statement of the physical fact that when friction can be neglected, the sum of the kinetic and potential energies of an isolated system is a constant. This proof can be generalized for motion in three dimensions. In case V depends on the three rectangular coordinates x, y, z, the corresponding force $\mathbf{F} = -\text{grad } V$ is a vector, with components

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad \text{and} \qquad F_z = -\frac{\partial V}{\partial z} \tag{1.1.5}$$

1.2 A LITTLE MATHEMATICS

With these definitions given, we can go back to our discussion of the twofold aspect of a physical problem. Let us take an example.

Suppose we have a mass m on the end of a spring. We find that to keep this mass displaced a distance x from its equilibrium position requires a force of Kx newtons. The farther we push it away from its equilibrium position, the harder the spring tries to bring it back. These are the physical facts about the system. The problem is to discuss its motion.

We see physically that the motion must be of an oscillatory nature, but to obtain any quantitative predictions about the motion we must have recourse to the second aspect, the mathematical method. We set up our equation of motion $F = m(d^2x/dt^2)$, using our physical knowledge of the force of the spring $F = -Kx$.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{1.2.1}$$

where ω^2 stands for K/m.

The physical statement corresponding to this equation is that the body's acceleration is always opposite in sign and proportional to its displacement, i.e., is always toward the equilibrium point $x = 0$. As soon as the body goes past this point in one of its swings, it begins to slow down; eventually it stops and then returns to the origin again. It cannot stop at the origin, however, for it cannot begin to slow down until it gets *past* the origin on each swing.

The trigonometric functions

Equation (1.2.1) is a differential equation. Its solution is well known, but since we shall meet more difficult ones later, it is well to examine our method of obtaining its solution, to see what we should do in other cases. We usually state that the solution to Eq. (1.2.1) is

$$x = a_0 \cos(\omega t) + a_1 \sin(\omega t) \tag{1.2.2}$$

but if we had no table of cos or sin, this statement would be of very little help. In fact, the statement that Eq. (1.2.2) is a solution of Eq. (1.2.1) is really only a definition of the symbols cos and sin. We must have more than just symbols in order to compute x for any value of t.

What we do—what is done in solving any differential equation—is to guess the answer and then see if it checks. In most cases the guess was made long ago and the solution is familiar to us, but the guess had nevertheless to be made.

So we shall guess the solution to Eq. (1.2.1). We shall make a pretty general sort of guess, that x can be expanded in a power series in t, and then see if some choice of the coefficients of the series will satisfy the equation. We set

$$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots = \sum_{n=0}^{\infty} a_n t^n$$

where the symbol Σ is shorthand for a sum of terms of the sort shown after the Σ sign, the limits of the sum being indicated below and above the Σ sign. For example,

$$\sum_{n=M}^{N} a_n f[nx] = a_M f[Mx] + a_{M+1} f[(M+1)x] + \cdots + a_{N-1} f[(N-1)x] + a_N f[Nx]$$

For this series, for x to be a solution of Eq. (1.2.1) we must have that d^2x/dt^2, which equals $2a_2 + 6a_3t + 12a_4t^2 + 20a_5t^3 + \cdots$, plus $\omega^2 x$, which equals $\omega^2 a_0 + \omega^2 a_1 t + \omega^2 a_2 t^2 + \omega^2 a_3 t^3 + \cdots$, equals zero for every value of t. The only way this can be done is to have the sums of the coefficients of each power of t each separately equal to zero, i.e., to have $a_2 = -\omega^2 a_0/2$, $a_3 = -\omega^2 a_1/6$, $a_4 = -\omega^2 a_2/12 = \omega^4 a_0/24$, etc. Therefore the series that satisfies Eq. (1.2.1) is

$$x = a_0\left(1 - \frac{\omega^2 t^2}{2} + \frac{\omega^4 t^4}{24} - \frac{\omega^6 t^6}{720} + \cdots\right) + a_1\left(t - \frac{\omega^2 t^3}{6} + \frac{\omega^4 t^5}{120} - \frac{\omega^6 t^7}{5{,}040} + \cdots\right)$$

By comparing this with (1.2.2) we see that what we actually mean by the symbols cos, sin is

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\ \sin z &= z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{5{,}040} + \cdots \end{aligned} \tag{1.2.3}$$

(in the problem $z = \omega t$) and that when we wish to compute values of cos or sin, we use the series expansion to obtain them. For instance,

$$\begin{aligned} &\cos 0 = 1 - 0 + 0 - \cdots = 1 \\ &\cos \tfrac{1}{2} = 1 - 0.125 + 0.003 - \cdots = 0.878 \\ &\cos 1 = 1 - 0.5 + 0.042 - 0.002 + \cdots = 0.540 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

We see that the mathematical solution involves certain arbitrary constants a_0 and a_1. These must be fixed by the physical conditions of the particular experiment we have made, and will be discussed later (Sec. 1.3).

Of course, we usually mean by the symbols cos z, sin z certain ratios between the sides of a right triangle whose oblique angle is z. To make our discussion complete, we must show that the trigonometric definitions correspond to the series given above. In any book on elementary calculus it is shown that the trigonometric functions obey the following relations:

$$\frac{d}{dz}\cos z = -\sin z \qquad \frac{d}{dz}\sin z = \cos z$$

$$\cos 0 = 1 \qquad \sin 0 = 0$$

By combining the first two of these relations, we see that both the sine and cosine of trigonometry obey the equation $d^2y/dz^2 = -y$, which is equivalent to (1.2.1). Taking into account the third and fourth relations, we see that the series solutions of this equation which correspond to the two trigonometric functions must be the ones given in (1.2.3).

Once this is known, we can utilize the trigonometric properties of the sine and cosine to simplify our discussion of the solution (1.2.2) [although, when it comes to computing values, the series (1.2.3) are always used]. For instance, we can say that x is a periodic function of time, repeating its motion every time that ωt increases by 2π or every time t increases by $(2\pi/\omega)$. The value of $2\pi/\omega$ is called the *period* of oscillation of the mass and is denoted by the symbol T_p. The number of periods per second, $\omega/2\pi$, is called the *frequency* of oscillation and is denoted by ν. The quantity $\omega = 2\pi\nu$, which equals the rate of increase of the angle ωt in the trigonometric functions $\cos(\omega t)$ and $\sin(\omega t)$, is called the *angular velocity* of the motion. Remembering the definition of ω, we see that

$$T_p = 2\pi\sqrt{\frac{m}{K}} \qquad \nu = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

Bessel functions

More complicated differential equations can be solved by means of the same sort of guess that we used above. For instance, we can solve the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{1}{x^2}\right)y = 0$$

by again guessing that $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ and setting this guess in the equation. Thus

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$\text{minus} \quad \frac{1}{x^2}y = -\frac{a_0}{x^2} - \frac{a_1}{x} - a_2 - a_3x - a_4x^2 - a_5x^3 - \cdots$$

$$\text{plus} \quad \frac{1}{x}\frac{dy}{dx} = \frac{a_1}{x} + 2a_2 + 3a_3x + 4a_4x^2 + 5a_5x^3 + \cdots$$

$$\text{plus} \quad \frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$$

must equal zero. Equating coefficients of powers of x to zero as before, we have

$$a_0 = 0 \qquad a_2 = -\frac{a_0}{3} = 0 \qquad a_3 = -\frac{a_1}{8}$$

$$a_4 = 0 \qquad a_5 = -\frac{a_3}{24} = \frac{a_1}{8 \cdot 24} \cdots$$

Therefore

$$y = a_1\left(x - \frac{x^3}{2 \cdot 4} + \frac{x^5}{2 \cdot 4 \cdot 4 \cdot 6} - \frac{x^7}{2 \cdot 4 \cdot 6 \cdot 4 \cdot 6 \cdot 8} \cdots\right)$$

We shall call this series a *Bessel function*, just as we call each of the series in (1.2.3) a trigonometric function. To save having to write the series out every time, we shall give it a symbol. We let

$$J_1(x) = \frac{1}{2}\left(x - \frac{x^3}{2 \cdot 4} + \frac{x^5}{2 \cdot 4 \cdot 4 \cdot 6} - \cdots\right)$$

just as we represent series (1.2.3) by the symbols cos and sin.

Essentially, this series is no more complicated than series (1.2.3). We can compute values of $J_1(x)$ in the same manner as we computed values of $\cos z$.

$$J_1(0) = 0 - \cdots = 0$$
$$J_1(\tfrac{1}{2}) = 0.250 - 0.008 + \cdots = 0.242$$
$$J_1(1) = 0.5 - 0.063 + 0.003 - \cdots = 0.440$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Once someone has obtained this series, given us the symbol $J_1(x)$, and computed its values, we can use the symbol with as great freedom as we use the symbol $\cos z$. We can say that the solution of (1.2.4) is $y = AJ_1(x)$, where the arbitrary constant A is to be determined by the physical conditions of the problem, as we shall discuss later.

There are other Bessel functions, in addition to the one we have been discussing. The general equation they satisfy,

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right)y = 0 \qquad (1.2.4)$$

is called the Bessel equation. The solution which is finite at $x = 0$ can be represented in terms of the series

$$J_n(x) = \frac{(\frac{1}{2}x)^n}{n!} - \frac{(\frac{1}{2}x)^{n+2}}{1!\,(n+1)!} + \frac{(\frac{1}{2}x)^{n+4}}{2!\,(n+2)!} - \cdots \qquad (1.2.5)$$

which is called the nth-order Bessel function of x. The solution $J_1(x)$, worked out in detail, is thus the first-order Bessel function.

Addition and differentiation of the series solution will show that there are two fairly simple relationships between Bessel functions of consecutive order,

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \qquad J_{n-1}(x) - J_{n+1}(x) = 2\frac{d}{dx}J_n(x)$$

or (1.2.6)

$$J_{n-1}(x) = \frac{d}{dx}J_n(x) + \frac{n}{x}J_n(x) \qquad J_{n+1}(x) = -\frac{d}{dx}J_n(x) + \frac{n}{x}J_n(x)$$

These are called *recursion formulas*. Together they correspond to the differential equation (1.2.4) for $J_n(x)$; successive use of Eqs. (1.2.6) results in

$$\frac{d^2}{dx^2}J_n = \frac{1}{2}\frac{d}{dx}J_{n-1} - \frac{1}{2}\frac{d}{dx}J_{n+1} = -J_n + \frac{n-1}{2x}J_{n-1} + \frac{n+1}{2x}J_{n+1}$$

$$= -J_n + \left[\frac{n-1}{2x} - \frac{n+1}{2x}\right]\frac{d}{dx}J_n + \left[\frac{n(n-1)}{x^2} + \frac{n(n+1)}{x^2}\right]J_n$$

which reduces to the equation for J_n. The recursion formulas also reflect the choice of the coefficient multiplying the x^n term in the series representation of Eq. (1.2.5).

To have these formulas apply for negative values of n as well as for positive values, we must define the functions for negative order:

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{for } n \text{ an integer}$$

The validity of the following useful formulas can be demonstrated by differentiation of the series and application of Eqs. (1.2.6).

$$\int J_1(x)\,dx = J_0(x) \qquad \int J_n(x)\,dx = 2\sum_{m=0}^{\infty} J_{n+2m+1}(x)$$

$$1 = \sum_{m=-\infty}^{\infty} J_{2m}(x) = J_0(x) + 2\sum_{m=1}^{\infty} J_{2m}(x)$$

$$x = 2\sum_{m=0}^{\infty}(2m+1)J_{2m+1}(x)$$

So far we have been talking about one solution of Eq. (1.2.4); since it is a second-order equation, there must be a second solution [just as Eq. (1.2.1) has two solutions]. In the case of the Bessel equation, however, the second solution, which is usually denoted by $N_n(x)$, becomes infinite at $x = 0$. Its properties will be taken up later. We can say here, however, that the complete solution of Eq. (1.2.4) is $AJ_m(x) + BN_m(x)$, where A and B are arbitrary constants.

Complex quantities

Another very useful way of dealing with the solution of (1.2.1) can be obtained by the following chain of reasoning: We utilize the series method to show that a solution of the equation $dy/dz = y$ is the series called the *exponential:*

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots$$

sometimes written exp z. By repeated differentiation we see that $x = Ce^{at}$ is a solution of the equation $(d^2x/dt^2) - a^2x = 0$. If a^2 were to equal $-\omega^2$, this equation would be the same as (1.2.1). Therefore we can say that a solution of (1.2.1) is $Ce^{-i\omega t}$, where $i = \sqrt{-1}$ (we could also use $e^{+i\omega t}$).

The function e^{-iz} is a complex number, with real and imaginary parts, and can be represented in the usual manner by a point on the "complex plane" whose abscissa is the real part of the function and whose ordinate is the imaginary part. It is also represented by the vector drawn from the origin to this point in the complex plane.

If we expand e^{-iz} in its series form,

$$e^{-iz} = \left(1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right) - i\left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)$$

we see immediately that the relation between the imaginary exponential and the trigonometric functions is

$$\begin{aligned} e^{-iz} &= \cos z - i \sin z \\ \cos z &= \tfrac{1}{2}(e^{iz} + e^{-iz}) \\ \sin z &= -\tfrac{1}{2}i(e^{iz} - e^{-iz}) \end{aligned} \tag{1.2.7}$$

This shows that the number e^{-iz} can be represented on the complex plane (Fig. 1.1) by a vector of unit length inclined at an angle $-z$ with respect to the real axis. It also shows that the complex number $a + ib$ can be expressed as $Ae^{i\Phi}$, where $A = |a + ib| = \sqrt{a^2 + b^2}$ is called the *magnitude* of $a + ib$, and Φ is called its *phase*. For $Ae^{i\Phi} = A\cos\Phi + iA\sin\Phi$, and since $\cos\Phi = a/A$ and $\sin\Phi = b/A$, we have $Ae^{i\Phi} = a + ib$. The number $a - ib = Ae^{-i\Phi}$ is called the *complex conjugate* of $a + ib$ and is denoted by an asterisk, $(a + ib)^*$. Thus, if $C = a + ib$ is a complex number, its magnitude is the square root of $C \cdot C^* = |C|^2 = A^2 = a^2 + b^2$. The factor $e^{i\Phi}$ signifies a rotation of the vector A through an angle Φ in the complex plane.

Thus we see that $Ce^{-i\omega t}$ is a solution of Eq. (1.2.1) and can be expressed as a combination of $\cos(\omega t)$ and $\sin(\omega t)$, as is demanded by Eq. (1.2.2). But this new solution is a complex number, and the results of physics are, in general, real numbers; so the new solution would seem to be of little value to us. The compactness and ease of manipulation of the complex notation, however, impel us to find a convention which will enable us to express physical quantities in terms of complex numbers. Two conventions are possible.

One convention is to write down the solution as being $Ce^{-i\omega t}$ and agree that we use only the *real part* of this solution as a representation of the physical quantity. The other convention is somewhat more sophisticated, but in the end somewhat more convenient; it is to consider energy and other quadratic functions of the variables as the basic physical quantities and require that they be real quantities in the convention. Since the square of the magnitude of a complex number $|C|^2 = A^2 = a^2 + b^2$ is always real and nonnegative, we can make the convention that when a complex number C is to be squared, we multiply by its complex conjugate $C^* = a - ib$. This latter convention is particularly convenient for quantities which depend on

time through the factor $e^{-i\omega t}$, for then the square of the magnitude of the quantity does not contain the exponential factor, and is thus independent of time. We postpone details as to how this second convention works until Chap. 6 [Eq. (6.2.18)]; by that time sufficient background material will have been presented so that we can see how to use it.

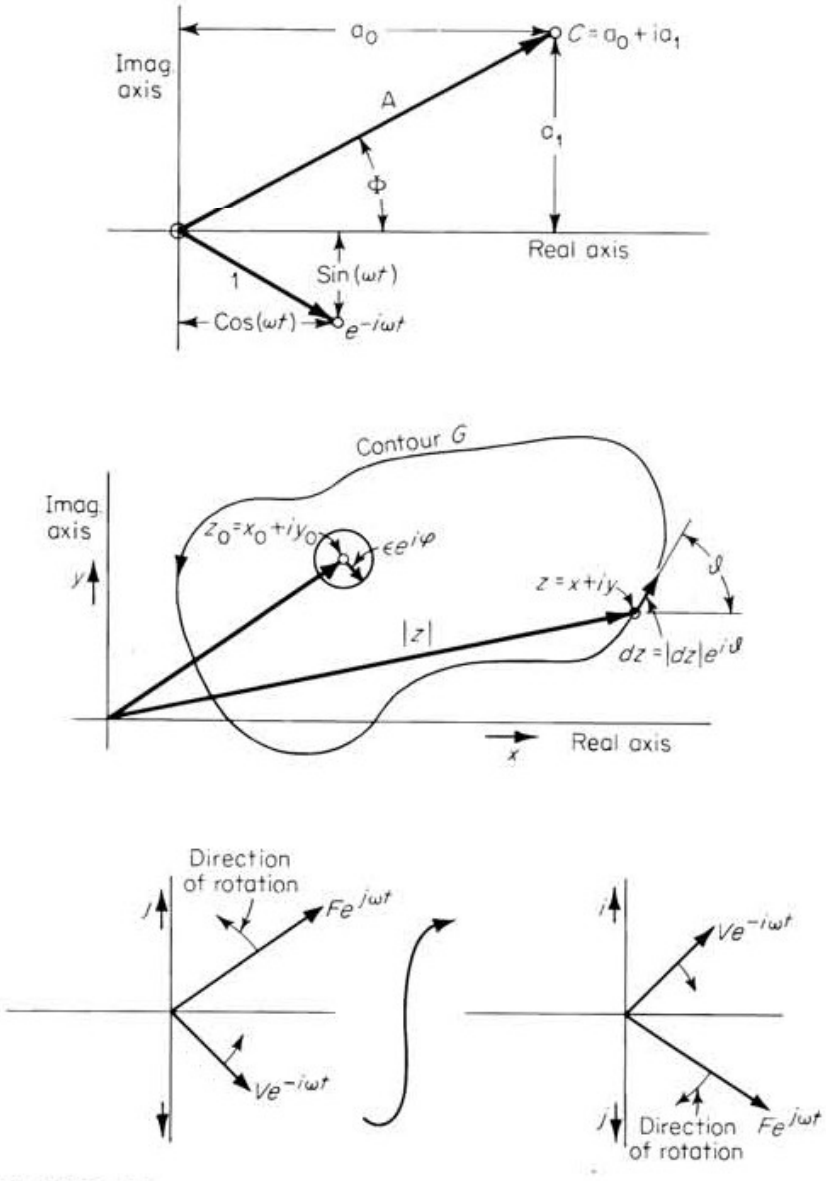


FIGURE 1.1
Representation of complex numbers on the complex plane. Integration around a contour G in the complex plane. Relation between i and j.

Until Chap. 6, therefore, we shall use the former convention, that the real part of the complex quantity has physical significance. It is possible to do this with the solution of any *linear* differential equation (i.e., equations containing only the first power of the unknown function and its derivatives); for if a complex function is a solution of a linear differential equation, both

its real and imaginary parts by themselves are also solutions. (Why is this not true for nonlinear differential equations?) We could, of course, make a convention that we use only the imaginary part of the function, for the imaginary part is also a solution of the equation, but the usual convention is to take the real part.

We can therefore express solution (1.2.2) as the real part of

$$x = Ce^{-i\omega t} \qquad C = a_0 + ia_1$$

This can be checked by the use of (1.2.7), for

$$Ce^{-i\omega t} = a_0 \cos(\omega t) + ia_1 \cos(\omega t) - ia_0 \sin(\omega t) + a_1 \sin(\omega t)$$

and the real part of this corresponds to (1.2.2). The advantages of this method will become apparent as we use it. For instance, since $C = Ae^{i\Phi}$, $A^2 = a_0^2 + a_1^2, \tan(\Phi) = a_1/a_0$, we see that the real part of $Ce^{-i\omega t} = Ae^{-i(\omega t - \Phi)}$ is $A \cos(\omega t + \Phi)$. Therefore we can express the solution of (1.2.1) in any of four ways:

$$x = a_0 \cos(\omega t) + a_1 \sin(\omega t) = A \cos(\omega t - \Phi)$$

or

$$x = Ce^{-i\omega t} = Ae^{-i(\omega t - \Phi)} \tag{1.2.8}$$

where our present convention requires that we take only the real part of the last expressions. The constant A is called the *amplitude* of oscillation of the mass, since it is the value of the maximum displacement of the mass from equilibrium.

This convention as to the use of complex quantities must be applied with discretion when we come to compute powers and energies, where squares of quantities enter. For instance, in a given problem the power radiated may turn out to be Rv^2, where v is the velocity of a diaphragm and R is a real number. The velocity may be represented by the expression $v = Ce^{-2\pi i\nu t}$, $C = a_0 + ia_1$, just as x is in Eq. (1.2.8). But to compute the average power radiated, we must take the real part of the expression before squaring and averaging,

$$Rv^2 = RA^2 \cos^2(2\pi\nu t - \Phi)$$

according to Eq. (1.2.8). Since the average value of $\cos^2$ is $\frac{1}{2}$, the average power radiated will be

$$\tfrac{1}{2}RA^2 = \tfrac{1}{2}R(a_0^2 + a_1^2) = \tfrac{1}{2}R\,|C|^2 = \tfrac{1}{2}R\,|v|^2$$

$$|v|^2 = (\text{real part of } v)^2 + (\text{imaginary part of } v)^2$$

If $z = x + iy$ is any complex number, the quantity $|z|$ is called the *magnitude* of z. It is the distance from the origin to the point in the complex plane corresponding to z, and it equals the square root of the sum of the squares of the real and imaginary parts of z. Therefore another important rule concerning the use of complex quantities in physical problems is the following: The *average value of the square* of a quantity represented by the

complex function $Ce^{-i\omega t}$ is equal to *one-half of the square of the magnitude* of C.

The angle Φ in Eq. (1.2.8) is called the *phase angle* of the complex quantity C, since it measures the angle of lag of the quantity $Ce^{-i\omega t}$ behind the simple exponential $e^{-i\omega t}$. We notice that the phase angle of i is 90°, that of -1 is 180°, and that of $-i$ is 270°, or 90° lead. If $x = Ce^{-i\omega t}$, the velocity $v = dx/dt = -i\omega Ce^{-i\omega t}$ leads the displacement x by 90°, as is indicated by the fact that $v = (-i)\omega x$.

In the present volume we use the letter i to stand for $\sqrt{-1}$ and the symbol $e^{-i\omega t}$ to express simple-harmonic dependence on time. Many books on electrical engineering use the symbol j instead of i and $e^{j\omega t}$ with the positive sign in the exponential. Since we intend to take the real part of the result, the choice of symbol and sign is but a convention; either convention is satisfactory if we are consistent about it.

As long as we are studying simple systems, with displacements a function only of time, there is little to choose between the two conventions; as a matter of fact, the positive exponential $e^{j\omega t}$ would be slightly preferable. However, as soon as we come to study wave motion, with displacement a function of position as well as of time, it turns out that the form involving the negative exponential $e^{-i\omega t}$ is rather more satisfactory than the positive. This will become apparent later.

Since we are going to deal with problems of wave motion in this book, we shall consistently use the negative exponential $e^{-i\omega t}$. And since this convention differs from the positive exponential used in most electrical engineering books, we shall use the symbol i instead of j, to emphasize the difference. It will, however, always be possible to *transform any of the formulas developed in this book into the usual electrical engineering notation by replacing every i in the formulas by $-j$.* We might, if we wish, consider i and j as the two roots of -1, so that

$$i = -j \qquad (i)^2 = (j)^2 = -1$$

In this notation, the impedance of a circuit with resistance and inductance in series will be $R - i\omega L$.

Another example of the interconnections between the various functions we have been describing, and how they can be extended by the use of the complex plane, is the relationship between the exponential function and the Bessel functions of Eq. (1.2.5). By expanding the exponential in a power series in z and t, and then by equating coefficients of the various powers of t, we can show that

$$\exp\left[\tfrac{1}{2}z\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(z) = J_0(z) + \sum_{n=1}^{\infty}\left[t^n + \left(\frac{-1}{t}\right)^n\right]J_n(z)$$

Next, by inserting $ie^{i\theta}$ for t on both sides of this equation, and by using Eq.

(1.2.7), we obtain a relationship between the exponential, the trigonometric, and the Bessel functions,

$$e^{iz\cos\theta} = J_0(z) + 2\sum_{n=1}^{n} i^n \cos(n\theta) J_n(z) \tag{1.2.9}$$

which will be of considerable use later.

Incidentally, for future reference, we should point out that the *hyperbolic* functions are related to the real exponential as the trigonometric functions are to the complex exponential.

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) = \cos(ix) \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}) = -i\sin(ix) \tag{1.2.10}$$

$$\cosh(ix) = \cos x \qquad \sinh(ix) = i\sin x \qquad \tanh x = \frac{\sinh x}{\cosh x} = -i\tan(ix)$$

Sometimes it is more convenient to use hyperbolic functions than trigonometric ones; as Eq. (1.2.10) has just demonstrated, one can easily transform back and forth from one to the other.

Other solutions

Power series are not always the best guesses for the solutions of differential equations. Now that we have defined the exponential function, we can sometimes express solutions of other equations in terms of exponentials. Consider the equation

$$\frac{d^2x}{dt^2} + n^2x = ae^{-i\omega t}$$

where, to represent a physical problem, we must use our convention on the right-hand side of the equation. This could be solved by expanding the exponential in a series and solving for the series for x as before. But since an exponential is in the equation, and since we know that Ce^{-int} is the solution of this equation when a is zero, it will be simpler to guess that

$$x = Ce^{-int} + Be^{-i\omega t}$$

Setting this in the equation, we have that $n^2x = n^2Ce^{-int} + n^2Be^{-i\omega t}$ plus $d^2x/dt^2 = -n^2Ce^{-int} - \omega^2Be^{-i\omega t}$ must equal $ae^{-i\omega t}$. This means that $B = a/(n^2 - \omega^2)$. If we have used our convention on the term $ae^{-i\omega t}$, we must use it on the answer. The real part of

$$x = Ce^{-int} + \frac{a}{n^2 - \omega^2}e^{-i\omega t}$$

is

$$x = a_0\cos(nt) + a_1\sin(nt) + \frac{a}{n^2 - \omega^2}\cos(\omega t) \tag{1.2.11}$$

if a is a real quantity. If a is complex and equals $De^{i\psi}$, we can write

$$x = A\cos(nt - \Phi) + \frac{D}{n^2 - \omega^2}\cos(\omega t - \psi)$$

We notice again that in (1.2.11) our solution has two arbitrary constants, a_0 and a_1. The arbitrariness corresponds to the fact that this solution must represent all the possible sorts of motion which the system can have when it is acted on by the forces implied by the equation. A mass on a spring can have different motions, depending on how it is started into motion at time $t = 0$. Therefore the particular values of the arbitrary constants a_0 and a_1 in (1.2.2) are determined entirely by the physical statements as to how the system was started into motion. These physical statements are called *initial conditions* and are usually stated by giving the position and velocity of the system at $t = 0$. More will be said on this point in the next section.

