

is a complex constant, independent of s . Applying formulas (9), (15), and (16) of Table 2.1 to this expansion, we have, for the deflection of the i th degree of freedom, in response to an impulse $1/\sqrt{m_j}$, delivered to the j th degree of freedom,

$$x_i(t) = \xi_{ji}^{(v)} \equiv \begin{cases} 0 & t < 0 \\ \sum_{n=1}^N e^{-\kappa_n t} [A_{ji}^{(n)} \cos(\omega_n t) + B_{ji}^{(n)} \sin(\omega_n t)] & t > 0 \end{cases} \quad (3.2.22)$$

which is a combination of the N different normal modes of oscillation of the system. We see that the n th mode oscillates with damped harmonic motion of frequency $\omega_n/2\pi$ and with damping constant κ_n .

If friction is neglected, all the κ 's become zero and Δ becomes Δ_0 , with roots $\pm i\omega_{0n}$ of Eq. (3.2.11). It can be shown that the constants $A_{ji}^{(n)}$ all go to zero, and the $B_{ji}^{(n)}$ becomes

$$B_{0ji}^{(n)} = \frac{1}{\omega_{0n}} \lim_{s \rightarrow -i\omega_{0n}} \left[\frac{(s^2 + \omega_{0n}^2) \Delta_{0ji}^{(c)}(s)}{\Delta_0(s)} \right] \equiv \frac{1}{\omega_{0n}} E_j(n) E_i(n) \quad (3.2.23)$$

where

$$[E_i(n)]^2 = \lim_{s \rightarrow -i\omega_{0n}} \left[\frac{(s^2 + \omega_{0n}^2) \Delta_{0ii}^{(c)}(s)}{\Delta_0(s)} \right]$$

and

$$\sum_{i=1}^N E_i(n) E_i(m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

so that

$$x_i(t) = \xi_{ji}^{(v)} \equiv \begin{cases} 0 & t < 0 \quad \text{zero-friction case} \\ \sum_{n=1}^N E_j(n) E_i(n) \left(\frac{1}{\omega_{0n}} \right) \sin(\omega_{0n} t) & t > 0 \end{cases} \quad (3.2.24)$$

The factor $E_i(n)$ is a measure of the participation of the i th degree of freedom in the n th normal mode of oscillation; $C_i = C E_i(n)$ is the solution of Eqs. (3.2.10) when $\omega = \omega_{0n}$. The factor $E_j(n)$ also measures the relative effectiveness of striking the j th degree of freedom, in exciting the n th normal mode.

It should also be apparent that when the initial conditions are that $x_j = 1$ and $x_i = 0$ ($i \neq j$) at $t = 0$ (initial velocities all zero), the subsequent motion of the system is

$$x_i(t) = \xi_{ji}^{(d)} \equiv \sum_{n=1}^N E_j(n) E_i(n) \cos(\omega_{0n} t) \quad t > 0 \quad (3.2.25)$$

for the zero-friction case. Combining Eqs. (3.2.24) and (3.2.25), we can write down the motion of the system when it is started into motion at $t = 0$

with $x_j = x_{0j}$ and $dx_j/dt = v_{0j}$ and allowed to oscillate freely thereafter.

$$\begin{aligned} x_i(t) &= \sum_{j=1}^N [x_{0j} \xi_{ji}^{(d)} + v_{0j} \xi_{ji}^{(v)}] \\ &= \sum_{n=1}^N E_i(n) \sum_{j=1}^N E_j(n) \left[x_{0j} \cos(\omega_{0n} t) + \frac{v_{0j}}{\omega_{0n}} \sin(\omega_{0n} t) \right] \end{aligned} \quad (3.2.26)$$

Once the limiting values of the ratios defined in Eqs. (3.2.23) are worked out, the degree of participation of the various normal modes is given by the factors $E_i(n)$, and the transient motion of the zero-friction case can be written down. The forced motion, in the zero-friction case, can be obtained by integrating $\xi_{ji}^{(v)}$, using a generalization of Eqs. (2.3.9).

The form of Eqs. (3.2.23) to (3.2.26) can be visualized in terms of a geometrical model, analogous to the normal modes of Eqs. (3.2.2) to (3.2.5). The quantities $E_i(n)$ can be considered the components of a unit vector in N -dimensional space, which defines the "direction" of the n th normal mode, the components measuring the configuration assumed by the system when it is oscillating with frequency $\omega_{0n}/2\pi$. The unit vectors are mutually perpendicular, since $\sum_i E_i(n) E_i(m) = 0$ unless $m = n$. The contribution of the initial displacement x_{0j} to the initial value of the n th normal mode is thus x_{0j} times the direction cosine $E_j(n)$, and thus the amplitude of vibration of the n th normal mode, because of initial displacements, is $\sum_j E_j(n) x_{0j}$, as indicated in Eq. (3.2.26). The contribution of the n th normal mode to the motion of the i th degree of freedom of the system is also measured by the direction cosine $E_i(n)$, so that, finally, the displacement of the i th degree of freedom, in response to initial displacements, is the double sum

$$x_i = \sum_{n=1}^N E_i(n) \sum_{j=1}^N E_j(n) x_{0j} \cos(\omega_{0n} t)$$

as in Eq. (3.2.26).