Contour integrals

In a number of cases discussed in this volume it will be necessary to use integrals of complex quantities of the general form $\int F(z)\,dz$, where $F = U + iV$ is a function of the complex variable $z = x + iy$. Since z can vary over the xy plane, it will in general be necessary to specify the path over which the integration is carried out. A natural extension of our usual definition of integration indicates that the integral is the limit of a series of terms, a typical term being the product of the value of $F(z)$ for a value of z on the path, and a quantity $dz = |dz|e^{i\vartheta}$, with length $|dz|$ and with phase angle ϑ determined by the direction of the tangent to the path at the point z. The integral is the limit of such a sum as $|dz|$ goes to zero. This is shown in Fig. 1.1. Another definition is that $\int F\,dz = \int (U + iV)(dx + i\,dy)$, where x and y are related in such a manner that the integral follows the chosen path.

When the path chosen is a closed one, the integral is called a *contour integral* (labeled $\oint$), and the chosen path is called a *contour*. Contour integrals have certain remarkable properties, proved in standard texts on theory of functions of a complex variable. We have space here to outline only some of these properties. For simplicity it will be assumed that the function $F(z)$ to be dealt with is a smoothly varying, reasonable sort of function over most of the z plane.

The value of a contour integral depends largely on whether the integrand $F(z)$ becomes zero or infinity for some value or values of z *inside* the contour. Near such points, $F(z)$ would take on the form $(z - z_0)^n R(z)$, where $R(z_0)$ is neither zero nor infinity and $R(z)$ is not discontinuous near z_0. When n is *not* an integer, positive or negative, the point z_0 is called a *branch point*, and the problem becomes more complicated than is necessary to discuss here. When n is a *negative integer*, the point z_0 is called a *pole* of $F(z)$, the pole for $n = -1$ being called a *simple pole* and that for $n = -2, -3, \ldots$ being called a *pole* of second, third, etc., *order*. The statement at the beginning of this

paragraph can now be made more specific: The value of the contour integral $\oint F(z)\,dz$ depends on the behavior of F at the poles and branch points that happen to be inside the contour.

When F has *no* branch points or poles inside the contour, the value of the contour integral is *zero*. This can be verified by a tedious bit of algebraic manipulation for those cases where F can be expressed as a convergent power series in z.

When F has a simple pole at $z = z_0$, but has no other pole or branch point inside the contour (Fig. 1.1), the contour integral can be shown to be equal to that around a vanishingly small circle drawn about z_0. Since z_0 is specified as being a simple pole, $F(z)$ has the form $R(z)/(z - z_0)$ near z_0, where $R(z)$ is continuous and finite in value near z_0. We can write the equation for the resulting circle as $z - z_0 = \epsilon e^{i\varphi}$ and $dz = i\epsilon e^{i\varphi}\,d\varphi$, where ϵ is vanishingly small. The contour integral then reduces to

$$\oint F(z)\,dz = R(z_0)\int_0^{2\pi} \frac{i\epsilon e^{i\varphi}\,d\varphi}{\epsilon e^{i\varphi}} = iR(z_0)\int_0^{2\pi} d\varphi = 2\pi i R(z_0)$$

where $R(z_0) = \lim_{z \to z_0} [(z - z_0)F(z)]$ is called the *residue* of $F(z)$ at its simple pole z_0.

When $F(z)$ has N simple poles, at $z_0, z_1, \ldots, z_N$, and no other poles or branch points, the contour integral $\oint F(z)\,dz$ equals $2\pi i$ times the *sum of the residues of F at the poles that are inside* the chosen contour. We note that the direction of integration around the contour is counterclockwise; changing the direction will change the sign of the result.

This is a remarkably simple result; in fact, it seems at first too simple to be true, until we realize that the function $F(z)$ we are dealing with is a very specialized form of function. F is not just any complex function of x and y; its dependence on these variables is severely limited by the requirement that it be a function of $z = x + iy$. It can have the form $bz^2 + (C/z)$ or $z \sin z$. but *not* the form $x \sin(z) + iy$ or $|z|$. For such a specialized function $F(z)$, which is called an *analytic* function, the positions of its branch points and poles, and its behavior near these branch points and poles, completely determine its behavior *everywhere else on the complex plane*. When this is understood, the result we have quoted for the value of a contour integral does not seem quite so surprising.

As an example of the rule of residues, we can take the integral

$$\oint \sin(z)\,dz/(z^2 - a^2)$$

where the contour is a circle of radius greater than a, with center at the origin $z = 0$. The integrand has two simple poles, at $z = a$ and $z = -a$, with residues $(1/2a) \sin a$ and $-(1/2a) \sin(-a)$. The result is therefore

$$\oint \frac{\sin z}{z^2 - a^2}\,dz = 2\pi i\left[\frac{\sin a}{a}\right]$$

A special case of the residue theorem can be stated in equation form:

$$\oint \frac{R(z)}{z - z_0}\, dz = 2\pi i R(z_0) \tag{1.2.12}$$

where the contour of integration does *not* enclose a branch point or pole of $R(z)$. By differentiating once with respect to z_0 on both sides of the equation, we obtain the equation

$$\oint \frac{R(z)}{(z - z_0)^2}\, dz = 2\pi i R'(z_0) \tag{1.2.13}$$

where $R'(z) = dR(z)/dz$. This indicates how we can evaluate contour integrals around poles of second order. Cases for poles of higher order may be dealt with by further differentiation of Eq. (1.2.13).

Infinite integrals

A class of integral often encountered may be evaluated in terms of contour integrals. Consider the integral

$$\int_{-\infty}^{\infty} F(z)e^{-izt}\, dz$$

where the integral is taken along the real axis. If $F(z)$ has no branch points or poles of higher order, if the simple poles of F are not on the real axis, and if $F(z)$ goes to zero as $|z|$ goes to infinity, the integral can be made into a contour integral by returning from $+\infty$ to $-\infty$ along a semicircle of infinite radius. When t is positive (to make it more general we should say, when the *real part* of t is positive, but let us assume that t is real), the exponential factor ensures that the integral along this semicircle is zero if the semicircle encloses the *lower* half of the z plane. For along this path, $z = \rho e^{i\varphi} = \rho \cos \varphi + i\rho \sin \varphi$, with ρ infinitely large and with φ going from 0 to $-\pi$, so that $\sin \varphi$ is negative. Therefore, along the semicircle,

$$e^{-izt} = e^{-i\rho t \cos \varphi + \rho t \sin \varphi}$$

As long as $\sin \varphi$ is negative this vanishes, because of the extremely large value of ρ. Therefore the addition of the integral around the semicircle turns it into a contour integral but does not change its value.

The final result is that the integral equals $-2\pi i$ times the sum of the residues of $F(z)e^{-izt}$ at every one of its poles located *below* the real axis. The negative sign is due to the fact that the contour in this case is in the *clockwise* direction.

When t is negative, we complete the contour along a semicircle enclosing the upper half of the z plane, which now has the vanishing integral, and the resulting value is $2\pi i$ times the sum of the residues at each of the poles *above* the real axis. The limitations on F have already been given.

As an example, we can write down the result

$$\frac{a^2}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-izt}}{z^2+a^2}\,dz=\begin{cases}(a/2)e^{at} & t<0\\ (a/2)e^{-at} & t>0\end{cases}$$

practically by inspection. Another result of interest later is

$$\frac{ia}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-izt}}{z+ia}\,dz=\begin{cases}0 & t<0\\ ae^{-at} & t>0\end{cases} \tag{1.2.14}$$

In some cases of interest, F has poles on the real axis. These cases can also be treated by making the convention that the integration is not exactly along the real axis but along a line an infinitesimal amount *above* the real axis. With this convention, and for functions $F(z)$ that have *only simple poles* and that vanish at $|z|\to\infty$, we have the general formula

$$\int_{-\infty}^{+\infty}F(z)e^{-izt}\,dz=\begin{cases}-2\pi i\begin{pmatrix}\text{sum of residues of}\\ Fe^{-izt}\text{ at all its}\\ \text{poles } on \text{ and } below\\ \text{the real axis}\end{pmatrix} & t>0\\ +2\pi i\begin{pmatrix}\text{sum of residues of}\\ Fe^{-izt}\text{ at all its}\\ \text{poles } above \text{ the}\\ \text{real axis}\end{pmatrix} & t<0\end{cases} \tag{1.2.15}$$

As an example, we can write down the very useful formulas

$$u(t)=\frac{-1}{2\pi i}\int_{-\infty}^{\infty}e^{-izt}\frac{dz}{z}=\begin{cases}0 & t<0\\ 1 & t>0\end{cases} \tag{1.2.16}$$

$$\frac{-a}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-izt}}{z^2-a^2}\,dz=\begin{cases}0 & t<0\\ \sin(at) & t>0\end{cases} \tag{1.2.17}$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-izt}}{z-a}\,dz=\begin{cases}0 & t<0\\ e^{-iat} & t>0\end{cases} \tag{1.2.18}$$

1.3 VIBRATORY MOTION

A final introductory topic, appropriate for this chapter, has to do with how we are to describe, mathematically, the motion of an acoustic system, and how such a description corresponds to the various measurements we can make on the system. Most systems of interest in acoustics vibrate; they move to and fro rather than just in one direction. With a few exceptions, such as the single pulse produced by an explosion, their mean-square velocity is much larger than the square of their mean velocity. In many cases we can ignore the mean velocity entirely and concentrate on analyzing their vibrations about their average position.

The energy equation

The oscillator of Eq. (1.2.1) is the simplest example; here the force on the mass point is proportional to its displacement and the motion is simple-harmonic, i.e., proportional to a sine or cosine of ωt. The displacement, as well as the velocity, averages to zero over a period of the oscillation. Next in order of complication would be a mass point moving in one dimension, acted on by any sort of force. Such a force may be of several different types. One type, like the force of the spring of Eq. (1.2.1), depends only on the value of the displacement x of the mass at the given instant; we label it $F(x)$ and call it the *internal* force since it depends only on the state of the oscillator. A force imposed from outside, on the other hand, would depend explicitly on time t, though it may also depend on x; we call this external force $G(x,t)$. In addition, there may be a force, or combination of forces, which depends on the velocity $v = dx/dt$ of the mass (the viscous drag of the air is such a force); we label this frictional force $D(v,t)$.

The differential equation of motion

$$m\frac{d^2x}{dt^2} - F(x) = G(x,t) + D(v,t) \tag{1.3.1}$$

then states that the acceleration of the mass is proportional to the sum of all the forces acting. A somewhat more enlightening form of this relationship can be obtained by integrating the equation with respect to x. As indicated in Eq. (1.1.4), the left-hand side can be integrated to give the sum $H(v,x)$ of the kinetic energy, $\frac{1}{2}mv^2$, and the potential energy, $-\int F\,dx = V(x)$; thus the energy equation is

$$H(v,x) = \tfrac{1}{2}mv^2 + V(x) = \int_0^t [G(x,t) + D(v,t)]\,v\,dt + E \tag{1.3.2}$$

where E is the value of H at $t = 0$. This states that x and v so adjust themselves at each succeeding instant that the change in H is equal to the change in the right-hand integral. The sum H of the kinetic and potential energy of the mass point, when explicitly written as a function of x and v, as was done here, is called the *total energy* of the system.

The effect of the frictional-force term D is such as to diminish H continually; the external force G may increase or decrease H. When *both* G and D are zero, when the mass moves under the influence of forces which depend *only* on the position of the mass, then H is equal to the constant value E. Therefore force $F(x)$, responsible for the potential energy $V(x)$, is called a *conservative* force, because energy is conserved when it alone acts; and a system acted on solely by conservative forces is called a conservative system. For the time being, we shall concentrate on the one-dimensional conservative system.

Periodic motion

The nature of the motion of such a system depends, of course, on the shape of the potential energy curve, V against x. If it has a minimum, where $dV/dx \equiv -F = 0$, the mass can stay at rest at this point of equilibrium (which can be placed for convenience at the origin $x = 0$). If it really is a minimum, then d^2V/dx^2 would be positive there, so that if the mass is displaced a small amount dx, the force on it, $(dF/dx)\,dx = -(d^2V/dx^2)\,dx$, would be in the

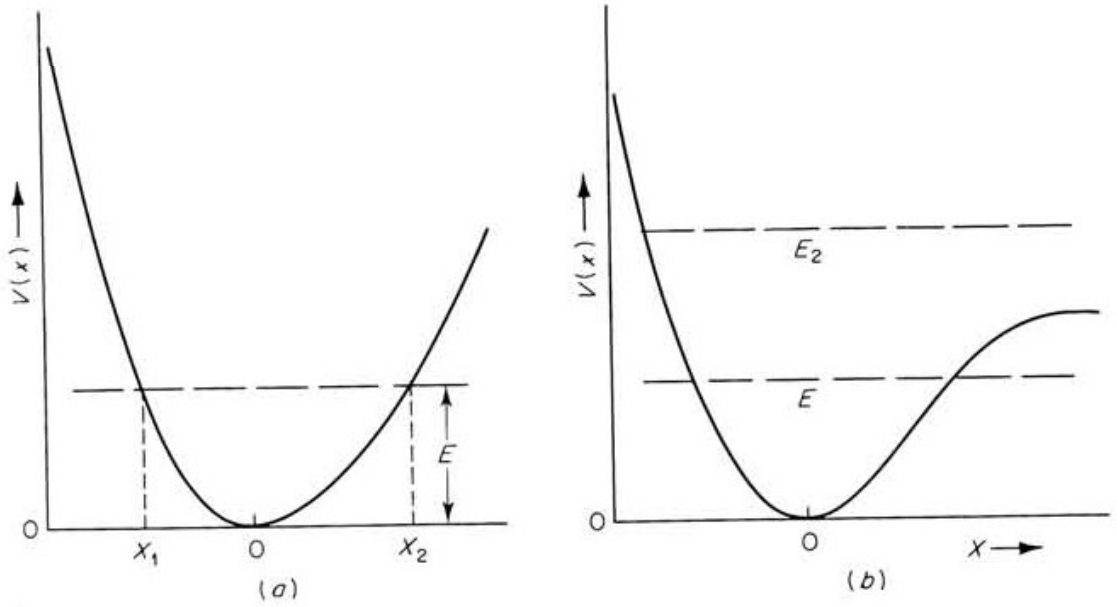


FIGURE 1.2
Potential-energy curves for bounded (curves *a* and *b*, E) and unbounded (curve *b*, E_2) motion of a one-dimensional oscillator.

opposite direction of dx, forcing the mass to return to equilibrium. Thus a minimum of the potential energy is a point of *stable* equilibrium.

In a conservative system, when D and G are zero, the energy equation, $\frac{1}{2}mv^2 + V(x) = E$ (a constant), indicates that at every instant the kinetic energy must adjust itself so that it equals $E - V(x)$. For a stable system the kinetic energy (and therefore the speed of the mass point) is a maximum at the minimum point of V, at $x = 0$; it is zero whenever $V = E$, for example, at x_1 and x_2 in Fig. 1.2*a*. We can make the relationship between v and x in a conservative system more specific by writing

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \qquad G \text{ and } D \text{ zero}$$

and more graphic by plotting v as a function of x, as in Fig. 1.3. The vx plane is called the *phase plane* (no relation to the phase angle Φ), and the curve of v against x, for a given value of E, is called a phase contour. The

position and velocity of the system at any given instant are represented by a point on the phase plane, called the system point; as time goes on this system point traces out a curve on the phase plane, the phase contour. The speed of traverse of the contour is determined by the ordinate of the point, for v is

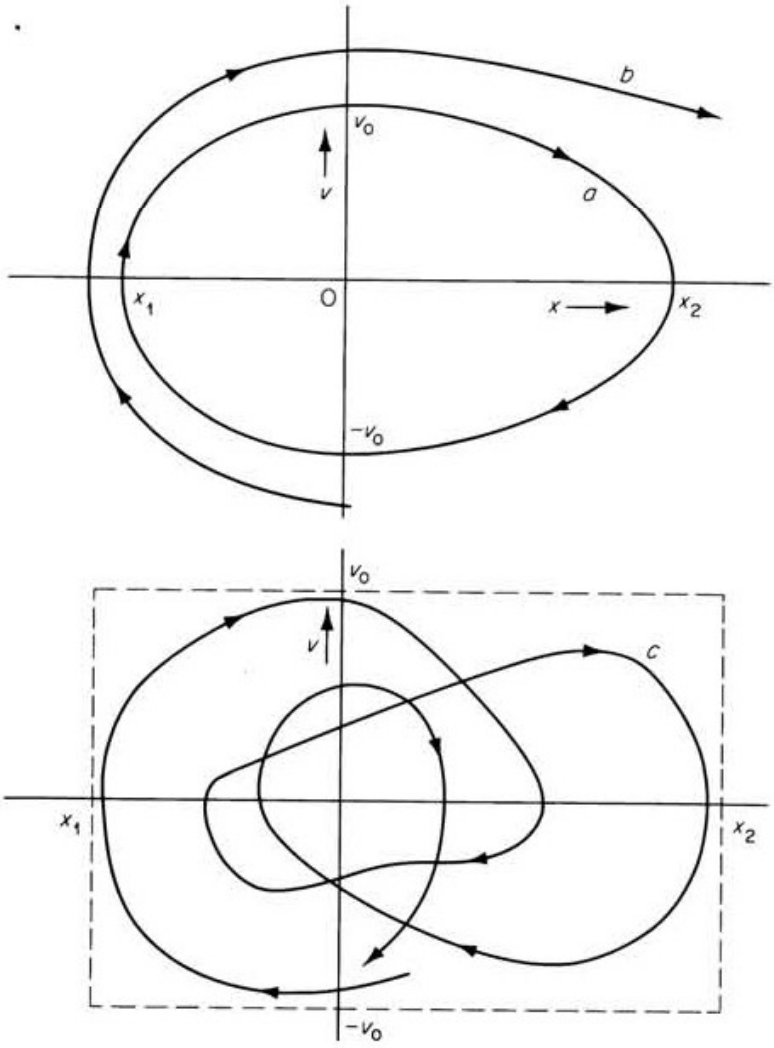


FIGURE 1.3
Phase contours, plotted on the phase plane for (*a*) bounded, periodic motion, (*b*) unbounded motion, and (*c*) nonperiodic, bounded motion.

the rate of change of x with time. The contour crosses the v axis ($x = 0$) at $v = \pm v_0$, where the speed is maximum; it crosses the x axis ($v = 0$) at $x = x_1$ and x_2, where $V(x) = E$ and the mass is farthest from the origin.

If the potential energy has only the one minimum at $x = 0$ and rises monotonically with increasing $|x|$, approaching infinity as $|x| \to \infty$ (as in Fig. 1.2*a*), the contour is a closed loop, and neither $|v|$ nor $|x|$ will ever become infinite. In this case the motion is *bounded* and, as we shall see shortly, is periodic. On the other hand, if $V(x)$ does not go to infinity as $x \to \infty$ (as

shown in Fig. 1.2*b*), then whenever E is larger than the asymptotic value of V (as with $E = E_2$), the contour is an open curve (as shown in contour b of Fig. 1.3), and the motion is unbounded.

To see that the bounded motion is periodic, we note that we can integrate for t. Since

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}[E - V(x)]} \tag{1.3.3}$$

then

$$t - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x} \frac{dx}{\sqrt{E - V(x)}}$$

gives t as a function of x; by inversion we can obtain x as a function of t. If the path of the system point on the phase plane is a closed loop, as with Fig. 1.3*a*, the point retraces its path periodically, once a cycle. The mean value of v over the cycle is zero, the mean values of v^2, of x, and of x^2 are finite and are the same for each cycle. Each traverse of the contour is exactly like all the others, which is what we mean when we call the motion *periodic*.

Thus, when the one-dimensional system is conservative, when G and D are zero, the total energy $H(v,x)$ of the system is constant, and the motion, if it is bounded, is periodic. From the symmetry of the phase contour with respect to the x axis, we see that the time required to go from x_1 to x_2 is equal to the time to go from x_2 back to x_1 [from Eq. (1.3.3)], so that the *period* T_p of the oscillation, the length of time it takes the mass point to go through a full cycle, is

$$T_p = 2(t_2 - t_1) = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}} \tag{1.3.4}$$

where $E = V(x_1) = V(x_2)$.

If the restoring force is linear, $F(x) = -Kx$, the potential energy $V = \frac{1}{2}m\omega_0^2 x^2$, where $\omega_0^2 = K/m$, is quadratic in x and Eq. (1.3.3) becomes

$$t - t_1 = \frac{1}{\omega_0} \int_{x_1}^{x} \frac{dx}{\sqrt{x_1^2 - x^2}} = \frac{1}{\omega_0} \cos^{-1} \frac{x}{x_1} \tag{1.3.5}$$

where we have set $E = \frac{1}{2}m\omega_0^2 x_1^2$, relating E and the maximum displacement x_1. Inversion of this result produces

$$x = x_1 \cos \omega_0 (t - t_1) \tag{1.3.6}$$

which is equivalent to Eq. (1.2.2). An oscillation of this sort, with displacement which is a sinusoidal function of time (and thus with velocity and acceleration which also are sinusoidal), is called *simple-harmonic* oscillation. As was shown in Sec. 1.2, and was shown again in the preceding paragraphs, the motion is periodic, with period $T_p = 2\pi/\omega_0 = 2\pi\sqrt{m/K}$.

Simple-harmonic motion is a particularly simple sort of periodic motion; the phase contour is an ellipse, or if the scale of the v coordinate of the phase plane is adjusted properly, it is a circle. If we plot v/ω_0 against x, we obtain

$$\frac{v}{\omega_0} = \frac{1}{\omega_0}\sqrt{\frac{2}{m}(E - \tfrac{1}{2}m\omega_0^2 x^2)} = \sqrt{x_1^2 - x^2} \tag{1.3.7}$$

which is the equation of a circle of radius $x_1 = \sqrt{2E/m\omega_0^2}$, with center at the origin. In this simple case the system point moves with uniform speed around a circle of radius proportional to the square root of the system's energy. The angular velocity ω_0 of the radius vector to the system point is, in this case, independent both of time and of the radius of the circle (and thus of E). This is not true of other periodic oscillations, when $V(x)$ is not a quadratic function of x.

Fourier series representation

When V is not proportional to x^2, the free oscillations of the mass point are still periodic as long as the motion is bounded, but x is not a simple sinusoidal function of t. However, the motion can be expressed in terms of a series of trigonometric functions, a generalization of (1.2.8).

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \Phi_n) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{-in\omega t} \tag{1.3.8}$$

where $A_n^2 = a_n^2 + b_n^2$, $a_n = A_n \cos\Phi_n$, $b_n = A_n \sin\Phi_n$

$C_0 = a_0$, $C_n = \frac{1}{2}(a_n + ib_n)$, $C_{-n} = \frac{1}{2}(a_n - ib_n)$ for $n \neq 0$

If all the C's are real, then all the b's are zero, $x(t)$ is real and an even function of t, that is, $x(-t) = x(t)$. If $C_{-n} = C_n^*$ (i.e., if all the a's and b's are real), then $x(t)$ is real but not necessarily symmetric about $t = 0$.

If we take $\Phi_0 = 0$ in Eq. (1.3.8), then $A_0 = a_0$ is the mean value of x, averaged over the cycle. The values of the amplitudes A_n and of the phase angles Φ_n (or of the C's) are, of course, determined by the value of E and the form of $V(x)$. How they are computed will be taken up shortly. Here we wish only to discuss the general properties of the series, assuming that the values of the A's and Φ's are known. As long as only integer values of n are included in the sums of Eq. (1.3.8), each term in the sum will return to its original value when t increases by $T_p = 2\pi/\omega$. The nth term will meanwhile have executed n cycles, but at the end of time T_p it will be back where it was originally, ready to start the $(n + 1)$st cycle. Therefore any series of the general form of Eq. (1.3.8) is periodic in time, with period $T_p = 2\pi/\omega$ (provided that the series converges, of course). Fourier first showed that the inverse is true; that any periodic motion which is physically feasible

(i.e., continuous and bounded) can be expressed in terms of such a series; therefore the series is called a *Fourier series.*

The quantity $1/T_p = \omega/2\pi$ is called the *fundamental* frequency of the oscillation; the higher terms, for $n > 1$, are called *harmonics.* Motion of the sort which can be represented by a Fourier series is the basis of all musical sounds. We shall see later in this volume that the vibrations of string and wind instruments can be (approximately) so represented and that the "tone quality" of the sounds produced by these instruments is determined to a great extent by the relative amplitudes of the various harmonics present in the series. Therefore the representation of periodic motion in terms of a Fourier series is more than a mathematical dodge to express a complicated function in terms of a series of simple functions. It corresponds somehow to the way we hear and distinguish musical sounds; specifying the relative amplitudes and phases of the different harmonics is an appropriate way to specify the tone quality of a musical sound (or a nonmusical one, as long as it is periodic). Experimentally, we use electric filters to analyze periodic sounds into their component harmonics, though the analyzers usually measure only the amplitudes A_n, and not the phase angles Φ_n.

The mathematical counterpart to the filter analyzer is based on the integral properties of the trigonometric functions,

$$\int_0^{T_p} \cos(n\omega t)\cos(m\omega t)\,dt = \tfrac{1}{2}T_p\delta_{nm} = \int_0^{T_p} \sin(n\omega t)\sin(m\omega t)\,dt$$

$$\int_0^{T_p} \cos(n\omega t)\sin(m\omega t)\,dt = 0 \qquad n \geqslant 0;\ m > 0 \tag{1.3.9}$$

where $T_p = 2\pi/\omega$, and the *Kronecker delta symbol* δ_{nm} is zero if $n \neq m$ and unity if $n = m$. Thus, if we multiply Eq. (1.3.8) by $\cos(m\omega t)\,dt$ and integrate over a cycle, the only term in the second form of the series which does not integrate to zero is the term with $n = m$, which integrates to $\frac{1}{2}T_p a_m$. Therefore, if we know the form of the curve of $x(t)$ as periodic function of t for one cycle, the coefficients a_n, b_n in the Fourier series expansion of $x(t)$ can be obtained by computing the integrals.

$$a_m = \frac{2}{T_p}\int_0^{T_p} x(t)\cos(m\omega t)\,dt \qquad b_m = \frac{2}{T_p}\int_0^{T_p} x(t)\sin(m\omega t)\,dt$$

$$a_0 = \frac{1}{T_p}\int_0^{T_p} x(t)\,dt \tag{1.3.10}$$

where $T_p = 2\pi/\omega$. Alternatively,

$$C_m = \frac{1}{T_p}\int_0^{T_p} x(t)e^{+im\omega t}\,dt$$

The electric filter analyzer essentially carries out these integrals when it measures the amplitude of the mth harmonic in a periodic sound.

To return to the one-dimensional, conservative system of Eq. (1.3.3), we have shown that its motion is periodic and that its period can be computed in terms of its potential energy and its total energy E. Its motion can be expressed in terms of a Fourier series of the sort shown in Eq. (1.3.8). How the coefficients a_n, b_n are computed from $V(x)$ and E will be taken up in Chap. 2.

Mean and mean-square displacement and velocity may be given in terms of the Fourier series coefficients.

$$\langle x \rangle = a_0 \qquad \langle x^2 \rangle = a_0^2 + \tfrac{1}{2} \sum_{n=1}^{\infty} A_n^2$$

$$\langle v \rangle = 0 \qquad \langle v^2 \rangle = \tfrac{1}{2} \sum_{n=1}^{\infty} n^2 \omega^2 A_n^2$$

We note that these averages depend on the amplitudes A_n, but not on the phases Φ_n. We note also that the ratio between $\langle v^2 \rangle$ and $\langle x^2 \rangle$ provides a rough measure of the degree of presence of the higher harmonics. The ratio $\langle v^2 \rangle / \langle x^2 \rangle$ is equal to ω^2 for simple-harmonic oscillations (for which $A_n = 0$ for $n > 1$). If a large number of harmonics are present in the motion, $\langle v^2 \rangle / \langle x^2 \rangle$ would be considerably large than ω^2.

We can now return to Eq. (1.2.9) to point out that the right-hand side is the Fourier series expansion of the periodic function $e^{iz \cos \theta}$. This, together with Eqs. (1.3.10) (setting $\theta = \omega t$), provides us with *integral representations* of the Bessel functions

$$J_n(z) = \tfrac{1}{2} \pi i^n \int_0^{2\pi} \exp (iz \cos \theta) \cos (n\theta)\, d\theta \tag{1.3.11}$$

giving the function $J_n(z)$ in terms of an integral of the exponential of a trigonometric function. These expressions will be of use in later calculations.

Nonperiodic vibration; autocorrelation

When the vibrating system has several degrees of freedom, such as the systems to be discussed in Chap. 3, the equations of motion are less simple than Eq. (1.3.1), and the motion is correspondingly more complicated. The complete description of the configuration of the system requires the specification of more than one displacement (let it be $N > 1$), some of the coordinates being components of linear displacements in several dimensions, and some possibly being angles of rotation of solid portions of the system. The nth displacement component, whether it be angle or distance, will have a corresponding moment of inertia or effective mass, which we can call m_n, and a corresponding set of torques or forces, F_n, G_n, and D_n, corresponding to Eq. (1.3.1). The displacements can be considered to be the N components of a vector displacement in N-dimensional space, which vector can be symbolized by $\mathbf{r}$. The corresponding N velocities can also be treated as components of a vector $\mathbf{v} = d\mathbf{r}/dt$.

The equations of motion for each degree of freedom, the generalization of Eqs. (1.3.1), are

$$m_n \frac{d^2x_n}{dt^2} - F_n(\mathbf{r}) = G_n(\mathbf{r},t) + D_n(\mathbf{v},t) \qquad n = 1, 2, \ldots, N$$

where the forces depend, in general, upon all the x's or v's, and perhaps on t for G_n and D_n. To obtain the energy equation we multiply the nth equation by $dx_n = v_n\,dt$, integrate, and sum over n, obtaining

$$H(\mathbf{r},\mathbf{v}) = \sum_{n=1}^{N} \tfrac{1}{2}m_n v_n^2 + V(\mathbf{r}) = \sum_{n=1}^{N} \int_0^t [G_n(\mathbf{r},t) + D_n(\mathbf{v},t)]v_n\,dt + E \quad (1.3.12)$$

where $V(\mathbf{r})$ is the N-dimensional potential energy, such that $\partial V/\partial x_n = -F_n(\mathbf{r})$. When all the G's and D's are zero, the system is conservative and the function H has the constant value E.

With most acoustical systems the potential function V has a minimum at the configuration of stable equilibrium of the system. If E is larger than this minimal value of V, the system will move in accord with the equations of motion. As long as E is not too large, the equation $E = V(\mathbf{r})$ defines a surface of finite extent, within which the N-dimensional vector $\mathbf{r}$ moves about. Because the kinetic energy of the system is a sum of the terms $\frac{1}{2}m_n(dx_n/dt)^2$, however, we cannot solve directly for the nth component v_n of velocity, as we were able to for one dimension in Eq. (1.3.3). At a given configuration, represented by vector $\mathbf{r}$, and for a total energy E, the magnitude of vector $\mathbf{v}$ is determined, but its direction is undetermined by the energy equation. The configuration point $\mathbf{r}$ will bounce around inside the surface $V(\mathbf{r}) = E$, but will not usually do it periodically, returning always along the same path; each bounce will usually start it off in a different direction.

Thus, if we plot the behavior of one component, the x_n and related v_n on the nth phase plane, for example, the contour will not be closed-loop as in Fig. 1.3*a*, but usually will be a complicated nonperiodic curve such as the one shown in Fig. 1.3*c*, confined within definite limits determined by E, but never repeating itself. See, for example, the curves of Fig. 3.3 for a system of two degrees of freedom. We shall see in Chap. 3 that the motion of a system with N degrees of freedom involves N different frequencies, usually mutually incommensurate, so that the combination is not periodic. In later chapters we shall see that an extended system has an infinite number of frequencies in its motion, so that the motion of any part may be quite complicated indeed.

Thus, for more complex systems, the displacement component x_n will not usually be a periodic function of time; so it cannot be represented by a Fourier series. In special cases, of course, it may be periodic, though seldom simple-harmonic; it may be as shown in Fig. 1.4*a*. In other cases the motion will be nearly periodic, as in Fig. 1.4*b*, but if the intercoupling is large, the motion will be completely aperiodic, though bounded, as shown in Fig. 1.4*c*.

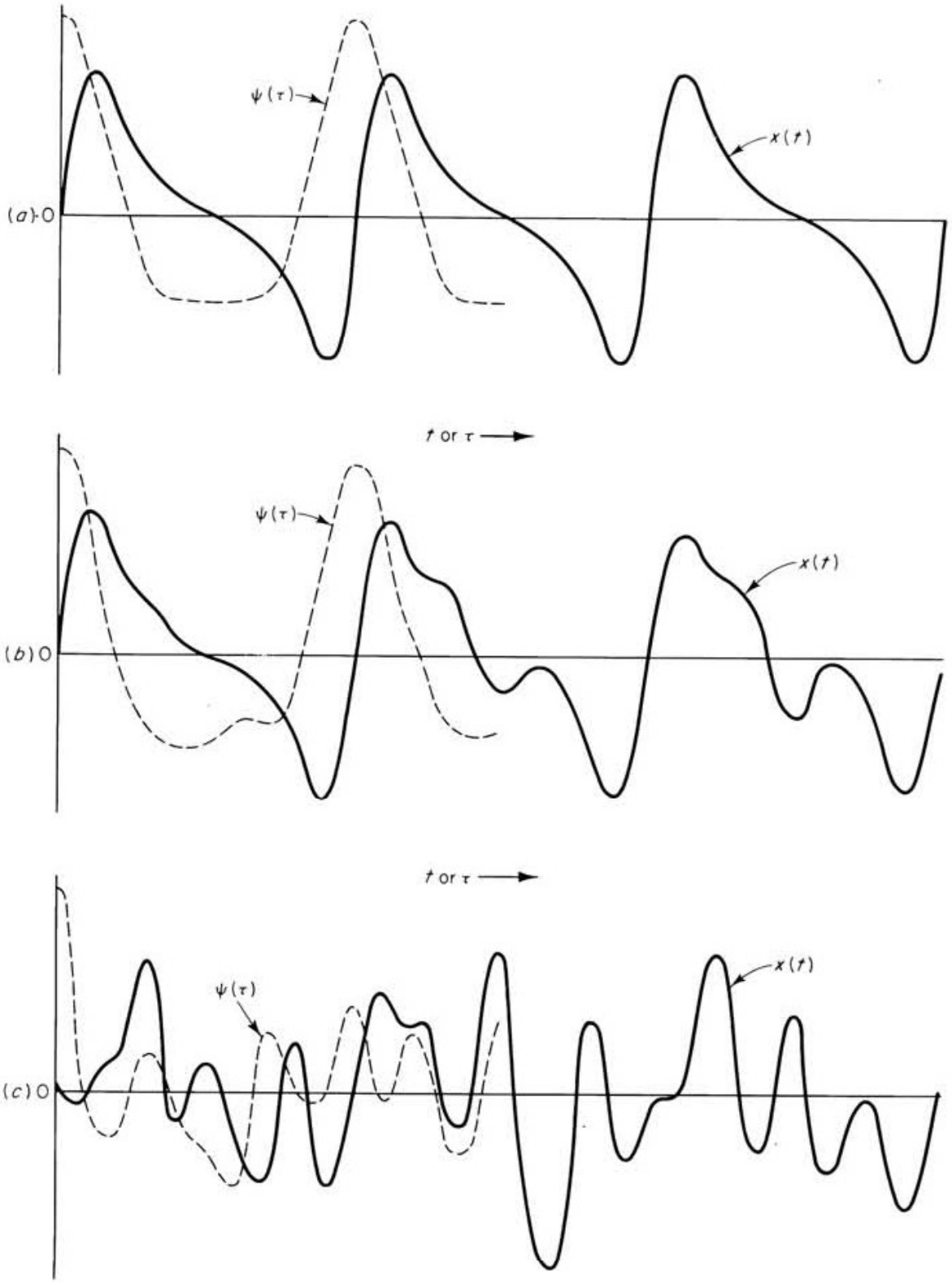


FIGURE 1.4
Solid lines are curves of displacement $x(t)$ versus time t; dashed curves are for corresponding autocorrelation index $\psi(\tau)$ against τ. Curves *a* are for periodic motion; curves *b* and *c* for bounded, nonperiodic motion.

The pressure fluctuation in the sound from a turbulent jet of air is also of this type; in fact, curves of this type are characteristic of steady nonperiodic sound. We should learn how to describe mathematically these motions, too.

Oscillations like those pictured in Fig. 1.4*b* and *c* are bounded, nonperiodic, and stationary. The average values of x or x^2 vary with time, but we can take their averages over a sufficiently long interval of time to even out these fluctuations.

$$\langle x \rangle = \frac{1}{T}\int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} x(t)\,dt \qquad \langle x^2 \rangle = \frac{1}{T}\int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} x^2\,dt$$

In the limit of very large T, these averages are independent of T. If the vibrations are stationary, they are independent of t also. There are two ways of describing such motion, one corresponding to the action of a frequency analyzer, the other amounting to a comparison of the motion at one instant with the motion a time τ later. Periodic motion repeats itself exactly, every successive cycle. Some nonperiodic motion tends to repeat itself, but not exactly, the degree of duplication departing further and further from the original as time goes on (an example is the curve of Fig. 1.4*b*).

This tendency toward duplication, or lack of it, can be measured by the *autocorrelation function* Υ, which is defined as follows:

$$\Upsilon(\tau) = \lim_{T\to\infty}\left[\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t)x(t+\tau)\,dt\right] \tag{1.3.13}$$

We see that $\Upsilon(0) = \langle x^2 \rangle$; thus it is sometimes useful to express correlation in terms of the dimensionless *autocorrelation index* $\psi = (\Upsilon/\langle x^2 \rangle)$.

$$\psi(\tau) = \lim_{T\to\infty} \frac{\displaystyle\int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t)x(t+\tau)\,dt}{\displaystyle\int_{-\frac{1}{2}T}^{\frac{1}{2}T} x^2(t)\,dt} \qquad \psi(0) = 1 \tag{1.3.14}$$

We can find the autocorrelation function for either periodic or nonperiodic functions of time. If x is periodic and can be represented by the Fourier series (1.3.8) (curve of Fig. 1.4*a*), its autocorrelation index is

$$\psi(\tau) = \frac{a_0^2 + \frac{1}{2}\displaystyle\sum_{n=1}^{\infty} A_n^2 \cos(n\omega\tau)}{a_0^2 + \frac{1}{2}\displaystyle\sum_{n=1}^{\infty} A_n^2} \tag{1.3.15}$$

which is itself a Fourier series, with its maximum values, unity, occurring at $\tau = 0, T_p, 2T_p, \ldots$ ($T_p = 2\pi/\omega$). If x is periodic, then $x(t)x(t+\tau)$ is exactly equal to $x^2(t)$ whenever $\tau = nT_p$; so $\Upsilon(\tau)$ will equal $\langle x^2 \rangle$ whenever $\tau = nT_p$, and thus $\Upsilon(\tau)$ will also be periodic in τ with period T_p. Note that Υ and ψ depend only on the amplitudes A_n of the Fourier series for x, not on the phase angles Φ_n; Υ and ψ always have their maxima at $\tau = nT_p$ if x is periodic.

Now suppose $x(t)$ is "not quite periodic," as with the curve of Fig. 1.4*b*. It almost repeats itself in the next cycle, but not quite, and it differs more and more from the original cycle as time goes on. In this case $x(t)x(t+\tau)$ will almost equal $x^2(t)$ if τ is a small integer times T_p, but will differ considerably from $x^2(t)$ if τ is large. Instead of being positive everywhere (as x^2 is), it will be as often negative as positive when τ is large. In this case the autocorrelation index will start at unity when $\tau = 0$ and will have a succession of maxima, spaced T_p apart, of decreasing magnitude until, for very large τ, $\psi(\tau) \to 0$. If there is very little correlation between x at one instant and x at the next instant as in the curve of Fig. 1.4*c*, then $\psi(\tau)$ will drop rapidly to zero as τ is increased. In fact, the rapidity with which $\Upsilon(\tau)$ drops to zero as τ is increased is a measure of the degree of randomness of the oscillations of x.

Therefore the autocorrelation function (or index) is one way of measuring the nature of bounded vibratory motion. It is not a complete measure of the vibrations, however, for it neglects the phase angles Φ. Thus a knowledge of $\Upsilon(\tau)$ is not sufficient to allow us to compute $x(t)$.

The Fourier integral

Another way of representing bounded vibrations is to analyze the motion into its various component frequencies. We did this already for periodic motion, with the Fourier series of Eq. (1.3.8). Nonperiodic motion cannot be represented in terms of a fundamental component plus a sequence of harmonics; it must be considered as a superposition of motions of all frequencies. Thus the *series* of harmonic terms must be generalized into an *integral* over all values of ω. Using the complex exponential form for the component simple-harmonic terms, the basic equations for the Fourier integral representation, corresponding to Eq. (1.3.12) for the series, are that

$$\text{If} \quad f(t) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t}\, d\omega \qquad \text{then} \qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt \quad (1.3.16)$$

Function $F(\omega)$ is called the *Fourier transform* of $f(t)$, and $f(t)$ is the *inverse* Fourier transform of $F(\omega)$. The first equation shows that any function of t (within certain limits, which we shall discuss shortly) can be expressed in terms of a superposition of simple-harmonic motion of all possible frequencies $\omega/2\pi$, each frequency having for its amplitude the magnitude $|F(\omega)|$ of the complex quantity $F(\omega)$ and having for its phase angle the phase angle of $F(\omega)$. A little thought will show that if $f(t)$ is to be real (when t is real), then $F(-\omega)$ must equal the complex conjugate of $F(\omega)$ [that is, $F(-\omega) = F^*(\omega) = |F|e^{-i\Phi}$, so that $F(\omega)F(-\omega) = |F|^2$ is real]. Quantity $|F(\omega)|^2$ is called the *spectrum density* of $f(t)$ at the frequency $\omega/2\pi$. An electric filter which cuts out all the frequencies in $f(t)$ except those between $\omega/2\pi$ and $(\omega + d\omega)/2\pi$ would deliver a measurable power proportional to $|F(\omega)|^2$ and also proportional to $d\omega/2\pi$, the width of the frequency band passed by the filter.

The second equation of (1.3.16) shows how the Fourier transform $F(\omega)$ can be computed if we know the mathematical form of $f(t)$. It is analogous to Eqs. (1.3.10) or (1.3.8) for the Fourier series coefficients. It is, of course, the reason we discussed the infinite integrals of Eqs. (1.2.14) to (1.2.18). The reciprocal relationship displayed in Eqs. (1.3.16) has meaning only if the infinite integrals converge, and are of practical interest only if they converge for real values of t and ω. Books on the theory of the Fourier transform prove that convergence will certainly occur if the integral of $|f(t)|^2$ over t from $-\infty$ to ∞ is finite, in which case

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = 2\pi \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega \tag{1.3.17}$$

and thus the integral of $|F(\omega)|^2$ is also finite.

This last relationship makes it look as though we should have difficulty in using the Fourier integral to represent stationary nonperiodic motion, of the sort we discussed in the previous subsection and portrayed in Fig. 1.4. If the vibration is stationary, $|x(t)|^2$ does not diminish as $t \to \infty$ and the integral of $|x|^2$ will not converge. We can get around the difficulty by a trick, however. Suppose we shut off $x(t)$ except for the time interval $-\frac{1}{2}T < t < \frac{1}{2}T$, and study the function

$$f(t) = \begin{cases} 0 & -\infty < t < -\frac{1}{2}T \\ x(t) & -\frac{1}{2}T < t < \frac{1}{2}T \\ 0 & \frac{1}{2}T < t < \infty \end{cases} \tag{1.3.18}$$

The integral of the square of this function will converge; in fact,

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = T\langle x^2 \rangle = 2\pi \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega \qquad T \text{ large} \tag{1.3.19}$$

where now

$$F(\omega) = \frac{1}{2\pi} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t) e^{i\omega t} \, dt$$

is a function which falls off sufficiently rapidly as ω goes to $\pm\infty$, so that the integral of $|F|^2$ converges as long as T is not infinite. Thus $|F(\omega)|^2/T$, which we can call the spectrum density per unit time, is a quantity whose integral over ω is equal to $(1/2\pi)\langle x^2 \rangle$ for T sufficiently large. This trick of analyzing only an interval T worth of $x(t)$ has its counterpart in measurement. The length of time required for the filter to separate out the components within a band $d\omega$ is longer, the narrower the bandwidth $d\omega$; however, we certainly cannot afford to wait forever, although the only way we could obtain a minutely detailed analysis would be to analyze it over an infinite length of time.

Thus the Fourier transform, with the use of the tricks of Eqs. (1.3.18) and (1.3.19), is another way of representing the bounded vibrations $x(t)$,

which emphasizes its frequency analysis rather than its temporal correlation, as $\Upsilon(\tau)$ did. It is a more complete representation than is Υ, for Eqs. (1.3.16) show that if $F(\omega)$ is known completely, then $x(t)$ can be reproduced in its entirety. This is usually more detailed than is useful, however; usually we are interested only in the spectrum density $|F(\omega)|^2$ and are willing to remain ignorant of the phase of F.

Actually, the two methods of representation are closely related. If we consider $\Upsilon(\tau)$, the autocorrelation function of $x(t)$, defined in Eq. (1.3.13) as a function which also can have a Fourier transform, by using the trick of Eq. (1.3.18), then this transform would be

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\,\Upsilon(\tau)\,d\tau = \frac{1}{2\pi T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} e^{i\omega\tau}\,d\tau \int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t)x(t+\tau)\,dt$$

$$= \frac{1}{2\pi T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} e^{-i\omega t}\,x(t)\,dt \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} e^{i\omega u}x(u)\,du \xrightarrow[T\text{ large}]{} \frac{2\pi\,|F(\omega)|^2}{T} \qquad (1.3.20)$$

where $F(\omega)$ is defined in Eq. (1.3.19) as the transform of $x(t)$ over the interval T. We thus reach the interesting and rather unexpected result that the Fourier transform of the autocorrelation function of $x(t)$ over the interval T is $2\pi/T$ times the spectrum density $|F(\omega)|^2$ of x, the square of the magnitude of the Fourier transform of x. Vice versa, if we know the spectrum density of x for a time interval T, its correlation function for the same interval is the inverse transform

$$\Upsilon(\tau) = \frac{2\pi}{T}\int_{-\infty}^{\infty} |F(\omega)|^2\,e^{-i\omega\tau}\,d\omega \qquad (1.3.21)$$

Thus $\Upsilon(0)$, which equals $\langle x^2\rangle$, is proportional to the frequency integral of $|F|^2$, which also is apparent from Eq. (1.3.19).

Properties of Fourier transforms

A few general properties of Fourier transforms will be useful when we come to use them later. In the first place, mathematically speaking, both t and ω can be considered to be complex quantities, and thus the integrals of Eqs. (1.3.16) are contour integrals, of the sort discussed in connection with Eqs. (1.2.15). From Eqs. (1.2.16) to (1.2.18) we can obtain Fourier transforms for several important functions of t.

$$\text{If } f(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad \text{then} \quad 2\pi F(\omega) = \frac{i}{\omega}$$

$$\text{If } f(t) = u(t)\sin(at) \quad \text{then} \quad 2\pi F(\omega) = \frac{a}{a^2 - \omega^2} \qquad (1.3.22)$$

$$\text{If } f(t) = u(t)e^{-iat} \quad \text{then} \quad 2\pi F(\omega) = \frac{i}{\omega - a}$$

where the imaginary part of ω must be positive in order that the integral for

F converge. In each of these cases $F(\omega)$ has a pole or poles on the real ω axis. Setting $\omega = w + i\epsilon$, the integrand $e^{i\omega t}f(t)$ of Eqs. (1.3.16) for F has an exponential factor $e^{iwt-\epsilon t}$, which ensures convergence of the integral as long as ϵ is positive. We noted, in the discussion of Eqs. (1.2.15), that the contour over which ω is integrated should go somewhat above the real axis of ω, which is the same thing as saying that ϵ should be positive.

This arrangement for bringing convergence to the integrals is most useful in representing functions $f(t)$, which are zero for negative values of t, for then we need only worry about convergence at $t \to \infty$. In fact, if we are only interested in representing functions $f(t)$ which are zero for $t < 0$ (as would be the case if the system were at rest till $t = 0$, when it is started into motion), it may be more convenient to rotate the real-imaginary axes of ω, setting $\omega = is \quad (s > 0)$, so that

$$F_L(s) = 2\pi F(is) = \int_0^\infty f(t)e^{-st}\,dt \qquad f(t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} F_L(s)e^{st}\,ds \quad (1.3.23)$$

Function $F_L(s)$ is called the *Laplace transform* of $f(t)$. As we can see, it is just a different way of writing the Fourier transform, which happens to be useful for those functions which are zero for $t < 0$. We use the Fourier transform in this book, since it is more immediately related to the physical process of frequency analysis. We also shall use the Laplace transform when it is more convenient. Tables of the Fourier and Laplace transforms of various functions have been published; we mention only a few here. A short table of Laplace transforms is given in Chap. 2 (Table 2.1).

Differentiation of the integral for $f(t)$ or integration by parts of the integral for $F(\omega)$ shows that if $F(\omega)$ is the Fourier transform of $f(t)$, then $-i\omega F(\omega)$ is the Fourier transform of df/dt, and $-\omega^2 F(\omega)$ is the transform of d^2f/dt^2. From this we can obtain the transform of a differential equation; the transform of

$$\frac{d^2x}{dt^2} + \omega_0{}^2 x = f(t) \qquad \text{is} \qquad (\omega_0{}^2 - \omega^2)X(\omega) = F(\omega)$$

where $X(\omega)$ is the Fourier transform of $x(t)$, and $F(\omega)$ is the transform of $f(t)$. If $f(t)$ is known, we can compute $X(\omega) = F(\omega)/(\omega_0{}^2 - \omega^2)$ and then compute the inverse transform $x(t)$ (or else look it up in a table of Fourier transforms).

The pathological, but extremely useful, *Dirac delta function* is defined as follows:

$$\delta(t) = \frac{d}{dt}u(t) \qquad \int_{-\infty}^{\infty} \delta(t-a)g(t)\,dt = g(a) \qquad (1.3.24)$$

where $u(t)$ is defined in Eqs. (1.3.22) and where $g(t)$ is any function which is finite and continuous at $t = a$. The curve for $\delta(t)$ is the limiting form for a

very high, very narrow peak, centered at $t = 0$, having unit area under the peak. Direct insertion in Eqs. (1.3.16) shows that

$$\begin{aligned} &\text{The Fourier transform of } \delta(t - t_0) \text{ is } \frac{1}{2\pi} e^{i\omega t_0} \\ &\text{The inverse transform of } e^{-i\omega_0 t} \text{ is } \delta(\omega - \omega_0) \end{aligned} \tag{1.3.25}$$

This shows us that the Fourier transform of the Fourier series

$$x(t) = \sum_{n=0}^{\infty} A_n e^{-i(n\omega_0 t - \Phi_n)}$$

is

$$X(\omega) = \sum_{n=0}^{\infty} A_n e^{i\Phi_n} \delta(\omega - n\omega_0) \tag{1.3.26}$$

Therefore the frequency analysis of a periodic motion results in a set of equally spaced isolated peaks, one for the fundamental frequency ω_0 and one for each harmonic ($\omega = n\omega_0$). The magnitude of these peaks is proportional to A_n, the respective Fourier amplitude.

We shall use the Fourier transform often in our subsequent discussions, because it is natural, in acoustics, to analyze sounds into their simple-harmonic motions. We shall use the Laplace transform rather less often, when we are interested in studying the transient effects caused by the sudden initiation of some driving force. But there are still other integral transforms, which we shall have occasional recourse to. For example, there is the so-called Hankel transform,

$$F_m(\mu) = \int_0^{\infty} f(w) J_m(\mu w) w \, dw \qquad f(w) = \int_0^{\infty} F_m(\mu) J_m(\mu w) \mu \, d\mu \tag{1.3.27}$$

which we shall use in connection with cylindrical waves.

Problems

1 A mass m slides without friction on a horizontal table. It is attached to a light string which runs, without friction, through a hole in the table. The other end of the string is pulled downward by a constant force F. The mass (which is too large to drop through the hole) is held at rest a distance D from the hole and then let go. Set up the equation of motion of the mass, and solve it. Is the motion periodic? If so, what is its frequency, and how does the frequency depend on D?

2 A bead of mass m slides without friction on a straight wire that is whirling in a horizontal plane, about one end, with a constant angular velocity ω radians per sec. It is found that the centrifugal "force" on the bead is $m\omega^2 r$, where r is the distance of the bead from the center of rotation. The direction of the force is away from the center. Set up the equation of motion of the bead, and solve it. Is the motion periodic, and if so, what is the frequency?

3 Show that a solution of the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + n^2y = 0$$

can be represented by the series

$$y = C\left(1 - \frac{n^2x^2}{2^2} + \frac{n^4x^4}{2^2 \cdot 4^2} - \frac{n^6x^6}{2^2 \cdot 4^2 \cdot 6^2} \cdots\right)$$

The series inside the parentheses is called the Bessel function $J_0(nx)$, where $J_0(z) = 1 - (z^2/2^2) + \cdots$. Compute values of $J_0(0)$, $J_0(\frac{1}{2})$, and $J_0(1)$ to three places of decimals.

4 Show that the solution of the equation

$$\frac{d^2y}{dx^2} + (1 - kx^2)y = 0$$

is $$y = a_0D_e(k,x) + a_1D_o(k,x)$$

where D_e and D_o are symbols for the following series:

$$D_e(k,x) = \cos x + \frac{k}{12}x^4 - \frac{7k}{360}x^6 + \frac{11k + 15k^2}{10{,}080}x^8 \cdots$$

$$D_o(k,x) = \sin x + \frac{k}{20}x^5 - \frac{13k}{2{,}520}x^7 + \frac{17k + 63k^2}{90{,}720}x^9 \cdots$$

Compute values of D_e and D_o to three places of decimals for $k = 1$ and for $x = 0$, $\frac{1}{2}$, and 1.

5 Show that the solution of the equation

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + \lambda y = 0 \qquad \text{is} \qquad y = a_0P_e(\lambda,x) + a_1P_o(\lambda,x)$$

where P_e and P_o are symbols for the following series:

$$P_e(\lambda,x) = 1 - \frac{\lambda}{2!}x^2 - \frac{\lambda(6 - \lambda)}{4!}x^4 - \frac{\lambda(6 - \lambda)(20 - \lambda)}{6!}x^6 - \cdots$$

$$P_o(\lambda,x) = x + \frac{2 - \lambda}{3!}x^3 + \frac{(2 - \lambda)(12 - \lambda)}{5!}x^5 + \frac{(2 - \lambda)(12 - \lambda)(30 - \lambda)}{7!}x^7 + \cdots$$

Compute values of P_e and P_o, to three places of decimals, for $\lambda = 0$, 1, and 2, for $x = 0$, $\frac{1}{2}$, and 1.

6 Show that the solution of the equation

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \left(1 - \frac{2}{x^2}\right)y = 0 \qquad \text{is} \qquad y = a_0j_1(x) + a_1n_1(x)$$

where j_1 and n_1 are symbols for the following functions:

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

What are the values of j_1 and n_1 at $x = 0, \frac{1}{2}$, and 1? What is the solution of the equation

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y = 0$$

7 What is the length of the line drawn from the origin to the point on the complex plane represented by the quantity $(a + ib)^{-1}$? What is the angle that this line makes with the real axis? What are the amplitudes and phase angles of the following quantities:

$$(a - ib)^{-\frac{1}{2}}; \qquad (a + ib)^{-1} + (c + id)^{-1}; \qquad (a + ib)^n e^{-i\omega t}$$

8 What are the real and imaginary parts, amplitudes, and phase angles of

$$\sqrt{a + ib}, \qquad \log(a + ib), \qquad e^{-2\pi i(\nu + i\mu)t}, \qquad e^{-ix}(1 + e^{a+ib})^{-1}$$

9 The *hyperbolic* functions are defined in a manner analogous to the trigonometric functions [Eq. (1.2.10)].

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}) \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z})$$

$$\tanh(z) = \frac{\sinh z}{\cosh z} = \frac{1}{\coth(z)} \qquad \text{(Tables I and II)}$$

Show that $\cosh(iy) = \cos y$; $\cos(iy) = \cosh y$; $\sinh(iy) = i\sin y$; $\sin(iy) = i\sin y$. Find the real and imaginary parts of $\cosh(x + iy)$, $\cos(x + iy)$, $\sinh(x + iy)$, $\sin(x + iy)$. What are the magnitudes and phase angles of these quantities?

10 Find the real part R and imaginary part X of $\tanh(x + iy)$. Show that

$$R^2 + [X + \cot(2y)]^2 = \frac{1}{\sin^2(2y)}$$

$$[R - \coth(2x)]^2 + X^2 = \frac{1}{\sinh^2(2x)}$$

Plot the curves on the complex plane for $\tanh(x + iy)$ when y is allowed to vary but $x = 0, \pi/10, \infty$; when x is allowed to vary but $y = 0, \pi/4, \pi/2$.

11 Where on the complex plane are the poles of the functions $1/\cosh z$; $\tan z$; $e^{iz}/z^2(z^2 - a^2)$; $\tan z/z(z^2 + a^2)$? Are they all simple poles? Calculate the residues at all the simple poles of these functions.

12 Compute the values of the integrals

$$\int_{-\infty}^{\infty} \frac{e^{-izt}}{z(z^2 + a^2)}\,dz \qquad \int_{-\infty}^{\infty} \tan(z) e^{-izt}\frac{dz}{z}$$

13 Transform the ordinary integral $\int_0^{2\pi} (1 - 2p\cos\theta + p^2)^{-1}\,d\theta$ to a contour integral over z by setting $e^{i\theta} = z$. What is the shape of the contour? Where are the poles of the integrand when p is real and smaller than unity? Calculate the value of the integral when $0 < p < 1$.

14 Calculate the value of the integral

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1 + x^2)(1 - 2x\cos\theta + x^2)}$$

15 Show that

$$-\frac{\omega}{2\pi}\int_{-\infty}^{\infty} e^{-izt}\left[\frac{1-(-1)^n e^{iz(n\pi/\omega)}}{z^2-\omega^2}\right]dz = \begin{cases} 0 & t<0 \\ \sin(\omega t) & 0<t<\dfrac{n\pi}{\omega} \\ 0 & \dfrac{n\pi}{\omega}<t \end{cases}$$

16 A mass m moves in the x direction under the influence of a potential energy $V(x) = Ax^2/(x+d)^2$. Plot the potential energy and use the plot to discuss the behavior of the system for different total energies E. Plot the phase contour for $E = \frac{3}{4}A$ and for $E = \frac{5}{4}A$. Show that, for $E < A$, the motion is periodic, with period $T_p = \sqrt{2m}\,Ad\pi/(A-E)^{\frac{3}{2}}$.

17 Plot the Fourier series $x = 10\cos(\omega t) + 5\cos(3\omega t) + \cos(5\omega t)$. Compute and plot the autocorrelation function $\Psi(\tau)$ for this x.

18 Compute the coefficients and phase angles for the Fourier series expansion of the following periodic function of t:

$$x(t) = \begin{cases} t & -\frac{1}{2}\pi < t < \frac{1}{2}\pi \\ \pi - t & \frac{1}{2}\pi < t < \frac{3}{2}\pi \\ t - 2\pi & \frac{3}{2}\pi < t < \frac{5}{2}\pi \\ \cdots\cdots & \end{cases}$$

19 A particular function $x(t)$ can be represented by the sum

$$x(t) = \sum_{n=1}^{N} \cos(\omega_n t - \Phi_n)$$

where no pair of the ω_n's are commensurate, and the Φ_n's are arbitrary functions of n. Show that x is nonperiodic but bounded, that the mean value of x^2 is $\frac{1}{2}N$, and that the maximum possible value of x^2 is N^2. What is the expression for the autocorrelation function for x?

20 Compute the Fourier transforms of the following functions: e^{-at^2}, $a^2/(a^2+t^2)$. What are the inverse transforms of $-1/2\pi\omega^2$ and $A/(\omega^4 - a^4)$?

21 Frequency analysis of a certain noise shows that its spectrum density $|F(\omega)|^2$ is independent of ω over the range $|\omega| < \omega_m$ and is zero for $|\omega| > \omega_m$. What is the autocorrelation function?

22 The autocorrelation function for pressure in a turbulent gas is $\Psi(\tau) = P_0{}^2 e^{-\omega_m|\tau|}$. What is the spectrum density?

23 If

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt \qquad \text{and} \qquad G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)e^{i\omega t}\,dt$$

show that the inverse transform of the product FG, $\displaystyle\int_{-\infty}^{\infty} F(\omega)G(\omega)e^{-i\omega t}\,d\omega$, is equal to $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$, which is called the *convolution integral* of f and g. What about the Fourier transform of $f(t)$ times $g(t)$? Demonstrate both these results in the special case where $F(\omega) = (\omega + ia)^{-1}$, $G(\omega) = (\omega + ib)^{-1}$, by performing all the integrals, using Eqs. (1.2.15) when necessary.

CHAPTER 2

THE LINEAR OSCILLATOR

2.1 FREE OSCILLATIONS

Now that we have discussed, to some extent, the mathematical methods that we shall need in our work, we shall come back to the physics.

The whole study of sound is a study of vibrations. Some part of a system has *stiffness*; when it is pulled away from its position of equilibrium and then released, the system oscillates. We shall first study the simplest possible sort of vibrations for the simplest sort of system, a mass m fastened to some sort of spring, so that it can oscillate back and forth in just one direction. A very large number of vibrators with which we deal in physics and engineering are of this type or are approximately like it. All pendulums (the "spring" here is the force of gravity) are like this, and all watch balance wheels. Loudspeaker diaphragms which are loaded so that their mass is concentrated near their center are approximately like this (at least at low frequencies), as are loaded tuning forks, etc. Even when an oscillating system is more complex than the simple oscillator, many of its properties are like it. Later, when we study these more complicated systems, we can simplify our discussion considerably by pointing out first the properties wherein the system behaves like the simple case and then showing where it differs.

In this chapter we shall simplify the problem still further; we shall investigate the motion of the one-dimensional oscillator with an internal stiffness force which is *linearly proportional* to x, $F_i(x) = -Kx$. As mentioned in Sec. 1.3, if the system is to perform bounded vibrations, F_i must be zero at some equilibrium point (which can be set at $x = 0$) and must be negative when x is positive and positive when x is negative, so as to push the mass back to $x = 0$. This usually means that F_i can be expanded in a Taylor series about the equilibrium point,

$$F_i(x) = -Kx + K_2x^2 + K_3x^3 + \cdots$$

If x is sufficiently small, the terms in x^2, x^3, etc., can be neglected compared with the linear term $-Kx$. Many acoustical vibrations are sufficiently small in amplitude for this linear approximation to be quite good. For instance,

an air molecule needs to vibrate with an amplitude of motion of only about a tenth of a millimeter to do its full part in transmitting away the racket generated at Times Square in New York on New Year's Eve. Seldom does the amplitude of oscillation of a loudspeaker diaphragm exceed a millimeter.

The general solution

We shall assume, then, that the "springiness" force acting on the simple oscillator can be expressed by the equation

$$F = -Kx \tag{2.1.1}$$

where x is the displacement of the mass from equilibrium, K is called the *stiffness constant* (its value depends on the sort of spring we use), and the minus sign indicates that the force opposes the displacement. This will be a very good assumption to make for most of the cases dealt with in this book. The reciprocal of the stiffness constant, $C_m = 1/K$, is called the *compliance* of the spring.

To start with, we shall consider that no other forces act on the oscillator. This is, of course, not a good assumption in many cases; usually there are frictional forces acting, and sometimes external forces come in. There are many cases, however, where the frictional force is negligible compared with the springiness force, and we shall treat these first, bringing in friction and external forces later.

This brings us to the equation of motion discussed in Sec. 1.2:

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \omega^2 = \frac{K}{m} = \frac{1}{mC_m} \tag{2.1.2}$$

We have already seen that the solution of this equation can be expressed, in terms of our convention, as

$$x = Ce^{-i\omega t} \qquad C = a_0 + ia_1$$

or as

$$x = a_0 \cos(\omega t) + a_1 \sin(\omega t) \tag{2.1.3}$$

We must now discuss the physical implications of this solution.

It has been shown in Sec. 1.2 that the motion of the body is periodic, having a frequency $\nu_0 = \omega/2\pi = (1/2\pi)\sqrt{K/m}$. This frequency is larger for a stiffer spring (for a larger K) and is smaller for a heavier mass.

Initial conditions

We have seen that the specific values of the constants a_0 and a_1 are determined by the way the mass is started into motion. Ordinarily, we start an oscillator into motion by giving it a push or by pulling it aside and letting it go, i.e., by giving it, at $t = 0$, some specified initial displacement and initial velocity. Once we have fixed these two *initial conditions*, the motion is completely determined from then on, unless we choose to interfere with it again. It is not hard to see that a_0 is the value of the initial displacement

and ωa_1 that of the initial velocity. Solution (2.1.3) can then be rewritten in the following forms:

$$x = x_0 \cos(2\pi\nu_0 t) + \frac{v_0}{2\pi\nu_0} \sin(2\pi\nu_0 t)$$

$$= A \cos(2\pi\nu_0 t - \Phi) \qquad \nu_0 = \frac{1}{2\pi}\sqrt{\frac{K}{m}} \tag{2.1.4}$$

$$A^2 = x_0^2 + \left(\frac{v_0}{2\pi\nu_0}\right)^2 \qquad \tan\Phi = \frac{v_0}{2\pi\nu_0 x_0}$$

where x_0 is the initial displacement, and v_0 the initial velocity of the mass.

These equations reemphasize the fact that *only* the initial value of x and of dx/dt need be given to determine the subsequent motion of an oscillator completely. Once x_0 and v_0 are specified, then, even if we did not have a completely worked-out solution of (2.1.2), we could find the initial value of the body's acceleration by inserting x_0 on the right-hand side of (2.1.2). The initial value of the third derivative can be obtained by differentiating (2.1.2) with respect to t and placing the value of v_0 on the right-hand side in place of dx/dt; and so on. If it is recalled that the behavior of a function is completely specified by its Taylor's-series expansion

$$f(t) = f(0) + t\left(\frac{df}{dt}\right)_{t=0} + \frac{t^2}{2}\left(\frac{d^2f}{dt^2}\right)_{t=0} + \frac{t^3}{6}\left(\frac{d^3f}{dt^3}\right)_{t=0} + \cdots$$

(within a certain range of t whose limits are of interest in specific problems, but which need not bother us here), we see that once x_0 and v_0 are given, thus fixing the values of all the higher derivatives at $t = 0$, the future motion is determined.

It is not hard to generalize this reasoning, to see that a body, acted on by any sort of force that depends on x and v, will have its motion specified completely just by assigning definite values to its initial position and velocity. The mathematical counterpart to this statement is the rule that the solution of any second-order differential equation (one having a second derivative term in it but no higher derivative) has two arbitrary constants in it.

Another very important physical fact which can be deduced from (2.1.4) is that the frequency of the oscillation depends *only* on K and m and *not at all* on x_0 or v_0. This means that for a given mass and a given spring, as long as the law of force of the spring is $F = -Kx$, the frequency of oscillation will be the same no matter how we start the system to oscillate, whether it oscillates with an amplitude of motion of 1 cm or 0.001 mm. This is a very important fact in its practical applications, for if the law of force of actual springs were not nearly $F = -Kx$, or if this property did not hold for solutions of Eq. (2.1.2), no musical instrument could be played in tune. Imagine trying to play a piano when the frequency of each note depended on how hard one struck the keys! Compare this with Probs. 1 and 16 of Chap. 1.

We might have found this fact for a number of cases by a long series of experimental observations, but our mathematical analysis tells us immediately that *every* mass acted on by a force $F = -Kx$ has this property. Oscillations of this type are called *simple-harmonic oscillations*, as noted in the discussion of Eq. (1.3.6).

Energy of vibration

We shall need an expression for the energy of a mass oscillating with simple-harmonic motion of amplitude A and frequency v_0. The energy is the sum of the potential and kinetic energies

$$E = \tfrac{1}{2}mv^2 + \int_0^x Kx\,dx = \tfrac{1}{2}mv^2 + \tfrac{1}{2}Kx^2$$

$$= 2\pi^2 m v_0^2 A^2 \sin^2(2\pi v_0 t - \Phi) + \tfrac{1}{2}KA^2 \cos^2(2\pi v_0 t - \Phi)$$

But $4\pi^2 v_0^2 = K/m$, so that

$$E = \tfrac{1}{2}KA^2[\sin^2(2\pi v_0 t - \Phi) + \cos^2(2\pi v_0 t - \Phi)]$$

or

$$E = \tfrac{1}{2}KA^2 = 2\pi^2 m v_0^2 A^2 = \tfrac{1}{2}mU^2 \qquad (2.1.5)$$

where $U = 2\pi v_0 A$ is the *velocity amplitude* of the motion.

The total energy is thus equal to the potential energy at the body's greatest displacement, $\frac{1}{2}KA^2$, or is equal to the kinetic energy at the body's greatest speed, $\frac{1}{2}mU^2$. Expressed in terms of v_0 and A, we see that E depends on the square of these two quantities.

2.2 DAMPED OSCILLATIONS

So far, we have not considered the effects of friction on oscillating systems. In general, friction does not play a very important role in the problems we shall consider in the first part of the book. If we show its effects on the simplest system with which we deal, we can deduce its effect, by analogy, on more complicated systems.

One variety of friction that is important in vibrational problems is the resistance to motion which the air surrounding the body manifests. Energy in the form of sound waves is sent out into the air. From the point of view of the vibrating system, this can be looked on as friction, for the energy of the system diminishes, being drained away in the form of sound. This resisting force depends on the velocity of the vibrator, and unless the velocity is large (much larger than those with which we shall usually deal), it is proportional to the velocity. It can be expressed mathematically as $-R_m(dx/dt)$, where the constant R_m is called the *resistance constant*. The total force on a simple oscillator acted on by both friction and springiness is

therefore $-R_m(dx/dt) - Kx$, and the equation of motion becomes

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 4\pi^2\nu_0^2 x = 0 \qquad k = \frac{R_m}{2m} \qquad 4\pi^2\nu_0^2 = \frac{K}{m} \tag{2.2.1}$$

The value of ν_0 is the frequency that the oscillator would have if the friction were removed (R_m were zero), and is called the *natural frequency* of the oscillator. Other resistive forces, representing the transformation of acoustic energy into heat, also are proportional to the velocity for small-amplitude motion; so Eq. (2.2.1) again holds. It is interesting to notice that the equation for the free oscillation of charge in a circuit containing inductance, resistance, and capacitance has the same form as (2.2.1). The inductance is analogous to the mass m, the resistance to the resistance factor R_m, and the inverse of the capacitance to the stiffness constant K.

The general solution

To solve Eq. (2.2.1) we make use of the exponential function again. We guess that the solution is Ce^{bt} and solve for b. Substitution in (2.2.1) shows that $(b^2 + 2kb + 4\pi^2\nu_0^2)e^{bt}$ must be zero for all values of the time. Therefore

$$b^2 + 2kb + 4\pi^2\nu_0^2 = 0 \qquad \text{or} \qquad b = -k \pm \sqrt{k^2 - 4\pi^2\nu_0^2}$$

In all the problems that we shall consider, the stiffness constant K is much larger than the resistance constant R_m (i.e., friction will not be big enough to make the motion much different from that discussed in the last section). Since $4\pi^2\nu_0^2$ is supposed to be larger than k^2, the square root in the expression for b is an imaginary quantity, and we had better write

$$b = -k \pm 2\pi i\nu_f \qquad \nu_f = \nu_0\sqrt{1 - \left(\frac{k}{2\pi\nu_0}\right)^2} \tag{2.2.2}$$

This means that, following our convention, we can write

$$x = Ce^{-kt - 2\pi i\nu_f t}$$

or

$$x = e^{-kt}[a_0 \cos(2\pi\nu_f t) + a_1 \sin(2\pi\nu_f t)] = A_0 e^{-kt}\cos(2\pi\nu_f t - \Phi) \tag{2.2.3}$$

The values of a_0 and a_1 are again fixed by the initial conditions for the oscillator; a_0 must equal the initial displacement x_0, and the initial velocity in this case is equal to $2\pi\nu_f a_1 - ka_0$, so that

$$a_1 = \frac{v_0 + kx_0}{2\pi\nu_f}$$

The solution is *not* periodic, since the motion never repeats itself, each swing being of somewhat smaller amplitude than the one before it. However, if k is quite small compared with ν_f, we can say that it is very nearly periodic. In any case, the frequency of the oscillations is ν_f, which is very nearly equal to ν_0 if k is small. It again turns out that the frequency is independent of the amplitude of the motion. Of course, strictly speaking, we should not use the word frequency in connection with nonperiodic motion. But when the damping is small, the motion is almost periodic, and the word will have some meaning, although a rather vague one.

There are several respects in which the motion of the damped oscillator differs from that of the simple oscillator. The most important difference is that the amplitude of motion of the damped oscillator decreases exponentially with time; it is A_0e^{-kt}, instead of being just A (A_0 is the initial value of the amplitude). The amplitude decreases by a factor $1/e$ in a time $1/k$ sec ($e = 2.718$). This length of time is a measure of how rapidly the motion is damped out by the friction, and is called the *modulus of decay* τ of the oscillations. The fraction of this decrease in amplitude which occurs in one cycle, i.e., the ratio between the period of vibration and the modulus of decay, is called the *decrement* δ of the oscillations. Another method of expressing this is in terms of the "Q of the system," where $Q = \omega_0 m/R_m$ is the number of cycles required for the amplitude of motion to reduce to $1/e^{\pi}$ of its original value. If these quantities are expressed in terms of the constants of the system (the small difference between ν_f and ν_0 being neglected), it turns out that

$$\begin{aligned} Q &= \frac{\omega_0 m}{R_m} = \frac{\pi\nu_0}{k} \qquad \omega_0 = 2\pi\nu_0 = \sqrt{\frac{K}{m}} \qquad k = \frac{R_m}{2m} \\ \tau &= \frac{1}{k} = \frac{Q}{\pi\nu_0} \qquad\qquad \delta = \frac{\pi}{Q} = \frac{k}{\nu_0} = \frac{1}{\tau\nu_0} \end{aligned} \tag{2.2.4}$$

The smaller R_m is, the larger Q and τ are, indicating that it takes a longer time for the oscillations to damp out, and the smaller δ is, indicating that the reduction in amplitude per cycle is smaller. These properties, of course, are independent of the way the oscillator is started into motion.

Another difference between the damped and the undamped oscillator is the difference in frequency. When $k/2\pi\nu_0$ is small, the expression for ν_f can be expanded by means of the binomial theorem, and all but the first two terms can be neglected.

$$\nu_f = \nu_0\sqrt{1 - \left(\frac{k}{2\pi k_0}\right)^2} = \nu_0 - \frac{k^2}{8\pi^2\nu_0} + \cdots \tag{2.2.5}$$

In most of the cases with which we shall be dealing, k and ν_0 have such values that even the second term in the series is exceedingly small, so that the change in frequency is usually too small to notice.

Energy relations

The subject of damped oscillations can be considered from a quite different point of view—that of energy loss. We must first develop an expression for the average energy of the system at any instant. We cannot use the formulas (2.2.5) because the amplitude of oscillation in the present case is not constant. The sum of the kinetic and potential energies of a body of mass m acted on by a springiness force Kx, whose displacement is given by the formula $A(t)\cos(2\pi\nu t - \Phi)$, is

$$\begin{aligned} E(t) &= \tfrac{1}{2}m\nu^2 + \tfrac{1}{2}Kx^2 \\ &= 2\pi^2 m\nu^2 A^2 - 2\pi m\nu \frac{dA}{dt} A \sin(2\pi\nu t - \Phi)\cos(2\pi\nu t - \Phi) \\ &\qquad + \tfrac{1}{2}m\left(\frac{dA}{dt}\right)^2 \cos^2(2\pi\nu t - \Phi) \end{aligned}$$

When averaging this value over a single oscillation, the second term on the right drops out. If A is very slowly varying, so that dA/dt is small compared with νA, then the third term can be neglected, and we have the approximate formula for the energy of motion and position (i.e., the energy that can be recovered, that is not yet irrevocably lost in heat):

$$E(t) \simeq \tfrac{1}{2}K[A(t)]^2 = 2\pi^2 m\nu^2[A(t)]^2 = \tfrac{1}{2}m[U(t)]^2 \qquad (2.2.6)$$

where the symbol $\simeq$ means "is approximately equal to." In the case of the damped oscillator, this "free energy" is $2\pi^2 m\nu_0^2 A_0^2 e^{-2kt}$, which diminishes exponentially with time.

The rate of loss of energy due to friction is equal to the frictional force opposing the motion $R_m v$ multiplied by the velocity v (since force times distance is energy, rate of change of energy is force times velocity). The rate of loss of energy is

$$\begin{aligned} P = R_m v^2 = [&4\pi^2 R_m \nu_f^2 \sin^2(2\pi\nu_f t - \Phi) \\ &+ 4\pi R_m \nu_f k \sin(2\pi\nu_f t - \Phi)\cos(2\pi\nu_f t - \Phi) \\ &+ R_m k^2 \cos^2(2\pi\nu_f t - \Phi)]A_0^2 e^{-2kt} \end{aligned}$$

Using the same approximations as before, we have for the average loss of energy per second

$$P = -\frac{dE}{dt} \simeq 2\pi^2 \nu_0^2 R_m A_0^2 e^{-2kt} = \tfrac{1}{2}R_m[U(t)]^2 \qquad (2.2.7)$$

If we had started out without formula (2.2.3) for the details of the motion but had simply said that the free energy at any instant was given by (2.2.6) and that the energy loss was given by (2.2.7), where R_m is small, we could have found the dependence of E on the time by means of (2.2.7). Eliminating U from Eqs. (2.2.6) and (2.2.7) results in $P \simeq (R_m/m)E = 2kE$,

so that we have

$$P = -\frac{dE}{dt} \simeq 2kE \qquad \text{or} \qquad \frac{dE}{dt} \simeq -2kE \tag{2.2.8}$$

The solution of this is

$$E \simeq E_0 e^{-2kt} = \tfrac{1}{2} m U_0^2 e^{-2kt} = 2\pi^2 \nu_0^2 m A_0^2 e^{-2kt}$$

which checks with Eq. (2.2.6). We see from this that the damping out of the motion is required by the fact that the energy is being lost by friction. We might point out here that the fraction of free energy lost per cycle is just $2k/\nu_f \simeq 2k/\nu_0 = 2\pi/Q$, where Q is given in Eq. (2.2.4).

In nearly every more complicated case of vibrations, the effect of friction is the same as in this simple case. The amplitude of vibration slowly decreases, and the frequencies of natural oscillation are very slightly diminished. Usually, the change of frequency is too small to be of interest.

2.3 FORCED OSCILLATIONS

It often happens that a system is set into vibration because it is linked in some way with another oscillating system (which we shall call the *driving system*). For instance, the diaphragm of a microphone vibrates because it is linked, by means of sound waves, to the vibrations of a violin string; and a loudspeaker diaphragm vibrates because it is linked to the current oscillations in the output circuit of an amplifier. The system picks up energy from the driving system and oscillates. In the two instances mentioned, and in many others, the driven system does not feed back any appreciable amount of energy to the driving system, either because the linkage between the two is very weak (as is the case with the violin and microphone) or else because the driving system has so much reserve energy that the amount fed back is comparatively negligible (as is the case with the amplifier and loudspeaker). In these cases the only property of the driving system that we need to know is that it supplies a periodic force which acts on the driven system. The more complicated case, where the feedback of energy cannot be neglected, will be considered later.

The general solution

For the present, we ask what happens to a simple oscillator when it is acted on by a periodic force $F\cos(2\pi\nu t)$, or $Fe^{-2\pi i\nu t}$, according to our convention. We wish to know what its motion is just after the force has been applied, and more important, what its motion is after the force has been acting for a long time. We are also interested in how this behavior depends on the frequency ν of the driving force (which does not have to be the same as the natural frequency ν_0 of the oscillator).

The total force on the oscillator is a combination of the springiness and the frictional and the driving force $-R_m v - Kx + Fe^{-2\pi i\nu t}$. The equation of motion is

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = ae^{-2\pi i\nu t}$$
$$a = \frac{F}{m} \qquad k = \frac{R_m}{2m} \qquad \omega = 2\pi\nu \qquad \omega_0^2 = \frac{K}{m} \tag{2.3.1}$$

We discussed a similar equation in Sec. 1.2, where we showed that a choice of two exponentials, one corresponding to the free vibration of the oscillator and the other to the forced motion, was a good guess for the solution. We try $x = Ce^{-kt-i\omega_f t} + De^{-i\omega t}$ in Eq. (2.3.1), where k and $\omega_f = 2\pi\nu_f$ are the damping and frequency parameters of free oscillation, as defined in Eqs. (2.2.2) and (2.2.5). The differential operator on the left side of Eq. (2.3.1) cancels out the free-oscillation term in x. For the second term, however, the equation is not satisfied unless

$$(-m\omega^2 - iR_m\omega + K)De^{-i\omega t} = Fe^{-i\omega t} \qquad \text{or} \qquad D = \frac{F}{-i\omega Z_m} \tag{2.3.2}$$

where
$$Z_m = -i\omega m + R_m + i\frac{K}{\omega} = R_m - iX_m = |Z_m|\, e^{-i\varphi}$$
$$X_m = \omega m - \frac{K}{\omega} \qquad \tan\varphi = \frac{X_m}{R_m} = \frac{m(\omega^2 - \omega_0^2)}{\omega R_m}$$

Some time after the motion has started, the free-oscillation term $Ce^{-kt-i\omega_f t}$ has died out, and the second term remains.

$$x \to De^{-i\omega t} = \frac{F}{-i\omega Z_m}\, e^{-i\omega t} \qquad t \gg \frac{1}{k}$$
$$v \to -i\omega De^{-i\omega t} = \frac{F}{Z_m}\, e^{-i\omega t} = \frac{F}{|Z_m|}\, e^{-i(\omega t-\varphi)}$$

The ratio between the driving force and the velocity of the steady-state driven motion, in amplitude and phase angle, is thus equal to the quantity $Z_m = R_m - iX_m$. The steady-state amplitude of motion $|D|$ is F/K for a static ($\omega = 0$) force. As the driving frequency is increased from zero, the amplitude $|D|$ increases (if R_m is not too large) from the static value $D_{st} = F/K$ to a peak (Fig. 2.2) at *resonance*, when $\omega^2 = \omega_0^2 - 2k^2 \simeq \omega_0^2$. The resonance amplitude is $F/2km\omega_f \simeq QD_{st}$, where $Q = \omega_0 m/R_m$ is discussed in connection with (2.2.4). For higher frequencies the amplitude drops back toward zero. Thus the Q of the oscillator is the ratio between the displacement amplitude at resonance to displacement for an equal-amplitude static force (of course, if $Q < 1$, there is no peak at resonance, $|D|$ is maximum at $\omega = 0$).

We recall that the usual electrical-engineering notation is obtained by substituting $-j$ for i; so we can see that the equation for Z_m is exactly analogous to the equation for the complex electric impedance of a series circuit, with the *mechanical resistance* R_m analogous to the electric resistance, the mass m analogous to the electric inductance, and the mechanical compliance $C_m = 1/K$ analogous to the electric capacitance. The quantity $X_m = \omega m - (K/\omega)$ can be called the *mechanical reactance* of the system. The units in which this mechanical impedance is expressed are *not* ohms, for the quantity is a ratio between force and velocity rather than between voltage and current. The symbolic analogy is close enough, however, to warrant the use of the same symbol Z with the subscript m to indicate "mechanical." The units of mechanical impedance are dyne-seconds per centimeter, or grams per second.

The solution of Eq. (2.3.1) can therefore be written in either of the two alternative forms

$$x = Ce^{-kt-2\pi i\nu_f t} - \frac{F}{2\pi i\nu Z_m} e^{-2\pi i\nu t}$$

or

$$x = e^{-kt}[a_0 \cos(2\pi\nu_f t) + a_1 \sin(2\pi\nu_f t)] + \frac{F}{2\pi\nu\,|Z_m|} \cos(2\pi\nu t - \vartheta)$$

where $\vartheta = \varphi + \frac{1}{2}\pi$. The constants a_0 and a_1 are determined, as before, by the initial position and velocity of the mass.

Transient and steady state

When the force is first applied, the motion is very complicated, being a combination of two harmonic motions of (in general) different frequencies. But even if the friction is small, the first term, representing the free, or "transient," vibrations, damps out soon, leaving only the second term, which represents simple-harmonic motion of frequency equal to that of the driving force (see examples of this in Fig. 2.1).

$$x \to -\frac{F}{2\pi i\nu Z_m} e^{-i\omega t} \quad \text{or} \quad x \to \frac{F}{2\pi\nu\,|Z_m|} \sin(2\pi\nu t - \varphi)$$

$$v \to \frac{F}{Z_m} e^{-i\omega t} \quad \text{or} \quad v \to \frac{F}{|Z_m|} \cos(2\pi\nu t - \varphi) \tag{2.3.3}$$

where $$\tan\varphi = -\cot\vartheta = \frac{\omega m - (K/\omega)}{R_m} \qquad \omega = 2\pi\nu$$

This part of the motion is called the *steady state*. We see that it is completely independent of the way in which the oscillator is started into motion, its amplitude, phase, and frequency depending only on the constants F and ν

of the force and on the oscillator constants m, R_m, and K. No matter how we start the oscillator, its motion will eventually settle down into that represented by (2.3.3).

Steady-state motion is motion of a system that has forgotten how it started.

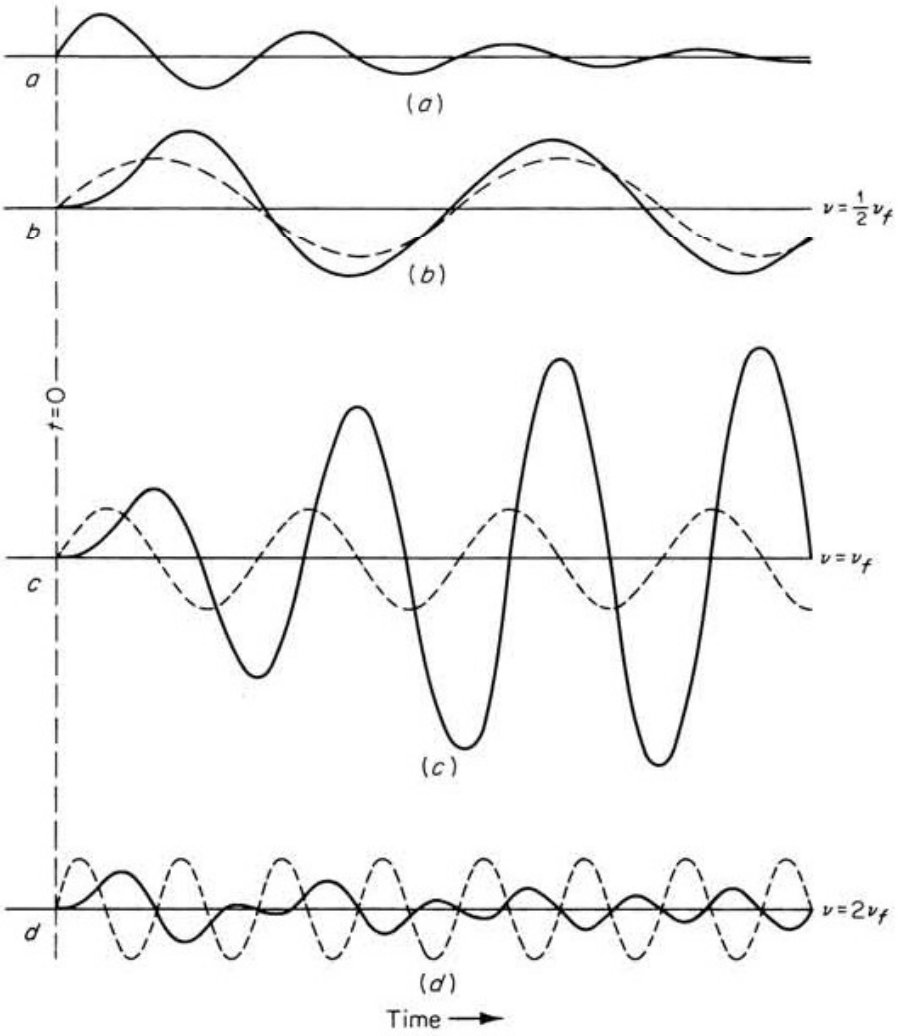


FIGURE 2.1
Forced motion of a damped-harmonic oscillator ($k = \omega_f/10$). Curve *a* shows free oscillations, and curves *b*, *c*, and *d* show forced oscillations due to sudden application of force at $t = 0$. Dotted curves give force as function of time; solid curves give displacement. Effect of transient is apparent at the left side of the curves; at the right side the steady state is nearly reached.

Impedance and phase angle

The amplitude and velocity amplitude are proportional to the amplitude of the driving force and are inversely proportional to the magnitude of the mechanical impedance Z_m. The analogy with electric circuits is therefore complete. The velocity corresponds to the current, the mass to the inductance, the frictional constant to the resistance, and the stiffness constant to the reciprocal of the capacitance.

The impedance is large except when $\nu = \nu_0$, but at this frequency, if the friction is small, it has a sharply defined minimum. Therefore the amplitude of motion in the steady state is small except when $\nu = \nu_0$, where it has a sharp maximum. The case where the frequency ν of the driving force equals the natural frequency ν_0 of the oscillator, when the response is

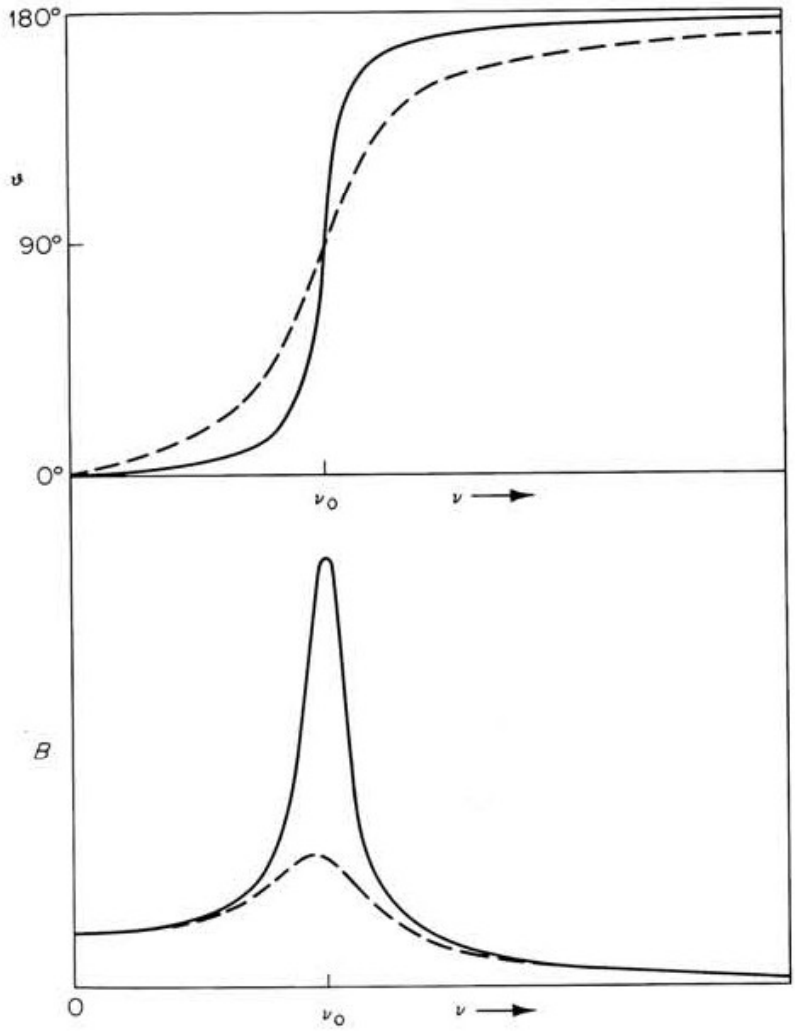


FIGURE 2.2
Phase ϑ and amplitude $B = |F/\omega Z_m|$ of forced motion as functions of the frequency ν of the driving force. The frictional constant R_m for the dotted curve is eleven times that for the solid curve, all other constants being equal.

large, is called the condition of *resonance*. The peak in the curve of amplitude against ν is sharp if R_m is small and is broad and low if R_m is large, as is shown in Fig. 2.2. This figure also shows that the steady-state motion of the oscillator is not very sensitive to the value of the frictional constant except in the range of frequencies near resonance. The dotted curve for the amplitude of motion is for a value of R_m eleven times that for the solid curve, yet the two are practically equal except in this frequency range.

The motion is not usually in phase with the force, the angle of lag of the

displacement behind the force being given by the angle ϑ, which is zero when $\nu = 0$, is $\pi/2$ when $\nu = \nu_0$, and approaches π as ν approaches infinity (indicating that the displacement is opposite in direction to the force). The angle of lag of the velocity behind the force, $\varphi = -(\pi/2) + \vartheta$, is analogous to the phase angle in ac theory. It is $-\pi/2$ when $\nu = 0$, zero when $\nu = \nu_0$, and $+\pi/2$ when ν is very large.

In other words, when the frequency ν of the driving force is much smaller than the natural frequency ν_0 of the oscillator, the amplitude is small and the displacement is in phase with the force. As ν is increased, the amplitude increases and gets more and more out of phase with the force, until at resonance the amplitude is very large (if R_m is small) and the velocity is in phase with the force. As ν is still further increased, the amplitude drops down and eventually becomes very small. For these large values of ν the displacement is opposed to the force. Figures 2.1 and 2.2 illustrate this behavior.

We use systems driven by periodic forces in two very different ways. One type of use requires the system to respond strongly only to particular frequencies (examples of this are the resonators below the bars of a xylophone, the strings of a violin, and the human mouth when shaped for a sung vowel). In this case we must make the friction as small as possible, for then the only driving frequency that produces a large response is that equal to the natural frequency of the driven system. The other type of use requires the system to respond more or less equally well to all frequencies (examples are the diaphragms of microphones and loudspeakers and the sounding board of a violin). In some cases we wish the steady-state amplitude to be independent of the frequency; in others we wish the amplitude of the velocity to be constant; and in still other cases we should like the acceleration to have a constant amplitude. One or another of these requirements can be met *within a certain range of frequencies* by making the stiffness, the friction, or the mass large enough so that its effects outweigh those of the other two in the desired range of frequency.

These three limiting types of driven oscillators are called *stiffness-controlled*, *resistance-controlled*, and *mass-controlled* oscillators, respectively. Their properties and useful ranges of frequency are

$$\begin{aligned}
&\textit{Stiffness-controlled: } K \text{ large} \qquad Z_m \simeq i\frac{K}{2\pi\nu} \qquad x \simeq \frac{F}{K}e^{-i\omega t} \\
&\qquad \nu \text{ considerably less than both } \frac{1}{2\pi}\sqrt{\frac{K}{m}} \text{ and } \frac{K}{2\pi R_m} \\
&\textit{Resistance-controlled: } R_m \text{ large} \qquad Z_m \simeq R_m \qquad \frac{dx}{dt} \simeq \frac{F}{R_m}e^{-i\omega t} \\
&\qquad \nu \text{ considerably less than } \frac{R_m}{2\pi m}\text{, larger than } \frac{K}{2\pi R_m}
\end{aligned} \tag{2.3.4}$$

Mass-controlled: m large $\qquad Z_m \simeq -2\pi i\nu m \qquad \dfrac{d^2x}{dt^2} \simeq \dfrac{F}{m} e^{-i\omega t}$

ν considerably larger than both $\dfrac{1}{2\pi}\sqrt{\dfrac{K}{m}}$ and $\dfrac{R_m}{2\pi m}$

It is to be noticed that every driven oscillator is mass-controlled in the frequency range well above its natural frequency ν_0, is resistance-controlled near ν_0 (though this range may be very small), and is stiffness-controlled for frequencies much smaller than ν_0. It simply requires the proper choice of mechanical constants to place one or another of these ranges in the desired place in the frequency scale. It is also to be noted that there is *always* an upper limit to the frequency range over which an oscillator is stiffness-controlled, a lower limit to the range over which one is mass-controlled, and both an upper and a lower limit to the range over which an oscillator can be resistance-controlled. We can move these limits about by changing the mechanical constants, but we never remove the limits entirely.

Energy relations

The average energy lost per second by the oscillator due to friction, when in the steady state, is $P = \frac{1}{2}R_m U^2 = \frac{1}{2}R_m(F/|Z_m|)^2$ [Eq. (2.2.7) of last section]. The rate of supply of energy from the driver to the driven oscillator is

$$\begin{aligned} vF\cos(2\pi\nu t) &= \frac{F^2}{|Z_m|}\cos(2\pi\nu t)\cos(2\pi\nu t - \varphi) \\ &= \frac{F^2}{|Z_m|}\cos(2\pi\nu t)[\cos\varphi\cos(2\pi\nu t) + \sin\varphi\sin(2\pi\nu t)] \end{aligned}$$

The average value of this energy supplied per second by the driver is $\frac{1}{2}(F^2/|Z_m|)\cos\varphi = \frac{1}{2}R_m(F/|Z_m|)^2$ (since $\cos\varphi = R_m/|Z_m|$), which equals the loss of energy to friction P. One can say that the amplitude and phase of the driven oscillator so arrange themselves that the energy delivered by the driver just equals the energy lost by friction. Note the correspondence with the discussion following Eq. (1.3.2).

Response to transient forces

A simple-harmonic driving force is, of course, a very specialized way of setting the simple oscillator in motion. Driving forces are much more likely to be nonperiodic. For example, the force setting the piano string into motion is the blow of a hammer, and the oscillatory forces acting on an airplane wing are the nonperiodic reactions of turbulent air. As long as the internal restoring force $F_i(x)$ is linear in x, however, this complication adds no basically new complication. For if all the terms on the left-hand side of Eq. (2.3.1) are linear in x or its derivatives, a superposition of solutions

is a solution. For example, if the driving force has two components, $F_e = F_1 e^{-i\omega_1 t} + F_2 e^{-i\omega_2 t}$, then the steady-state solution of (2.3.1) is the sum $(-F_1/i\omega_1 Z_1)e^{-i\omega_1 t} + (-F_2/i\omega_2 Z_2)e^{-i\omega_2 t}$ of the responses for each component separately, Z_1 being the impedance of the system, given in Eq. (2.3.2), for $\omega = \omega_1$, and Z_2 is the impedance appropriate for the driving frequency $\omega_2/2\pi$. Since, as we saw in Sec. 1.3, any function of t can be built up as a sum or integral of simple-harmonic terms, the corresponding motion of the linear oscillator is the corresponding sum or integral of the responses to each of the components separately.

This additive property of the solution works only for the linear oscillator; in Chap. 14 we shall see that it does not work if either or both the frictional and elastic restoring forces are nonlinear. In the linear case, the equation for a nonperiodic driving force $f(t)$,

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + Kx = f(t)$$

has a Fourier transform which can be expressed in terms of the transforms X and F of x and f, respectively. From page 31 we see that we obtain

$$(-m\omega^2 - iR_m\omega + K)X(\omega) \equiv -i\omega Z(\omega)X(\omega) = F(\omega) \qquad (2.3.5)$$

where $$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{i\omega t}\,dt \qquad F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt$$

[Eqs. (1.3.16)] and where $Z(\omega)$ is the impedance defined in Eq. (2.3.2) for a simple-harmonic driving force of frequency $\omega/2\pi$. Therefore, for the linear oscillator, the response to any driving force can be computed by first analyzing the force into its frequency components $F(\omega)$, finding the response $X(\omega)$ for each component separately, and finally combining these by the Fourier integral

$$x(t) = \int_{-\infty}^{\infty} X(\omega)e^{-i\omega t}\,d\omega = \int_{-\infty}^{\infty} \frac{F(\omega)}{-i\omega Z(\omega)}\,e^{-i\omega t}\,d\omega \qquad (2.3.6)$$

to make up the complete response $x(t)$.

Examples of the Fourier transform method

Suppose, for example, the mass is given a push centered at $t = 0$, represented by the formula

$$f(t) = \begin{cases} \dfrac{a}{2}\,e^{at} & t < 0 \\ \dfrac{a}{2}\,e^{-at} & t > 0 \end{cases}$$

where the mean duration of the push is proportional to $1/a$, and the magnitude of the push has been adjusted so that its total impulse $\int f\,dt$ is unity.

The Fourier transform of this function is

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt = \frac{a/4\pi}{a+i\omega} + \frac{a/4\pi}{a-i\omega} = \frac{a^2/2\pi}{a^2+\omega^2}$$

Thus, according to Eq. (2.3.5), the Fourier transform of the displacement is

$$X(\omega) = \frac{F(\omega)}{-m\omega^2 - iR_m\omega + K} = \frac{-(a^2/2\pi m)}{(\omega+ia)(\omega-ia)(\omega-\omega_f+ik)(\omega+\omega_f+ik)}$$

where $k = R_m/2m$, $\omega_f^2 = \omega_0^2 - k^2$, and $\omega_0^2 - K/m$ are the damping constant, the squares of the resonance and natural frequencies (times 2π) discussed earlier in this section.

We are now in a position to use Eq. (1.2.15) to compute $x(t)$, the inverse transform of $X(\omega)$, from Eq. (2.3.6). The integrand of the integral for x, as a function of the complex variable ω, has four poles, at $\omega = \pm ia$ and $\omega = \pm\omega_f - ik$, one above the real axis and the other three below. The residue of $X(\omega)e^{-i\omega t}$ at $\omega = ia$ is $ae^{at}/4\pi im[\omega_f^2 + (k+a)^2]$, that at $\omega = -ia$ is $-ae^{-at}/4\pi im[\omega_f^2 + (k-a)^2]$, and those at the other two poles are $-a^2e^{-kt-i\omega_f t+i\Phi}/4\pi m\omega_f A$ and $a^2e^{-kt+i\omega_f t-i\Phi}/4\pi m\omega_f A$, respectively, where $a^2 + (\omega_f + ik)^2 = Ae^{i\Phi}$ (A being real). Consequently, the instructions of Eq. (1.2.15) produce

$$x(t) = \begin{cases} \dfrac{ae^{at}}{2m[\omega_f^2 + (k+a)^2]} & t < 0 \\ \dfrac{ae^{-at}}{2m[\omega_f^2 + (k-a)^2]} + \dfrac{a^2}{m\omega_f A}e^{-kt}\sin(\omega_f t - \Phi) & t > 0 \end{cases}$$

Since $A^2 = (a^2 + \omega_f^2 + k^2)^2 - 4k^2a^2$ and $\sin\Phi = 2k\omega_f/A$, it is not difficult to verify that $x(t)$ and dx/dt are both continuous at $t = 0$, though df/dt is discontinuous at $t = 0$. The displacement x increases exponentially during the time $t < 0$, when the applied force is smoothly increasing. At $t = 0$ there is a discontinuity in the rate of change of the force, which generates a transient free vibration in x, exhibiting the damped oscillations of Eq. (2.2.3).

When $a \to \infty$, the force $f(t)$ becomes the Dirac delta function of Eq. (1.3.24), corresponding to an instantaneous unit impulse applied at $t = 0$. In this case $F(\omega) \to 1/2\pi$, and the solution becomes

$$x(t) = \begin{cases} 0 & t < 0 \\ \dfrac{1}{m\omega_f}e^{-kt}\sin(\omega_f t) & t > 0 \end{cases} \tag{2.3.7}$$

which could, of course, have been obtained from Eqs. (2.2.3) (for initial displacement zero, initial velocity $1/m$).

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When the driving force starts at $t = 0$ and the oscillator is at rest before this, the specialized techniques of the Laplace transform [see Eq. (1.3.22)] are useful. For example, suppose the driving force is

$$f(t) = \begin{cases} 0 & t < 0 \\ \sin(ut) & t > 0 \end{cases}$$

which has a transform

$$F(\omega) = F(is) = \frac{1}{2\pi}\int_0^\infty e^{-st}\sin(ut)\,dt = \frac{u/2\pi}{u^2 + s^2} = \frac{-(u/2\pi)}{(\omega + u)(\omega - u)}$$

The Fourier transform of the displacement is then

$$X(\omega) = \frac{F(\omega)}{-i\omega Z(\omega)} = \frac{u/2\pi m}{(\omega + u)(\omega - u)(\omega - \omega_f + ik)(\omega + \omega_f + ik)}$$

with all its poles on or below the real axis of ω. Thus, according to Eqs. (1.2.15), $x(t) = 0$ for $t < 0$ (as, of course, it should, since no force acts on the system until $t = 0$). The inverse transform for $t > 0$ is

$$x(t) = \frac{1}{mA}\left[\sin(ut - \vartheta) - \frac{u}{\omega_f}e^{-kt}\sin(\omega_f t - \Phi)\right] \tag{2.3.8}$$

where $\omega_f^2 - (u - ik)^2 = Ae^{i\vartheta}$ $\qquad \tan\vartheta = \dfrac{2uk}{(\omega_0^2 - u^2)} = \dfrac{uR_m}{K - mu^2}$

$$(\omega_f + ik)^2 - u^2 = Ae^{i\Phi} \qquad \tan\Phi = \frac{2k\omega_f}{\omega_f^2 - k^2 - u^2}$$

$$A^2 = (\omega_f^2 + k^2 - u^2)^2 + 4u^2k^2 = \left(\frac{u}{m}\right)^2 |Z(u)|^2$$

Curves for x against t, for different values of $u/2\pi$, the driving frequency, are shown in Fig. 2.1. At $t = 0$ the oscillator starts from zero displacement and velocity, with linearly increasing acceleration at first; its motion for a while is a combination of forced and transient oscillations; eventually, the transient term [the second term in (2.3.8)] dies out and leaves the steady-state motion (the first term). The angles ϑ and Φ are those discussed in Eqs. (2.3.2) and (2.3.3). This time we have used $u/2\pi$ for the driving frequency, instead of ν or $\omega/2\pi$, since we are now using ω as the variable in the Fourier transform.

There is still another way of looking at Eq. (2.3.6), which will have some application later. We insert the definition of $F(\omega)$ into Eq. (2.3.6) and invert the order of integration.

$$x(t) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}\,d\omega}{-i\omega Z(\omega)}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{i\omega\tau}d\tau = \int_{-\infty}^{\infty} f(\tau)\,d\tau\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega(t-\tau)}}{-i\omega Z(\omega)}d\omega\right]$$

To elucidate the implications of this expression, we return to Eq. (2.3.7) for the response of an oscillator to a unit impulsive driving force. Formally speaking, this can be represented by the inverse Fourier transform of the product of $1/-i\omega Z(\omega)$ and the Fourier transform of a delta function, which is unity. If an impulsive force $f(\tau)\,\delta(t-\tau)$ is applied at $t=\tau$, the Fourier transform of the force is

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\tau)e^{i\omega t}\,\delta(t-\tau)\,dt=\frac{1}{2\pi}f(\tau)e^{i\omega\tau}$$

and the response of the oscillator is $f(\tau)g(t-\tau)$, where

$$g(t-\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega(t-\tau)}}{-i\omega Z(\omega)}\,d\omega=\begin{cases}0 & t<\tau\\ \dfrac{1}{m\omega_f}e^{-k(t-\tau)}\sin\left[\omega_f(t-\tau)\right] & t>\tau\end{cases}$$

$$m\frac{d^2g}{dt^2}+R_m\frac{dg}{dt}+Kg=\delta(t-\tau)$$

is just the quantity in brackets in the double integral for $x(t)$.

In other words, the effect of each portion $f(\tau)$ of the force, for each separate time interval $d\tau$, can be computed separately, and the separate responses can then be combined,

$$x(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)\,d\tau \tag{2.3.9}$$

to obtain the complete response. Instead of analyzing the force into its frequency components, we can analyze it into its *time components* $f(\tau)\,\delta(t-\tau)$, solve the equation of motion for each impulsive force separately, and then combine the solutions to obtain $x(t)$. Again, this is possible only because the equation of motion is linear in x.

To finish this discussion, we write down the general rule for finding the response of the simple oscillator to a driving force $f(t)$, which starts at $t=0$ and has a Fourier transform $F(\omega)=\dfrac{1}{2\pi}\displaystyle\int_0^{\infty}e^{i\omega t}f(t)\,dt$, as long as $F(\omega)$ has only simple poles (none of them will be above the real axis if $f=0$ for $t<0$). Using Eqs. (1.2.15), we have

$$x(t)=0 \qquad \text{for } t<0$$

$$x(t)=\frac{\pi}{m}\frac{1}{i\omega_f}e^{-kt}\left[e^{i\omega_f t}F(-\omega_f-ik)-e^{-i\omega_f t}F(\omega_f-ik)\right]$$

$$-\begin{pmatrix}2\pi i \text{ times the sum of the residues of}\\ Fe^{-i\omega t}/(-i\omega Z) \text{ at all simple poles of}\\ F \text{ on and below the real axis of } \omega\end{pmatrix} \qquad \text{for } t>0 \tag{2.3.10}$$

$$k=\frac{R_m}{2m}\qquad \omega_0^2=\frac{K}{m}\qquad \omega_f^2=\omega_0^2-k^2$$

The first term is the transient motion of the system due to the sudden onset of $F(t)$ at $t = 0$. The terms due to the poles of $F(\omega)$ are a sort of generalized steady-state motion and depend on the specific form of $f(t)$.

The corresponding expression for $v(t)$ is

$$v(t) = 0 \qquad \text{for } t < 0$$

$$v(t) = \frac{\pi}{m\omega_f} e^{-kt}[(\omega_f + ik)e^{i\omega_f t}F(-\omega_f - ik) + (\omega_f - ik)e^{-i\omega_f t}F(\omega_f - ik)] - \begin{pmatrix} 2\pi i \text{ times the sum of the residues of} \\ Fe^{-i\omega t}/Z \text{ at all poles of } F \text{ on or below} \\ \text{the real axis of } \omega \end{pmatrix} \qquad \text{for } t > 0$$

Further discussion requires choosing specific forms for f and F.

The two examples we have worked out have displayed both transient and steady-state terms.

Autocorrelation of the response

In addition to impulses and other suddenly applied forces, acoustic systems are often subjected to stationary, bounded, nonperiodic driving forces. The pressure fluctuations caused by turbulent flow and the random impulses imparted to automobile springs, while moving over a rough road, are examples. The spectrum density $|F(\omega)|^2$ and autocorrelation function $\Upsilon(\tau)$, discussed in Sec. 1.3, are appropriate means of describing such forces; we can also use them to describe the response of the acoustic system.

A typical driving force of this kind is the random fluctuation called "white noise," which is a superposition of all frequencies below some natural limit $\omega_n/2\pi$, the amplitude of each frequency component below this limit being roughly equal. A white-noise driving force will thus have a spectrum density $|F(\omega)|^2$, that is independent of ω for $|\omega| < \omega_n$ and drops to zero for $|\omega| > \omega_n$. In such a random noise, the phase of each frequency component, and thus the phase angle of $F(\omega)$, is a random function of ω, any value from 0 to 2π being equally likely. Thus the magnitude of $F(\omega)$ is a rather simple function of ω, but its phase cannot be expressed in terms of any simple, continuous function. Because of this irregular dependence of phase angle Φ on ω, it is impossible to carry out the inverse Fourier transform and express a white-noise force $f(t)$ as an analytic function of t.

We can approximate white-noise behavior by taking the sum (instead of the integral) of a number of sinusoidal terms, with randomly chosen frequencies and phase angles. The curve for $f(t)$ shown in Fig. 2.3, for example, has been computed from a series of 10 unit-amplitude terms,

$$f(t) = \sum_{m=1}^{10} \cos(\omega_m t - \Phi_m)$$

where the Φ's were chosen at random from the range 0 to 2π and the ω's were random choices from the range 0 to 10π. Thus the plotted curve for $f(t)$ approximates white noise with a "natural limit frequency" of 5. No periodicity is apparent; once in a while the components nearly all reinforce each other and a peak occurs, but these peaks are not periodically related. Most of the time the curve meanders aimlessly above and below the t axis.

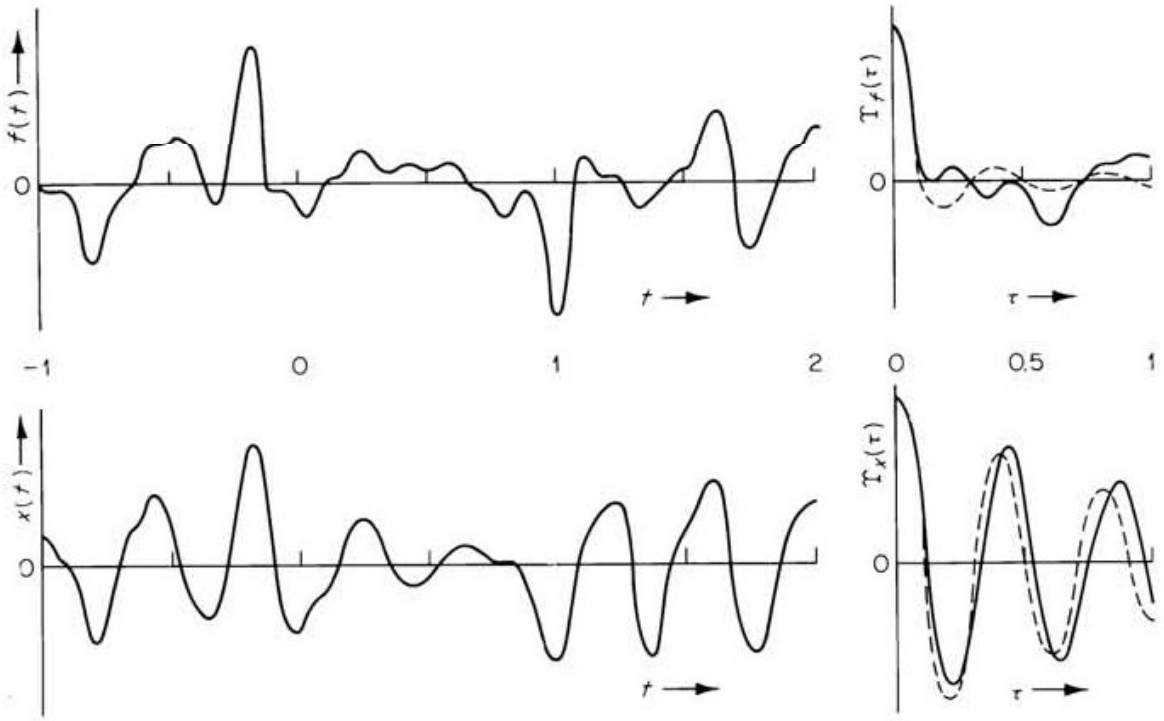


FIGURE 2.3
White-noise force $f(t)$, its autocorrelation function $\Upsilon_f(\tau)$; corresponding response $x(t)$ of simple oscillator and its autocorrelation function $\Upsilon_x(\tau)$. Solid curves are for the sample curve shown; dotted curves are the mean autocorrelation function for white noise.

As we pointed out in Sec. 1.3, the appropriate way to describe such a random force is in terms of its autocorrelation function

$$\Upsilon_f(\tau) = \lim_{T\to\infty}\left[\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} f(t)\,f(t+\tau)\,dt\right] \tag{2.3.11}$$

and its related Fourier transform [Eq. (1.3.20)]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\Upsilon_f(\tau)\,d\tau = \frac{2\pi\,|F(\omega)|^2}{T}$$

which is proportional to the spectrum density $|F|^2$ of $f(t)$. Although the phase angle of the Fourier transform $F(\omega)$ of $f(t)$ is a random function of ω, the spectrum density $|F|^2$ is a fairly smooth function of ω, which can be

measured and approximated in terms of simple analytic functions. Similarly with the autocorrelation function: dealing with squares of amplitudes avoids difficulties with irregularities of phase. For example, the $f(t)$ of Fig. 2.3 corresponds roughly to a density $|F|^2$ which is constant ($F = F_0$) for the frequency range $-\omega_n < \omega < \omega_n$ ($\omega_n = 10\pi$ in the figure) and is zero for $|\omega| > \omega_n$. The autocorrelation function of $f(t)$ is therefore

$$\Upsilon_f(\tau) = \frac{2\pi F_0^2}{T}\int_{-\omega_n}^{\omega_n} e^{-i\omega\tau}\, d\omega = \frac{4\pi\omega_n F_0^2}{T}\frac{\sin(\omega_n\tau)}{\omega_n\tau} \tag{2.3.12}$$

which is the dotted line in the right-hand top part of Fig. 2.3. The solid line was obtained by integrating the product $f(t)f(t+\tau)$ as read from the plot, over t from $t = -1$ to $t = +1$ [i.e., for $T = 2$ in Eq. (2.3.11)]. This is, of course, only approximately equal to the Υ_f for a function which has uniform spectrum density from $-\omega_n$ to $+\omega_n$, given by the dotted line, but the correspondence is fairly good.

The white-noise forces encountered in practice do not usually have quite as sharp a cutoff in their spectrum density at $\omega = \omega_n$. All frequencies are represented in the pressure fluctuations of turbulence, for example, though the density diminishes in magnitude for ω larger than some ω_n. A better approximation for the spectrum density would be

$$|F(\omega)|^2 = F_0^2 \exp\frac{-\omega^2}{2\omega_n^2}$$

which has an autocorrelation function

$$\Upsilon_f(\tau) = \frac{2\pi F_0^2}{T}\int_{-\infty}^{\infty} e^{-i\omega\tau-(\omega^2/2\omega_n^2)}\, d\omega = (2\pi)^{\frac{3}{2}}\frac{\omega_n F_0^2}{T}\, e^{-\frac{1}{2}\omega_n^2\tau^2} \tag{2.3.13}$$

having a "peak" at $\tau = 0$, of "width" $2/\omega_n$, dropping exponentially to zero on both sides of the peak, without any of the oscillations of the Υ_f of Eq. (2.3.12). The small amount of high frequencies present in this variety of white noise makes the plot of $f(t)$ more jagged than that shown in Fig. 2.3; this high frequency cancels out the last trace of oscillations in Υ_f.

Having now seen how a randomly fluctuating force can be described in terms of its spectrum density and its autocorrelation function, we must next determine the response of a simple oscillator to such a driving force. Reference to Eq. (2.3.5) shows that the spectrum density for the motion of the system is

$$\begin{aligned} |X(\omega)|^2 &= \frac{|F(\omega)|^2}{\omega^2|Z|^2} \\ &= \frac{|F(\omega)|^2/m^2}{(\omega+\omega_f+ik)(\omega-\omega_f+ik)(\omega+\omega_f-ik)(\omega-\omega_f-ik)} \end{aligned} \tag{2.3.14}$$

where ω_f and k are defined in Eqs. (2.2.1) and (2.2.2) and after Eq. (2.3.10).

In most cases of practical interest the "cutoff" frequency $\omega_n/2\pi$ for the random force is considerably larger than the resonance frequency $\omega_f/2\pi$ of the oscillator, so that, over the range of ω, where $|X(\omega)|^2$ is nonnegligible, $|F(\omega)|^2$ is practically equal to F_0^2, and we may substitute F_0^2 for $|F(\omega)|^2$ in Eq. (2.3.14).

Several interesting results are immediately apparent. In the first place, as long as the driving force is essentially white noise over the range covering ω_f, then the spectrum density $|X(\omega)|^2$ of the oscillator displacement is determined primarily by the properties of the oscillator (its ω_f and k or Q), and not by the presence or absence of frequency components in $f(t)$, for frequencies much larger than $\omega_f/2\pi$. In fact, the only effect of the white noise on the spectrum density of the displacement is on its magnitude, which is proportional to F_0^2. In the second place, unless the frictional term $k = R_m/2m$ is larger than $\omega_0 = \sqrt{K/m}$, the resonance peak in $1/|Z|^2$ ensures that the spectrum density $|X|^2$ of the displacement has a peak at $\omega = \omega_f$. If k is much smaller than ω_f ($Q \gg 1$), the frequencies near $\omega_f/2\pi$ predominate in the motion of the oscillator; the system has picked out those frequencies in the driving force which are near its own resonance frequency. This, of course, has an effect on the form of the autocorrelation function $\Upsilon_x(\tau)$ of the oscillator's displacement, which can be seen as soon as we compute it.

$$\Upsilon_x(\tau) = \frac{2\pi F_0^2}{T}\int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}\, d\omega}{\omega^2\,|Z(\omega)|^2} = \left(\frac{\pi^2 F_0^2}{m^2 k \omega_0 \omega_f T}\right) e^{-k\tau}\cos(\omega_f\tau - \theta) \quad (2.3.15)$$

where $\omega_f + ik = \omega_0 e^{i\theta}$, or $\sin\theta = k/\omega_0 = \frac{1}{2}Q$, and T is very large compared with $2\pi/\omega_f$. The expression given is for positive values of τ; for negative values, change the sign of τ and of θ in the exponential and the cosine. This function Υ_x has zero slope at $\tau = 0$, drops down to zero value at $\tau = \pm[(\frac{1}{2}\pi + \theta)/\omega_f]$, exhibiting damped oscillations beyond this, showing that, when k is small, there is considerable long-range correlation in $x(t)$. Indeed, the behavior of Υ_x is similar to the free vibrations of x, as a mental analysis of the system's response will demonstrate it should be. The factor in parentheses, multiplying the damped cosine, is of course the mean-square amplitude of motion of the oscillator.

The curve for $x(t)$ in Fig. 2.3 is a typical response curve for a simple oscillator with $\omega_f = 5\pi$ and $k = 0.5$ to a random driving force of the sort shown above it. The increased emphasis of the frequencies near $\nu_f = 2.5$ (period $= 0.4$) is apparent, but it is also apparent that $x(t)$ is far from being periodic in time; the nonperiodic driving force continually interferes with whatever resonance oscillations it may start. When one calculates the autocorrelation function, the random disturbances average out, and Υ_x resembles the undisturbed vibration of the oscillator. The solid curve for Υ_x is obtained by integrating the curve for $x(t)$ for $T = 2$; it checks fairly well with the dotted curve, obtained from Eq. (2.3.15), considering the smallness of the value of T.

Use of the Laplace transform

In many cases we do not know, or do not wish to know, what has happened to the system in the past. We know the position and velocity at $t = 0$ and the nature of the forces acting after $t = 0$, and we wish to compute the motion thereafter. In such a case it is somewhat easier to use the modification of the Fourier transform called the Laplace transform. It was defined in Eqs. (1.3.23). The basic properties may be obtained from those of the Fourier transform. The following equations will be of use later: $F_L(s)$ is the Laplace transform of $f(t)$ if

$$F_L(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad f(t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} F_L(s)e^{st}\,ds$$

where $\omega = is$ is the corresponding variable for the Fourier transform.

To solve the equation for the forced motion of a simple oscillator, for $t > 0$, when the force is suddenly applied at $t = 0$,

$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + Kx = f(t)u(t)$$

we take its Laplace transform,

$$(ms^2 + Rs + K)X_L(s) - mx'(0) - (ms + R)x(0) = F_L(s)$$

where $x(0)$ is the initial displacement of the oscillator and $x'(0)$ is its initial velocity. We can then solve for $X_L(s)$, the Laplace transform of $x(t)$.

$$X_L(s) = \frac{(1/m)F_L(s) + x'(0) + (s + 2k)x(0)}{(s + k)^2 + \omega_f^2} \tag{2.3.16}$$

where $k = R/2m$, and $\omega_f^2 = (K/m) - k^2$ as before [Eq. (2.3.10) et seq.]. Since $x(0)$ and $x'(0)$ are present, this formula already includes initial conditions.

When we take the inverse transform of this expression, the initial values $x(0)$ and $x'(0)$ are numerical constants, independent of s, so that the transient part of the solution can be obtained from (9), (15), and (16) of Table 2.1.

$$\begin{aligned} x_{tr} &= \text{inverse transform of } \frac{x'(0) + kx(0) + (s + k)x(0)}{(s + k)^2 + \omega_f^2} \\ &= \frac{x'(0) + kx(0)}{\omega_f}e^{-kt}\sin(\omega_f t) + x(0)e^{-kt}\cos(\omega_f t) \end{aligned} \tag{2.3.17}$$

which holds for the transient part for $t > 0$. To this must be added the inverse transform of $F_L(s)/[(s + k)^2 + \omega_f^2]$ to obtain the complete expression for x for $t > 0$. It is not difficult to see that x_{tr} reduces to $x(0)$ at $t = 0$, and that its time derivative reduces to $x'(0)$.

In most cases of interest the inverse transform of the F_L term (the steady-state motion) can be obtained by appropriate manipulation of Table 2.1. For example, if the mass is hit by a hammer just after $t = 0$, being given an impulse I_0, then the driving force is $f(t) = I_0\,\delta(t - \epsilon)$ $(\epsilon \to 0)$, and $F_L(s) = I_0 e^{-\epsilon s}$ $(\epsilon \to 0)$. The initial values $x(0)$ and $x'(0)$ are the displacement and velocity of the oscillator just before the hammer stroke. Formula

TABLE 2.1

FUNCTION	LAPLACE TRANSFORM
(1) $fa(t)$	$aF_L(s)$
(2) $f(at)$	$(1/a)F_L(s/a)$
(3) $f'(t) = df/dt$	$sF_L(s) - f(0)$
(4) $f''(t) = d^2f/dt^2$	$s^2F_L(s) - f'(0) - sf(0)$
(5) $\int_0^t f(x)dx$	$(1/s)F_L(s)$
(6) $t^n f(t)$	$(-1)^n(d^nF_L/ds^n)$
(7) $(1/t)f(t)$	$\int_s^\infty F_L(q)dq$
(8) $f(t-a)$ $t > a$; 0 $t \leqslant a$	$e^{-as}F_L(s)$
(9) $e^{bt}f(t)$	$F_L(s - b)$
(10) $\int_0^t f(x)g(t-x)\,dx$	$G_L(s)F_L(s)$
(11) $\delta(t - a)$ [Eq. (1.3.25)]	e^{-as}
(12) $u(t - a)$ [Eq. (1.3.22)]	$(1/s)e^{-as}$
(13) $e^{bt}u(t)$	$1/(s - b)$
(14) $t^n e^{bt}u(t)$	$n!/(s - b)^{n+1}$
(15) $\sin(bt)u(t)$	$b/(s^2 + b^2)$
(16) $\cos(bt)u(t)$	$s/(s^2 + b^2)$
(17) $t^n \sin(bt)u(t)$	$\frac{n!}{2i}[(s - ib)^{-n-1} - (s + ib)^{-n-1}]$
(18) $t^n \cos(bt)u(t)$	$\frac{n!}{2}[(s - ib)^{-n-1} + (s + ib)^{-n-1}]$
(19) $2\sum_{n=0}^{\infty} f(t - 2na - a)u(t - 2na - a)$	$F_L(s)/\sinh(as)$
(20) $2\sum_{n=0}^{\infty} (-1)^n f(t - 2na - a)u(t - 2na - a)$	$F_L(s)/\cosh(as)$
(21) $2\sum_{n=0}^{\infty} e^{-(2n+1)b}\,\delta(t - 2na - a)$	$1/\sinh(as + b)$

(15) of Table 2.1 can then be used to complete the formula for the displacement of the oscillator after the hammer blow,

$$x(t) = \frac{I_0 + mx'(0) + mkx(0)}{m\omega_f} e^{-kt} \sin(\omega_f t) + x(0)e^{-kt} \cos(\omega_f t) \quad (2.3.18)$$

the velocity just after the hammer blow being greater than that just before by the amount I_0/m.

In some cases the F_L term must be expanded in partial fractions before the inverse transforms can be found in Table 2.1. The following partial fraction expansions will be useful later:

$$\frac{1}{(s-a)(s-b)} = \frac{1}{(s-a)(a-b)} + \frac{1}{(s-b)(b-a)}$$

$$\frac{1}{(s-a)(s-b)(s-c)} = \frac{1}{(s-a)(a-b)(a-c)} + \frac{1}{(s-b)(b-a)(b-c)} + \frac{1}{(s-c)(c-a)(c-b)} \quad (2.3.19)$$

$$\frac{1}{(s-a)^2(s-b)} = \frac{1}{(s-a)^2(a-b)} - \frac{1}{(s-a)(a-b)^2} + \frac{1}{(s-b)(b-a)^2}$$

. . . . . .

The last expression has a pole of second order at $s = a$; its inverse transform would have a term proportional to te^{-at}.

For example, if $f(t) = \sin(ut)u(t)$, so that $F_L = u/(s^2 + u^2)$, the steady-state term in X_L can be expanded into two terms, one inversely proportional to $(s + k)^2 + \omega_f^2$, the other to $s^2 + u^2$; the resulting inverse transform is, of course, equal to Eq. (2.3.8).

Problems

1 A vibrator consists of a 100-g weight on the end of a spring. The spring's restoring force is proportional to the weight's displacement from equilibrium; if the weight is displaced 1 cm, this force is 10,000 dynes. The frictional force opposing its motion is proportional to its velocity and is 100 dynes when its velocity is 1 cm per sec. What is the modulus of decay of the oscillator? What is its decrement? What "frequency" do the vibrations have? What frequency would they have if there were no friction? If the weight were originally at rest and then were struck so that its initial velocity was 1 cm per sec, what would be its subsequent motion? What would be its maximum displacement from equilibrium?

2 The diaphragm of a loudspeaker weighs 1 g, and the displacement of its driving rod 1 mm from equilibrium requires a force of 1 million dynes. The frictional force opposing motion is proportional to the diaphragm's velocity and is 300 dynes when the velocity is 1 cm per sec. If it is assumed that the diaphragm moves like a simple oscillator, what will be its natural frequency, and what its modulus of decay? The driving rod is driven by a force of 100,000 $\cos(2\pi\nu t)$ dynes. Plot curves of real and imaginary parts of the mechanical impedance of the diaphragm as function of frequency, from $\nu = 0$ to $\nu =$ 1,000 cps.

3 The diaphragm of Prob. 2 is driven by a force of 100,000 $\cos(2\pi\nu t)$ dynes. Plot a curve of the amplitude of motion of the diaphragm as function of frequency, from $\nu = 0$ to $\nu = 1{,}000$ cps. Over what frequency range is this loudspeaker mass-controlled?

4 What is the mechanical impedance of a mass m without spring or friction? What is the impedance of a spring without mass? What will be the angle by which the displacement of the mass lags behind an oscillating force? What will be the angle of lag of the velocity of the mass behind the force? What are the corresponding angles for the spring?

5 The driving rod of the loudspeaker of Prob. 2 is driven by a force of $10{,}000[15 \sin (200\pi t) - 10 \sin (600\pi t) + 3 \sin (1000\pi t)]$ dynes. Plot the displacement of the diaphragm during one cycle, and compare it with the curve for the force.

6 The sharpness of resonance of a forced damped oscillator is given by the "half-breadth of the resonance peak," the difference between the two frequencies for which the amplitude of oscillation is half that at the resonance frequency ν_0. Prove that if the natural period of oscillation is negligibly small compared with 2π times the modulus of decay (i.e., if $k/4\pi$ is small compared with ν_0), then this half breadth equals $\sqrt{3}/\pi$ times the reciprocal of the modulus of decay of the oscillator. What is the half breadth for the diaphragm of Prob. 2? What would the frictional force have to be in order that the half breadth may be 20 cycles?

7 A mass m is attached to the lower end of a spring of stiffness constant K. The upper end of the spring is moved up and down with an amplitude $Be^{-i\omega t}$, and the frictional force on the mass is proportional to the relative velocity of the mass and the upper end of the spring (dr/dt), where $r = x - Be^{-i\omega t}$. Show that the equation of motion of the mass is

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + Kx = (K - i\omega R_m)Be^{-i\omega t}$$

that the steady-state motion of the mass is

$$x = \frac{R_m + i(K/\omega)}{R_m - i[\omega m - (K/\omega)]}Be^{-i\omega t}$$

and that the phase lag of x behind the displacement of the top of the spring is $\tan^{-1}\{[\omega m - (K/\omega)]/R_m\} + \tan^{-1}(K/\omega R_m)$. What is the amplitude of motion of x? What are the phase and amplitude of x at very low frequencies? At very high frequencies?

8 A simple mechanical system of impedance $Z_m = R_m - i[\omega m - (K/\omega)]$ has applied to it a force

$$F(t) = \begin{cases} 0 & t < 0 \\ \sin(at) & 0 < t < \dfrac{n\pi}{a} \\ 0 & \dfrac{n\pi}{a} < t \end{cases}$$

Use the results of Prob. 15 of Chap. 1 and of Eq. (2.3.10) to compute the displacement of the system.

9 Using the integral formula

$$\int_{-\infty}^{\infty} \exp\left[-i\omega t - \frac{t^2}{2T^2}\right]dt = \sqrt{\frac{\pi}{2}}\,T\exp(-\tfrac{1}{2}\omega^2T^2)$$

calculate the response of a simple linear oscillator with mass, resistance, and stiffness to a transient driving force

$$f(t) = F_0 \exp\left[-\tfrac{1}{2}\left(\frac{t}{T}\right)^2\right]$$

10 A simple oscillator of mass m, stiffness constant K, and resistance R is acted on by a stationary, nonperiodic driving force $f(t)$ having an autocorrelation function

$$\Psi_f(\tau) = \langle F^2 \rangle \exp(-a|\tau|)$$

where $\langle F^2 \rangle$ is the mean-square amplitude of the force. Use Eq. (1.3.20) to compute the spectrum density $|F(\omega)|^2$ of the driving force over a long time period T. Show that the spectrum density of the amplitude of the steady-state motion of the simple oscillator, under the influence of the force, is

$$|X(\omega)|^2 = \frac{(aT/2\pi^2 m^2)\langle F^2 \rangle}{(\omega^2 + a^2)[(\omega^2 - \omega_f^2)^2 + 2(\omega^2 + \omega_f^2)k^2 + k^4]}$$

where k and ω_f are given in Eqs. (2.2.1) and (2.3.10). Discuss (and sketch) the shape of this frequency-response curve as a function of ω, relating the heights and widths of its peaks when $k \ll \omega_0 \ll a$, $k \simeq \omega_0 \ll a$, and $k \ll \omega_0 \ll a$.

11 A force $f(t) = Qte^{-bt}u(t)$ is applied to a simple oscillator of mass m, stiffness K, and resistance R, initially at rest. Use the Laplace transform to calculate the displacement of the mass as a function of time for $t > 0$.

12 A periodically pulsating force

$$f(t) = F_0 \sum_{n=0}^{\infty} \delta\left[t - \frac{\pi}{\omega_f}(2n + 1)\right] \qquad \text{[see Eq. (1.3.24) for } \delta\text{]}$$

is applied to a simple oscillator, where the ω_f of the force is adjusted to be equal to the ω_f of the oscillator, as defined in the discussion preceding Eqs. (2.3.7). Show that, between the Nth and the $(N + 1)$st pulse, when $N \gg \omega_f/k$, so that steady state has been reached, the displacement of the oscillator is

$$x(t) \simeq \frac{F_0/m\omega_f}{1 - \exp(-2\pi k/\omega_f)} e^{-ktN} \sin(\omega_f t_N)$$

$$\text{for } 0 < t_N = t - \frac{\pi}{\omega_f}(2N + 1) < \frac{2\pi}{\omega_f}$$

CHAPTER

3

COUPLED LINEAR OSCILLATORS

3.1 TWO DEGREES OF FREEDOM

Perhaps the next step in a purely logical presentation of the subject would be to discuss the motion of a one-dimensional oscillator under the influence of a nonlinear stiffness force. However, the effects of a nonlinear force are complicated enough, and the techniques used in solving nonlinear equations differ sufficiently from those developed in Chap. 2, so that it will be preferable to defer nonlinear problems until a later chapter. Instead, we shall turn to the study of the behavior of two or more simple linear oscillators coupled together by forces which also are linear.

Examples of coupled oscillators

The simplest coupled system is that shown in Fig. 3.1*a*, two masses constrained to move in one direction and coupled by three springs to each other and to rigid supports. If the motion is longitudinal and of small

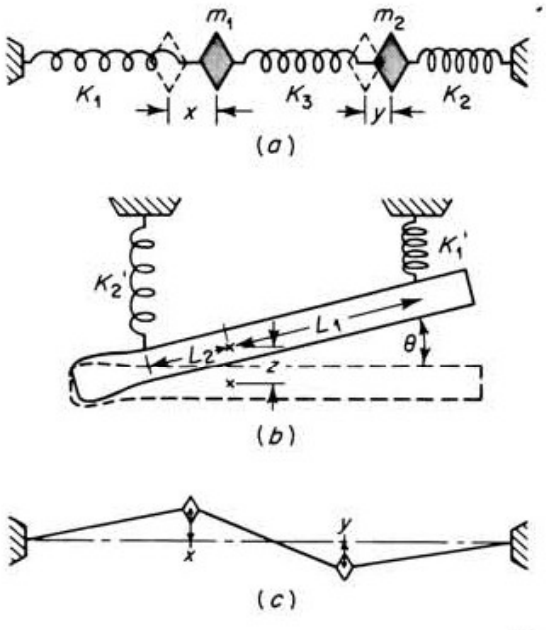


FIGURE 3.1
Examples of coupled oscillators.

enough amplitude so that the springs' restoring forces are linear functions of their extension, the equations of motion for the two masses are

$$\begin{aligned} m_1 \frac{d^2x}{dt^2} &= -K_1x - K_3(x - y) = -K_{11}x + K_{12}y \\ m_2 \frac{d^2y}{dt^2} &= -K_2y - K_3(y - x) = -K_{22}y + K_{21}x \end{aligned} \tag{3.1.1}$$

where K_1, K_3, K_2 are the stiffness constants of the three springs, and $K_{11} = K_1 + K_3$, $K_{22} = K_2 + K_3$, $K_{12} = K_{21} = K_3$. If we hold m_2 at $y = 0$, then the restoring force on m_1 is $-K_{11}x$, and thus m_1 will oscillate with isolated simple-harmonic motion of frequency $\nu_1 = \omega_1/2\pi$, where $\omega_1^2 = K_{11}/m_1$. Similarly, the frequency of isolated vibration of m_2 is $\omega_2/2\pi$, where $\omega_2^2 = K_{22}/m_2$. To simplify the wording of our discussion, we shall assume that m_1 has the higher frequency, $\omega_1 \geqslant \omega_2$.

The transverse oscillations of two masses on a string under tension, as in Fig. 3.1*c*, correspond to equations of motion like (3.1.1), as long as the displacements of the masses are small compared with the length of the string. Similarly, the system shown in Fig. 3.1*b*, where a bar of mass M and moment of inertia I about its center of mass moves up and down and rotates in a vertical plane about its center of mass. Here again there are two degrees of freedom, the vertical displacement of the center of mass, z, and the angle θ of rotation of the bar. The equations of motion are

$$M \frac{d^2z}{dt^2} = -K_1'(z + L_1\theta) - K_2'(z - L_2\theta)$$

$$I \frac{d^2\theta}{dt^2} = -L_1K_1'(z + L_1\theta) + L_2K_2'(z - L_2\theta)$$

if the displacements are small enough. These equations are similar to Eqs. (3.1.1), M and I taking the place of m_1 and m_2 and $K_1' + K_2'$, $L_1^2K_1' + L_2^2K_2'$, and $L_2K_2' - L_1K_1'$ taking the place of K_{11}, K_{22}, and $K_{21} = K_{12}$, respectively.

Analysis of the motion

Both of the equations of motion (3.1.1) involve the displacements of both oscillators; the equation for x_1 has a term $K_{12}x_2$, and the equation for x_2 has a similar one, $K_{21}x_1$, where $K_{12} = K_{21}$. The coupling spring, which these terms represent, enables momentum and energy to pass from one oscillator to the other and back again. Instead of each oscillator performing simple-harmonic motion with its own frequency of isolated motion, ν_1 or ν_2, as the case may be, both oscillators exhibit a combination of two frequencies in their motion.

But if the general motion of the linear system is a combination of periodic motions of different frequencies, it should be possible, by starting the system properly, to suppress all but one of these frequencies and have the system perform simple-harmonic motion. To see whether this is possible and, if it is, to find the possible frequencies of such motion, we try inserting $x = Ae^{-i\omega t}$ and $y = Be^{-i\omega t}$ into Eqs. (3.1.1). Performing the differentiation and then dividing out the exponential, we obtain

$$(K_{11} - m_1\omega^2)A = K_{12}B \qquad (K_{22} - m_2\omega^2)B = K_{21}A$$

These two equations can be solved for A/B and ω; therefore it is possible for the system to perform simple-harmonic motion if it is started just right (i.e., if A/B has the right value). For the two equations to be mutually consistent, ω^2 must satisfy the quadratic equation

$$(m_1\omega^2 - K_{11})(m_2\omega^2 - K_{22}) = K_{12}^2 \qquad \text{or} \qquad (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) = \mu^4 \tag{3.1.2}$$

where $\omega_1^2 = (2\pi\nu_1)^2 = K_{11}/m_1$ and $\omega_2^2 = (2\pi\nu_2)^2 = K_{22}/m_2$ define the frequencies of isolated oscillation, ν_1 and ν_2, of the two masses (we have assumed, to simplify the discussion, that $\nu_1 \geqslant \nu_2$), and $\mu^4 = K_{12}^2/m_1m_2$ is a measure of the coupling between the two oscillators.

The two roots of Eq. (3.1.2)

$$\begin{aligned} \omega_+^2 &= \tfrac{1}{2}(\omega_1^2 + \omega_2^2) + \tfrac{1}{2}\sqrt{(\omega_1^2 - \omega_2^2)^2 + 4\mu^4} \\ \omega_-^2 &= \tfrac{1}{2}(\omega_1^2 + \omega_2^2) - \tfrac{1}{2}\sqrt{(\omega_1^2 - \omega_2^2)^2 + 4\mu^4} \end{aligned} \tag{3.1.3}$$

determine the frequencies $\nu_+ = \omega_+/2\pi$ and $\nu_- = \omega_-/2\pi$ at which the system can oscillate with simple-harmonic motion. We note that, if there is no coupling ($\mu = 0$), ω_+ equals the larger of the isolated angular velocities, ω_1, and ω_- equals the smaller, ω_2. When $\mu \neq 0$, the oscillators are no longer isolated, and ω_+ is a bit larger than ω_1 and ω_- a bit smaller than ω_2. Since $K_{12} = K_{21}$ is always smaller than K_{11} or K_{22}, and thus μ is always smaller than ω_1 or ω_2, ω_-^2 can never be negative.

We have thus shown that if the system is started into motion with just the right relationship between the displacements of m_1 and m_2 (the right value of A/B), it can oscillate with simple-harmonic motion of frequency $\omega_-/2\pi$. The appropriate amplitude ratio is given by the equation $(\omega_1^2 - \omega_-^2)A = \mu^2\sqrt{m_2/m_1}\,B$, or what is the same thing, $\mu^2\sqrt{m_1/m_2}\,A = (\omega_2^2 - \omega_-^2)B$, since ω_- is a root of Eq. (3.1.2). In other words, a possible motion of the system is represented by the following equations:

$$x = \sqrt{\frac{1}{m_1}}\,\mu^2\,C_-e^{-i\omega_- t} \qquad y = \sqrt{\frac{1}{m_2}}\,(\omega_1^2 - \omega_-^2)C_-e^{-i\omega_- t}$$

Another possible simple-harmonic motion has frequency $\omega_+/2\pi$, for which the displacements are

$$x = \sqrt{\frac{1}{m_1}}\,(\omega_+^2 - \omega_2^2)C_+e^{-i\omega_+ t} \qquad y = -\sqrt{\frac{1}{m_2}}\,\mu^2 C_+ e^{-i\omega_+ t}$$

where the complex constants C_+ and C_- may have any magnitude or phase angle.

These are very specialized kinds of motion of the system; in each case a particular relationship between the amplitudes and phases of the two masses is required. For example, if the system is to oscillate with frequency $\omega_-/2\pi$, the two masses must move in phase (we have assumed that $\omega_1 \geqslant \omega_2$) and the amplitude of motion of m_1 must be $\sqrt{m_2/m_1}[\mu^2/(\omega_1^2 - \omega_-^2)]$ times that of mass m_2. When $\mu \to 0$, this ratio tends to zero; in the limit of small coupling, m_2 carries most of the energy of the system, and m_1 moves only a relatively small amount (as though m_2 were the driver and m_1 a driven oscillator). When $\mu = 0$, this mode of oscillation corresponds to m_2 vibrating with its isolated frequency $\omega_2/2\pi$ and m_1 being at rest.

Correspondingly, the mode of oscillation with frequency $\omega_+/2\pi$ has the two masses moving with opposing displacements, with

$$y = -\sqrt{m_1/m_2}[\mu^2/(\omega_+^2 - \omega_2^2)]x$$

and if μ is small, m_1 has the majority of the energy and m_2 acts like the driven oscillator.

Neither of these vibrations is the general sort of motion which the system would exhibit if it were started into motion in an arbitrary manner. The more general motion, however, is just a sum of both modes. Since Eqs. (3.1.1) are linear in both x and y, a sum of solutions is also a solution. Consequently, the general motion of the system is given by the real part of

$$\begin{aligned} x &= \sqrt{1/m_1}[(\omega_+^2 - \omega_2^2)C_+e^{-i\omega_+ t} + \mu^2 C_- e^{-i\omega_- t}] \\ &= A_+ \cos(\omega_+ t - \Phi_+) + \frac{\mu^2\sqrt{m_2/m_1}}{\omega_1^2 - \omega_-^2} A_- \cos(\omega_- t - \Phi_-) \\ y &= \sqrt{1/m_2}[-\mu^2 C_+ e^{-i\omega_+ t} + (\omega_1^2 - \omega_-^2)C_- e^{-i\omega_- t}] \\ &= -\frac{\mu^2\sqrt{m_1/m_2}}{\omega_+^2 - \omega_2^2} A_+ \cos(\omega_+ t - \Phi_+) + A_- \cos(\omega_- t - \Phi_-) \end{aligned} \tag{3.1.4}$$

where the phases and amplitudes of C_+ and C_-, or the values of A_+, A_-, and Φ_+, Φ_-, are adjusted to fit the initial conditions for both x and y.

Transfer of energy

For example, if m_1 is held motionless a distance A from equilibrium and m_2 is held at its equilibrium position, and both are released at $t = 0$, the subsequent motion is

$$\begin{aligned} x &= \frac{A}{\omega_+^2 - \omega_-^2}[(\omega_+^2 - \omega_2^2)\cos(\omega_+ t) + (\omega_2^2 - \omega_-^2)\cos(\omega_- t)] \\ &= A\cos[\tfrac{1}{2}(\omega_+ - \omega_-)t]\cos[\tfrac{1}{2}(\omega_1 + \omega_2)t] \\ &\quad - A\frac{\omega_1^2 - \omega_2^2}{\omega_+^2 - \omega_-^2}\sin[\tfrac{1}{2}(\omega_+ - \omega_-)t]\sin[\tfrac{1}{2}(\omega_1 + \omega_2)t] \end{aligned} \tag{3.1.5}$$

$$\begin{aligned} y &= \frac{\mu^2\sqrt{m_1/m_2}}{\omega_+^2 - \omega_-^2}A[-\cos(\omega_+ t) + \cos(\omega_- t)] \\ &= 2A\frac{\mu^2\sqrt{m_1/m_2}}{\omega_+^2 - \omega_-^2}\sin[\tfrac{1}{2}(\omega_+ - \omega_-)t]\sin[\tfrac{1}{2}(\omega_1 + \omega_2)t] \end{aligned}$$

where we have used Eqs. (3.1.2) and (3.1.3) several times to simplify the formulas. Displacement y is zero at $t = 0$ (as required by the initial conditions), but it does not stay zero long. If ω_+ does not differ much from ω_-, the two cosine terms in y take several cycles to get out of phase. The second formula for y shows that when $\omega_+ - \omega_- \ll \omega_+$, the motion of m_2 may be described, roughly, as that of an oscillator with frequency equal to the average of $\omega_+/2\pi$ and $\omega_-/2\pi$, having an amplitude of motion proportional to the magnitude of $\sin[\frac{1}{2}(\omega_+ - \omega_-)t]$, a relatively slowly varying, periodic function. At $t = 0$, m_1 has all the energy and m_2 is at rest. As time goes on, energy is fed to m_2, and its amplitude of motion increases to its maximum value, $2\mu^2 A\sqrt{m_1/m_2}/(\omega_+^2 - \omega_-^2)$, after which the energy flows back again to m_1.

An approximate expression for the energy possessed by each of the masses can be obtained by using the second forms of Eqs. (3.1.5). It is, of course, not strictly correct to speak of the energy of only part of a system, nor is it correct to use Eq. (2.2.5), appropriate for the energy of a simple oscillator, for an oscillator with sinusoidally varying amplitude. But if the variation in amplitude is slow (i.e., if $\omega_+ - \omega_- \ll \omega_+$), the following formulas are a fairly good approximation:

$$\begin{aligned} E_1 &\simeq \tfrac{1}{2}m_1 A^2[\tfrac{1}{2}(\omega_+ + \omega_-)]^2\left\{1 - \frac{4\mu^4}{(\omega_1^2 - \omega_2^2)^2 + 4\mu^4}\sin^2[\tfrac{1}{2}(\omega_+ - \omega_-)t]\right\} \\ E_2 &\simeq \tfrac{1}{2}m_1 A^2\frac{\mu^4(\omega_+ + \omega_-)^2}{(\omega_1^2 - \omega_2^2)^2 + 4\mu^4}\sin^2[\tfrac{1}{2}(\omega_+ - \omega_-)t] \end{aligned} \tag{3.1.6}$$

for the mean "energies of vibration" of m_1 and m_2, respectively. We see that the energy surges back and forth from m_1 to m_2 with a period $4\pi/(\omega_+ - \omega_-)$, part of m_1's energy going to m_2 and then back again. If $\omega_1 = \omega_2$, then m_2 will drain all the energy from m_1 before returning it.

The case of weak coupling

We have already mentioned that when the coupling spring is weak, so that $\mu^2 \ll \omega_1^2$ or ω_2^2, the system behaves like two separate simple oscillators, each slightly influencing the motion of the other. If m_1 has the majority of the energy, m_2 acts like a driven oscillator, and vice versa. To show this more clearly, we work out the first approximations to Eqs. (3.1.3) when μ^2 is small.

$$\begin{aligned} \omega_+^2 &\simeq \omega_1^2 + \frac{\mu^4}{\omega_1^2 - \omega_2^2} \quad \text{or} \quad \omega_+ \simeq \omega_1 + \frac{\mu^4}{2\omega_1(\omega_1^2 - \omega_2^2)} \\ \omega_-^2 &\simeq \omega_2^2 - \frac{\mu^4}{\omega_1^2 - \omega_2^2} \quad \text{or} \quad \omega_- \simeq \omega_2 - \frac{\mu^4}{2\omega_2(\omega_1^2 - \omega_2^2)} \end{aligned} \tag{3.1.7}$$

where $\omega_1/2\pi$ is, as we said, the larger of the two isolated frequencies of oscillation and $\omega_2/2\pi$ is the smaller.

Now suppose we set the system into oscillation with frequency $\omega_+/2\pi$. The displacements will be, to the first approximation,

$$x \simeq A_+ \cos(\omega_+ t - \Phi_+) \qquad y \simeq -\frac{K_{12}A_+}{m_2(\omega_+^2 - \omega_2^2)} \cos(\omega_+ t - \Phi_+)$$

and since $K_{12}/m_2 \ll \omega_+^2 - \omega_2^2$, the amplitude of motion of m_2 is considerably smaller than that of m_1. If coupling constant K_{12} were zero, m_1 would be oscillating by itself, with its own frequency, $\omega_1/2\pi$. If K_{12} is small but not zero, m_1 still has most of the energy, but its frequency of oscillation is slightly increased by the coupling, becoming $\omega_+/2\pi$. Mass m_2 behaves like a driven oscillator, vibrating with the frequency $\omega_+/2\pi$ of the driver and having an amplitude of motion equal to $K_{12}A_+$ divided by $m_2(\omega_2^2 - \omega_+^2)$. However, $K_{12}A_+ = F$ is the amplitude of the force applied to m_2, caused by the motion of m_1, and $m_2(\omega_2^2 - \omega_+^2) = -i\omega_+ Z_2$, where $Z_2 = i(K_{22}/\omega_+) - i\omega_+ m_2$ is m_2's mechanical impedance [Eqs. (2.3.2)]. Therefore m_2's motion is just the steady-state motion of a simple undamped oscillator, driven by m_1.

The other mode of oscillation corresponds to the slower oscillator, m_2, being the driver. Its natural frequency is decreased slightly by the coupling; the fact that m_1 is moving changes the effective restoring force on m_2. As before, the amplitude of motion of m_1 is equal to $K_{12}A_-$, the amplitude of the driving force on m_1, divided by $m_1(\omega_1^2 - \omega_-^2)$, which is $-i\omega_-$ times the mechanical impedance of m_1-plus-spring at frequency $\omega_-/2\pi$. In each case the driver, the oscillator having the majority of the energy, has its frequency of motion slightly modified by the presence of the second oscillator, coupled to it.

Of course, when the coupling is large, the interchange of energy between the oscillators is more rapid, and it no longer is appropriate to call one the driver and the other the driven.

So far we have been assuming that ω_1 is larger than ω_2 and, recently, have been assuming that $\mu^2 \ll \omega_1^2 - \omega_2^2$. What happens when $\omega_1 = \omega_2$, when the two oscillators are identical? Clearly, Eqs. (3.1.7) no longer hold. In this case, when $\mu^2 \ll \omega_1^2 = \omega_2^2$, Eqs. (3.1.3) become, to the first approximation,

$$\omega_+^2 = \omega_1^2 + \mu^2 \qquad \omega_+ \simeq \omega_1 + \frac{\mu^2}{2\omega_1} \qquad \omega_- \simeq \omega_1 - \frac{\mu^2}{2\omega_1} \tag{3.1.8}$$

Thus the coupled system still cannot oscillate with the isolated frequencies $\omega_1/2\pi$ of either oscillator by itself; its frequency of oscillation is either a bit larger or a bit smaller than this.

The reason for the continued difference between the natural frequencies is not hard to find. If m_1 were the driver and oscillated at a frequency $\omega_1/2\pi$, equal to the resonance frequency of m_2, the impedance of m_2 at this frequency would be zero and m_2 could only reach steady state if it could absorb an infinite amount of energy from m_1 (since we neglect friction in this example). But m_1 does not have an infinite amount of energy to donate to m_2; so it has to change its frequency enough so that m_2 is no longer in exact resonance and the system can reach steady state with a reasonable amount of energy given to m_2.

The motion with frequency $\omega_+/2\pi$ is

$$x = A_+ \cos(\omega_+ t - \Phi_+) \qquad y \simeq -\sqrt{\frac{m_1}{m_2}}\, A_+ \cos(\omega_+ t - \Phi_+)$$

corresponding to an equal sharing of the energy of the system (energy of $m_1 = \frac{1}{2}m_1 A^2 \omega_+^2 =$ energy of m_2). The same would be true if the system were oscillating with frequency $\omega_-/2\pi$, except that the two masses would move in phase instead of having opposed motions.

On the other hand, if we try to start the system with frequency $\omega_1/2\pi = \omega_2/2\pi$ by, for example, setting m_1 into motion with m_2 initially at rest, the subsequent motions are

$$\begin{aligned} x &= \tfrac{1}{2}A[\cos(\omega_- t) + \cos(\omega_+ t)] = A \cos\frac{\mu^2 t}{2\omega_1} \cos(\omega_1 t) \\ y &= \sqrt{\frac{m_1}{4m_2}}\, A[\cos(\omega_- t) - \cos(\omega_+ t)] \\ &= \sqrt{\frac{m_1}{m_2}}\, A \sin\frac{\mu^2 t}{2\omega_1} \sin(\omega_1 t) \end{aligned} \tag{3.1.9}$$

This is really an oscillation consisting of equal components of the two frequencies $\omega_+/2\pi$ and $\omega_-/2\pi$, but when $\mu \ll \omega_1$, we can consider both masses as oscillating with frequency $\omega_1/2\pi$, each having a periodically varying amplitude, with the comparatively long period $2\pi\omega_1/\mu^2$. Although m_1 starts off with all the energy, in a time $\pi\omega_1/\mu^2$ it will be at rest and m_2 will have all the energy, and so on, energy being fed back and forth from one mass to the other, ad infinitum.

3.2 NORMAL MODES OF OSCILLATION

Two noncoupled linear oscillators have two natural frequencies of oscillation, one exhibited when one mass is in motion, the other when the other mass is moving. We have just seen that two coupled oscillators also have two frequencies of oscillation, but to perform motion exhibiting just one of these frequencies, *both* masses must be moving, with a particular relationship between the motions. This relationship can be given a geometrical interpretation, which will help to visualize the behavior of more complex systems, to be considered later. But let us start with a simple example.

Normal coordinates

Figure 3.2 is a vertical view of a mass m, suspended at the lower end of a long stiff bar of rectangular cross section, represented in the figure by the shaded rectangle. The mass can thus move in two directions, each parallel to the plane of the paper. The displacement of the mass from equilibrium can be expressed either in terms of the coordinates X, Y, in line with the bar's cross section, or else in terms of x, y, also with origin at the equilibrium point, but inclined at an angle α with respect to X, Y.

Suppose we find that a displacement X, in the direction of the greater cross-sectional dimension of the bar, is resisted by a force $-K_+X$, pointed in the negative X direction, and that a displacement Y, perpendicular to this, is opposed by a force $-K_-Y$, antiparallel to the Y axis. Therefore, if the mass m is set in motion in the X direction, it will keep swinging in the X direction, with a frequency $\omega_+/2\pi = (\frac{1}{2}\pi)\sqrt{K_+/m}$; likewise, oscillations in the Y direction will remain in the Y direction and will have frequency $\omega_-/2\pi = (1/2\pi)\sqrt{K_-/m}$. Coordinates X and Y are the "normal" coordinates to describe the motion.

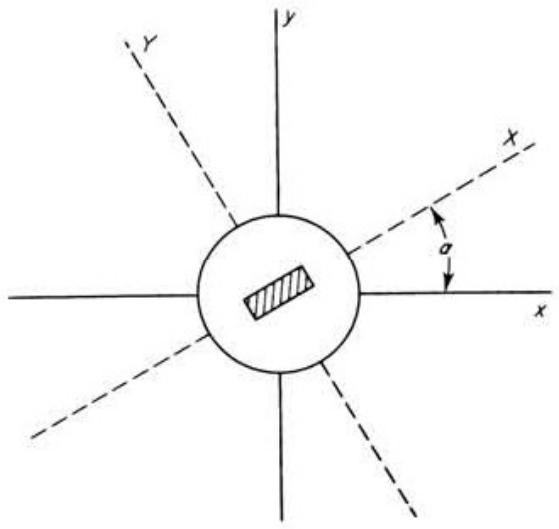


FIGURE 3.2
Vertical view of a two-dimensional oscillator, a mass on the lower end of stiff bar of rectangular cross section.

But suppose we wish to set the mass into oscillation in the x direction, at an angle α with respect to the X axis. What will its subsequent motion be? First we have to find out what force acts on m when it is displaced a distance x in this direction. This force can be calculated by transformation of axes. We find that the restoring force caused by the displacement x is not in the direction of the displacement; it has a component $-(K_+ \cos^2\alpha + K_- \sin^2\alpha)x$ in the x direction, but it also has a component $-(K_+ - K_-)(\sin\alpha\cos\alpha)x$ in the y direction, perpendicular to x (assuming still that the displacement is small enough so that the force is linearly proportional to the displacement). The angle between the displacement and the force is small if the bar is nearly rectangular, so that K_+ is nearly equal to K_-, but the two are not parallel unless $\alpha = 0$, 90, 180, or 270° if $K_+ \neq K_-$.

This is also true for a displacement along the y axis; the restoring force will then have a component $-(K_+ \sin^2\alpha + K_- \cos^2\alpha)y$ along y and a component $-(K_+ - K_-)(\sin\alpha\cos\alpha)y$ along x. The equations of motion of the mass, expressed in terms of the coordinates x, y, are thus

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -(K_+ \cos^2\alpha + K_- \sin^2\alpha)x - (K_+ - K_-)(\sin\alpha\cos\alpha)y \\ m\frac{d^2y}{dt^2} &= -(K_+ \sin^2\alpha + K_- \cos^2\alpha)y - (K_+ - K_-)(\sin\alpha\cos\alpha)x \end{aligned} \tag{3.2.1}$$

which are just Eqs. (3.1.1) for coupled oscillators, if we set

$$K_{11} = K_+ \cos^2\alpha + K_- \sin^2\alpha \qquad K_{22} = K_+ \sin^2\alpha + K_- \cos^2\alpha$$
$$K_{12} = K_{21} = -(K_+ - K_-)\sin\alpha\cos\alpha$$

Therefore, for the system of Fig. 3.2, the use of axes at an angle α to the normal coordinates converts a problem equivalent to two uncoupled oscillators into one equivalent to two coupled oscillators. The change occurs because a displacement in any direction other than "normal" produces a restoring force not in line with the displacement, and therefore gives rise to motion which is not in a straight line through the origin. Figure 3.3 shows the motion, both as plots of x and y as functions of time, and also as the path of m in the xy plane.

Transformation of coordinates

It would be well to work all this out in vector notation. The potential energy for a small displacement away from equilibrium, in terms of the normal coordinates X, Y, is [Eq. (1.1.3)]

$$V(X,Y) = \tfrac{1}{2}K_+X^2 + \tfrac{1}{2}K_-Y^2 \tag{3.2.2}$$

The corresponding force vector $\mathbf{F} = -\text{grad } V$ has component $-(\partial V/\partial X) = -K_+X$ along X and component $-(\partial V/\partial Y) = -K_-Y$ in the Y direction.

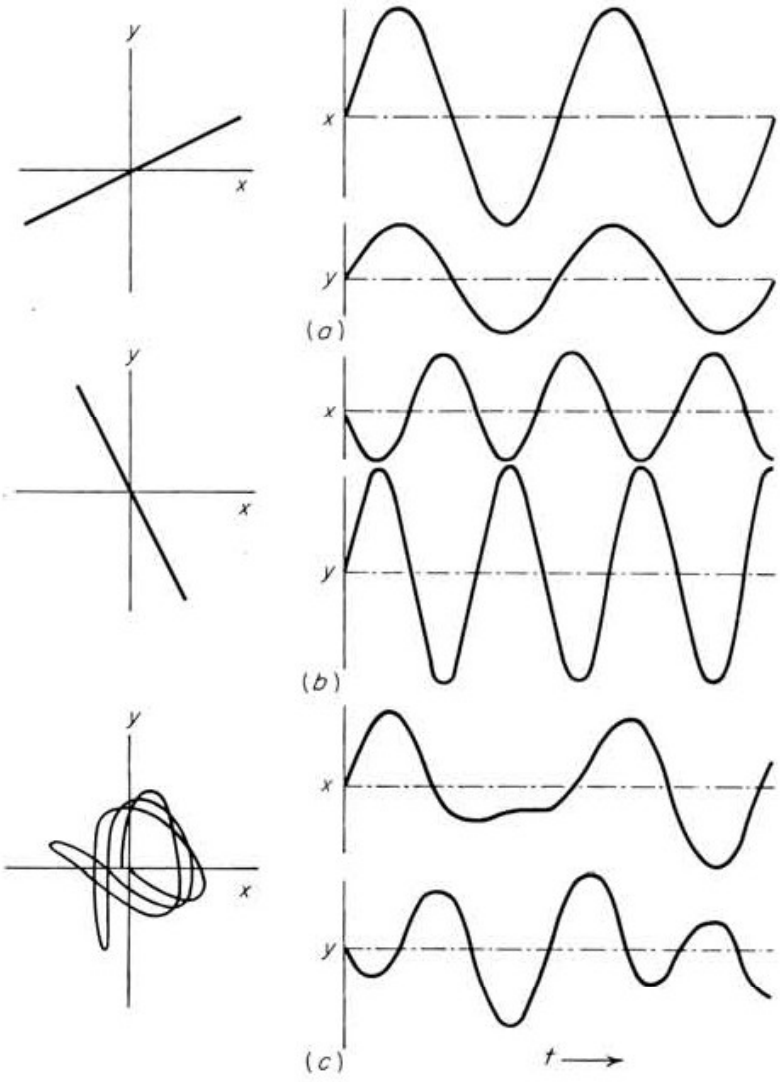


FIGURE 3.3
Motion of two coupled oscillators. Curves to the right show the displacements x and y as functions of time; those to the left show the path of the point representing the system in the xy plane (configuration space). Cases *a* and *b* show the two normal modes of vibration when the system point travels along a normal coordinate. Case *c* shows the general type of motion.

If we change to new axes, at an angle α to the normal coordinates, so that

$$\begin{aligned} x &= X\cos\alpha - Y\sin\alpha \qquad & y &= X\sin\alpha + Y\cos\alpha \\ X &= x\cos\alpha + y\sin\alpha \qquad & Y &= -x\sin\alpha + y\cos\alpha \end{aligned} \tag{3.2.3}$$

then simple substitution shows that the potential energy is

$$V(x,y) = \tfrac{1}{2}(K_+\cos^2\alpha + K_-\sin^2\alpha)x^2 + (K_+ - K_-)(\sin\alpha\cos\alpha)xy + \tfrac{1}{2}(K_+\sin^2\alpha + K_-\cos^2\alpha)y^2 \tag{3.2.4}$$

The components of force $\mathbf{F} = -\text{grad } V$ in the x and y directions are thus

$$F_x = -\frac{\partial V}{\partial x} = -(K_+ \cos^2 \alpha + K_- \sin^2 \alpha)x - (K_+ - K_-)(\sin \alpha \cos \alpha)y$$

$$F_y = -\frac{\partial V}{\partial Y} = -(K_+ \sin^2 \alpha + K_- \cos^2 \alpha)y - (K_+ - K_-)(\sin \alpha \cos \alpha)x$$

The vector equation of motion of the mass m is thus

$$m\frac{d^2\mathbf{R}}{dt^2} = -\text{grad } V \tag{3.2.5}$$

where $\mathbf{R}$ is the vector displacement of the mass from equilibrium.

If we express this equation in terms of the normal coordinates X, Y, the component equations are

$$m\frac{d^2X}{dt^2} + m\omega_+{}^2X = 0 \qquad m\frac{d^2Y}{dt^2} + m\omega_-{}^2Y = 0$$

where $\omega_+{}^2 = K_+/m$ and $\omega_-{}^2 = K_-/m$, which are the equations for two uncoupled harmonic oscillators. But if we insist on using the x, y coordinates to describe the motion, the component equations (3.2.1) are those for a pair of coupled oscillators. Evidently, we can change from coupled to uncoupled oscillators by rotating the axes into line with the normal axes of the bar. Use of Eqs. (3.1.3), with the special values

$$\omega_1{}^2 = \omega_+{}^2 \cos^2 \alpha + \omega_-{}^2 \sin^2 \alpha \qquad \omega_2{}^2 = \omega_+{}^2 \sin^2 \alpha + \omega_-{}^2 \cos^2 \alpha$$
$$\mu^2 = (\omega_-{}^2 - \omega_+{}^2) \sin \alpha \cos \alpha \tag{3.2.6}$$

appropriate for the system of Fig. 3.2, results in the following values of natural frequencies for the system:

$$\omega_+{}^2 \equiv \tfrac{1}{2}(\omega_+{}^2 + \omega_-{}^2) + \tfrac{1}{2}[(\omega_+{}^2 - \omega_-{}^2)^2(\cos^2 \alpha - \sin^2 \alpha)^2 + 4(\omega_+{}^2 - \omega_-{}^2) \sin^2 \alpha \cos^2 \alpha]^{\frac{1}{2}} = \omega_+{}^2$$

and similarly for $\omega_-{}^2$, which always come out equal to the frequencies of the normal coordinates, independent of the angle α of rotation of the axes. Also use of Eqs. (3.1.4), resulting in

$$x = \sqrt{\frac{1}{m}}\,(\omega_+{}^2 - \omega_-{}^2)(\cos^2 \alpha C_+ e^{-i\omega_+ t} - \sin \alpha \cos \alpha\, C_- e^{-i\omega_- t})$$

$$y = \sqrt{\frac{1}{m}}\,(\omega_+{}^2 - \omega_-{}^2)(\sin \alpha \cos \alpha\, C_+ e^{-i\omega_+ t} + \cos^2 \alpha C_- e^{-i\omega_- t})$$

reduces these equations to those of transformation of coordinates [Eqs.

(3.2.3)] when we set

$$X = \sqrt{\frac{1}{m}}\,(\omega_+^2 - \omega_-^2)\cos\alpha C_+ e^{-i\omega_+ t} = A_+ e^{-i\omega_+ t}$$

$$Y = \sqrt{\frac{1}{m}}\,(\omega_+^2 - \omega_-^2)\cos\alpha C_- e^{-i\omega_- t} = A_- e^{-i\omega_- t}$$

where C_+ and C_- are arbitrary constants. Thus each of the components of the motion, along each of the normal coordinates, has just one frequency, $\omega_+/2\pi$ or $\omega_-/2\pi$, respectively.

In other words, it is possible to describe the motion most simply in terms of a pair of normal coordinate axes, the component of motion along either of which exhibits just one of the natural frequencies of the system. Components of the motion along any other axes exhibit both frequencies. The motion is, of course, the same, no matter which axes are used; we have only changed our way of describing it by transforming axes.

More complex systems

The methods discussed for analyzing the motion of two coupled oscillators can be extended to systems of more than two oscillators. One determines the ways in which the system can oscillate with simple-harmonic motion and then builds up the free oscillations of the system as combinations of these various normal modes of oscillation. The complexities of the mathematics of this direct approach make it preferable to use the machinery of the Laplace transform to work out the general solution, even though this machinery obscures the physical picture to some extent.

We consider a system of N degrees of freedom, held together by linear forces which tend to bring each of the N displacements back to equilibrium. Thus each of the N coordinates ξ_i ($i = 1, 2, \ldots, N$), which together specify the configuration of the system, can be given as displacements from equilibrium. Some of the coordinates may be displacements, in some specified direction, of some portion of the system, measured in units of distance; others could be angular displacements of some portion from its equilibrium direction; and so on. To the ith coordinate corresponds an effective mass or moment of inertia, which can be labeled m_i. The coordinates themselves, the units in which they are measured, and the units of the corresponding mass factors are chosen so that the total kinetic energy of the system is the sum of terms of the form $\frac{1}{2}m_i(d\xi_i/dt)^2$.

The internal forces and torques which hold the system together are linear functions of the displacements as long as these displacements are all small enough. Of course, the displacement of one degree of freedom may tend to make other degrees of freedom move away from their equilibrium positions; this is the usual situation with coupled systems. In mathematical language, if $x, y, z, \ldots$ are the displacements from equilibrium of the various

degrees of freedom, expressed as distances or angles or any other units, the force (or torque) acting on the x coordinate, tending to make it return to equilibrium, is

$$F_{ex} = -A_{xx}x - A_{xy}y - A_{xz}z - \cdots$$

The coefficients A_{xx}, etc., can be called stiffness coefficients. If the forces are conservative, they can be obtained by partial differentiation of a potential energy function

$$V = \tfrac{1}{2}A_{xx}x^2 + \tfrac{1}{2}A_{xy}xy + \cdots + \tfrac{1}{2}A_{yx}yx + \tfrac{1}{2}A_{yy}y^2 + \cdots$$

where $A_{yx} = A_{xy}$ and $F_{ex} = -(\partial V/\partial x)$, etc. As mentioned, the kinetic energy of the system is

$$\tfrac{1}{2}m_x\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}m_y\left(\frac{dy}{dt}\right)^2 + \cdots$$

If there are frictional forces, they will be proportional to the rates of change of the various displacements and, for small displacements, should be linear functions

$$D_x = -R_x\frac{dx}{dt} - \cdots$$

Therefore, for small displacements, the equations of motion are

$$\begin{aligned} m_x\frac{d^2x}{dt^2} &= -R_x\frac{dx}{dt} - A_{xx}x - A_{xy}y - \cdots \\ m_y\frac{d^2y}{dt^2} &= -R_y\frac{dy}{dt} - A_{yx}x - A_{yy}y - \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{xy} &= A_{yx} \qquad \cdots \end{aligned} \tag{3.2.7}$$

We now simplify these equations by reducing all coordinates, whether distances or angles, to the dimensionality of distance multiplied by the square root of mass. We set $x_1 = \sqrt{m_x}\,x$, $x_2 = \sqrt{m_y}\,y$, etc. We also reduce the stiffness coefficients to dimensions of frequency squared ($1/t^2$) by setting $U_{11} = A_{xx}/m_x$, $U_{12} = A_{xy}/\sqrt{m_x m_y} = U_{21}$, etc. The frictional resistances also are changed, by defining $k_1 = R_x/2m_x$, etc. In terms of these reduced coordinates and coefficients, the equations of motion of the system take on the simplified form

$$\frac{d^2x_i}{dt^2} + 2k_i\frac{dx_i}{dt} = -\sum_{j=1}^{N} U_{ij}x_j \qquad i = 1, 2, \ldots, N \tag{3.2.8}$$

To begin with, we shall neglect the frictional forces, setting the k's equal to zero. In this case the total energy of the system,

$$\begin{aligned} E &= \tfrac{1}{2}m_x\left(\frac{dx}{dt}\right)^2 + \cdots + V \\ &= \tfrac{1}{2}\sum_{i=1}^{N}\left(\frac{dx_i}{dt}\right)^2 + \tfrac{1}{2}\sum_{i,j=1}^{N} x_i U_{ij} x_j \end{aligned} \tag{3.2.9}$$

is constant. As before, we ask whether it is possible for the system to perform simple-harmonic motion. We set $x_i = C_i e^{-i\omega t}$ and try to find solutions for the N simultaneous equations,

$$\sum_{j=1}^{N} U_{ij}C_j - \omega^2 C_i = 0 \qquad i = 1, 2, \ldots, N \tag{3.2.10}$$

The frequencies for which such solutions are possible are given by the roots of the equation $\Delta_0(-i\omega) = 0$, called the *secular equation*, where $\Delta_0(-i\omega)$ is the determinant of the coefficients of the C's in Eqs. (3.2.10).

$$\Delta_0(-i\omega) \equiv \begin{vmatrix} U_{11} - \omega^2 & U_{12} & U_{13} & \cdot & \cdot & \cdot \\ U_{21} & U_{22} - \omega^2 & U_{23} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 \tag{3.2.11}$$

This is an Nth-order equation for the quantity ω^2, having N roots, ${\omega_{01}}^2, \ldots, {\omega_{0N}}^2$, usually all distinct. It can be shown that, since all the coefficients U_{ii} are positive and since the nondiagonal coefficients U_{ij} are symmetric ($U_{ij} = U_{ji}$), none of the roots ${\omega_{0n}}^2$ can be negative.

Once the roots of the secular equation have been found, we could substitute them back into Eqs. (3.2.10) to find the particular combination of amplitudes C_j which allows the system to vibrate in a single normal mode, with one frequency $\omega_{0n}/2\pi$, and then find the combination of normal modes which satisfies specified initial conditions. This procedure is often tedious and does not lead to better understanding; so we shall turn to a discussion of the forced motion of the system, using the Laplace transform as the means of solution.

Forced motion

Suppose the system defined by Eqs. (3.2.8) is acted on by applied forces, the ith degree of freedom (distance or angle) being acted on by the applied force (or torque, as the case may be) F_{ai}. In reduced form the force would become $f_i = F_{ai}/\sqrt{m_i}$, with dimensions $l\sqrt{m}/t^2$, and the equations of motion are

$$\frac{d^2x_i}{dt^2} + 2k_i\frac{dx_i}{dt} + \sum_{j=1}^{N} U_{ij}x_j = f_i(t) \tag{3.2.12}$$

If the applied forces are simple-harmonic, so that $f_i = F_i e^{-i\omega t}$, the steady-state motion of the system can be obtained in terms of the determinant

$$\Delta(s) = \begin{vmatrix} s^2 + 2k_1 s + U_{11} & U_{12} & \cdots \\ U_{21} & s^2 + 2k_2 s + U_{22} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \tag{3.2.13}$$

and the cofactors $\Delta_{ij}^{(c)}(s) = \partial\Delta/\partial U_{ij}$,

$$x_i = \sum_{j=1}^{N} F_j \left[\frac{\Delta_{ji}^{(c)}(-i\omega)}{\Delta(-i\omega)}\right] e^{-i\omega t} \tag{3.2.14}$$

for steady-state motion.

The equation $\Delta(s) = 0$ has $2N$ roots for s. Because s occurs in the diagonal terms only and since $U_{ii} \geqslant 0$ and $U_{ij} = U_{ji}$, these roots occur in pairs,

$$\Delta(s) = 0 \qquad \text{for } s = -\kappa_n \pm i\omega_n;\ n = 1, 2, \ldots, N \tag{3.2.15}$$

where, if all the frictional coefficients k_i are zero, all the κ_n's are zero and the ω_n's are equal to the roots ω_{0n} of the determinant of Eq. (3.2.11). Expansion of determinant $\Delta(s)$,

$$\Delta(s) = s^{2N} + 2(\sum k_i)s^{2N-1} + (\sum U_{ii})s^{2N-2} + \cdots$$

indicates that the real and imaginary parts of the roots of $\Delta(s)$ are related to the frictional and elastic coefficients by

$$\sum_n \kappa_n = \sum_i k_i \qquad \sum_n (\kappa_n{}^2 + \omega_n{}^2) = \sum_i U_{ii} \tag{3.2.16}$$

and also that we can express $\Delta(s)$ as a product of N factors,

$$\Delta(s) = [(s + \kappa_1)^2 + \omega_1{}^2][(s + \kappa_2)^2 + \omega_2{}^2] \cdots [(s + \kappa_N)^2 + \omega_N{}^2] \tag{3.2.17}$$

Thus we can write Eqs. (3.2.14), for the steady-state motion of a system driven by simple-harmonic forces of frequency $\omega/2\pi$, as

$$x_i = \sum_{j=1}^{N} \frac{F_j \Delta_{ji}^{(c)}(-i\omega) e^{-i\omega t}}{[\omega_1{}^2 - (\omega + i\kappa_1)^2] \cdots [\omega_N{}^2 - (\omega + i\kappa_N)^2]}$$

If the frictional coefficients k_i are small, the real parts κ_n of the roots are small, and the steady-state motion will exhibit resonance. Whenever ω nearly equals one of the values ω_N of free vibration of the system, one of the factors in the denominator of the expressions for x_i will become small and the x_i's will become large, as illustrated in Fig. 2.2 for the simple oscillator.

To find the response of the system to a more general driving force, we use the Laplace transform. If the system is at rest and the driving forces are all zero when $t < 0$, then the transform of Eq. (3.2.12) is

$$(s^2 + 2k_i s)X_i + \sum_j U_{ij} X_j = F_i(s) \tag{3.2.18}$$

where F_i is the transform of the known function f_i, and X_i is the transform of the unknown function x_i.

$$F_i = \int_0^\infty e^{-st} f_i(t)\, dt \qquad X_i(s) = \int_0^\infty e^{-st} x_i(t)\, dt$$

The simultaneous equations (3.2.18) can be solved in terms of the determinant $\Delta(s)$ already defined, and its cofactors $\Delta_{ij}^{(c)}(s)$,

$$X_i(s) = \sum_{j=1}^{N} F_j(s) \left[\frac{\Delta_{ji}^{(c)}(s)}{\Delta(s)} \right] \tag{3.2.19}$$

Cofactor $\Delta_{ji}^{(c)}(s)$ is $(-1)^{j-i}$ times the $(N-1)$st-order determinant formed by removing the jth row and the ith column from the determinant $\Delta(s)$ of Eq. (3.2.13). Use of the factored form of $\Delta(s)$, given in Eq. (3.2.17), shows that the transforms $X_i(s)$ have poles when s equals any of the roots $-\kappa_n \pm i\omega_n$ of the secular equation $\Delta(s) = 0$. Usually, all $2N$ roots are distinct, in which case all these poles are simple poles. There may, of course, be additional poles if any of the transforms F_i have poles. The cofactors $\Delta_{ji}^{(c)}(s)$ have zeros but no poles.

To calculate the inverse transforms of the X_i's, we use an extension of Eqs. (2.3.19). If $G(s)$ is a function which is finite and continuous for all finite values of s, and if $H(s)$ is the simple product $(s-a_1)(s-a_2)\cdots(s-a_M)$, with roots a_n all differing from each other, the ratio $G(s)/H(s)$, which has M simple poles at the locations $s = a_n$, may be expanded in terms of its residue at each pole.

$$\frac{G(s)}{H(s)} = \sum_{n=1}^{M} \frac{G(a_n)}{s - a_n} \lim_{s \to a_n} \frac{s - a_n}{H(s)} \tag{3.2.20}$$

This partial fraction expansion enables us to express the transforms $X_i(s)$ in terms of quantities contained in Table 2.1 so that an inverse transform can be found.

For example, if the jth degree of freedom of the system is given a sudden blow, at $t = 0$, of an impulse equal to $1/\sqrt{m_j}$, so that $f_j(t) = \delta(t)$ and all the other f's are zero, the transforms of the applied forces are

$$F_j(s) = 1 \qquad F_i(s) = 0 \qquad i \neq j$$

and the expansion of the transforms of the displacements become

$$X_i(s) = \frac{\Delta_{ji}^{(c)}(s)}{(s + \kappa_1 + i\omega_1)(s + \kappa_1 - i\omega_1)\cdots(s + \kappa_N - i\omega_N)}$$

$$= 2 \sum_{n=1}^{N} [(s + \kappa_n)(A_{ji}^{(n)} + \omega_n B_{ji}^{(n)})][(s + \kappa_n)^2 + \omega_n{}^2]^{-1} \tag{3.2.21}$$

where

$$A_{ji}^{(n)} + iB_{ji}^{(n)} = \lim \left[\Delta_{ji}^{(c)}(s) \frac{s + \kappa_n + i\omega_n}{\Delta(s)} \right] \qquad s \to -\kappa_n - i\omega_n$$