The introduction of friction complicates this simple model of mutually perpendicular normal-mode axes. The Laplace transform technique, however, still works; the inverse transform of Eq. (3.2.19) is the solution. Examples of the procedure will be given in the problems.

3.3 LINEAR ARRAY OF OSCILLATORS

A system of some interest is the linear array of equal masses m , free to move longitudinally, coupled together by springs, all of stiffness constant $K = \frac{1}{4}m\omega_0^2$, with equilibrium spacing l , as shown in Fig. 3.4. Here the coupling is between neighboring masses only; mass 1 can affect mass n only through the intermediacy of the ones between. Hence this system exhibits wave motion; displacement of mass 1 moves mass 2 after a delay, and this in

turn moves mass 3, and so on. The delay is longer, the larger the mass m , and is shorter, the stiffer the spring; so we shall not be surprised to find that the velocity of waves along the array is proportional to $l\sqrt{K/m}$ (the square root is the power of K/m which has the dimensions of t^{-1}).

Propagation of momentum and energy

To demonstrate the wavelike nature of the motion, let us suppose that, up to $t = 0$, all the masses are at rest at their respective positions of equilibrium, each a distance l from its neighbor. And further, suppose that, at $t = 0$, the left-hand end of the spring between masses 0 and 1 is started into

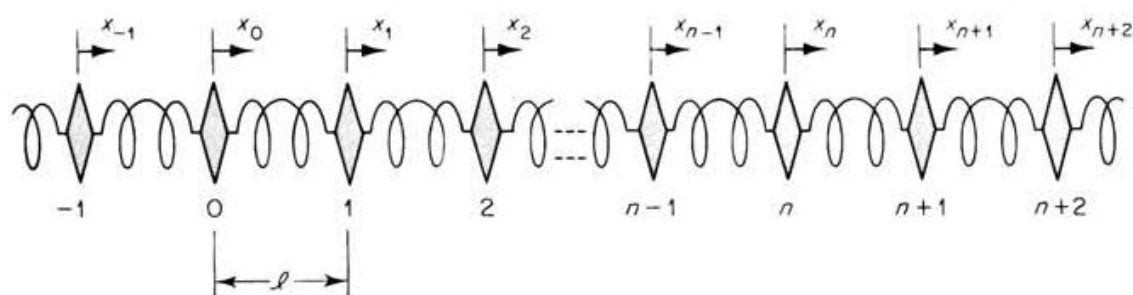


FIGURE 3.4

Infinite linear array of masses m , coupled by springs of stiffness $K = \frac{1}{4}m\omega_0^2$. Displacement of n th mass from equilibrium is x_n . Equilibrium spacing is l .

motion with uniform velocity u (we shall concentrate for the time being on what happens to the right of mass 0, so we can imagine that mass 0 is detached from the spring to its right). The motion of the spring end will compress the 0–1 spring, thus producing a force f_1 on mass 1, so that it will start moving; this in turn will start mass 2 into motion, and so on. Eventually, mass 1 and, somewhat later, mass 2 will be moving with velocity u , as is the spring end at 0; the new distance of separation will be l' instead of l . One after the other the successive masses will be started from rest and brought to velocity u .

The motion is plotted in Fig. 3.5 as curves of displacements of the successive masses as functions of time as the ordinate. The curve for point 0 is a straight line, of slope $1/u$, corresponding to its specified uniform motion with velocity u . The curve for the n th mass is nearly vertical (corresponding to no motion) for some time after $t = 0$, because it takes a while for the disturbance to reach the n th mass. The exact motion of this n th mass is rather complicated, as we shall see later, but, roughly speaking, it is at rest until shortly before a time T (where T is proportional to n), and shortly after time T it is moving with velocity u , along with the $n - 1$ masses to the left of it. A wave of starting up has passed along the array, with one mass after the other starting into motion, the starting of each successive mass being delayed a time $\tau = T/n$ behind the previous one.

A time delay of motion proportional to distance is the essential characteristic of wave motion. The velocity c of the wave motion exhibited here is equal to the distance nl , from the origin, of the mass just starting into motion at time T , divided by T .

$$c = \frac{nl}{T} = \frac{l}{\tau}$$

The value of the wave velocity c is determined by the balance between the force transmitted by the springs and the momentum gained by the array per

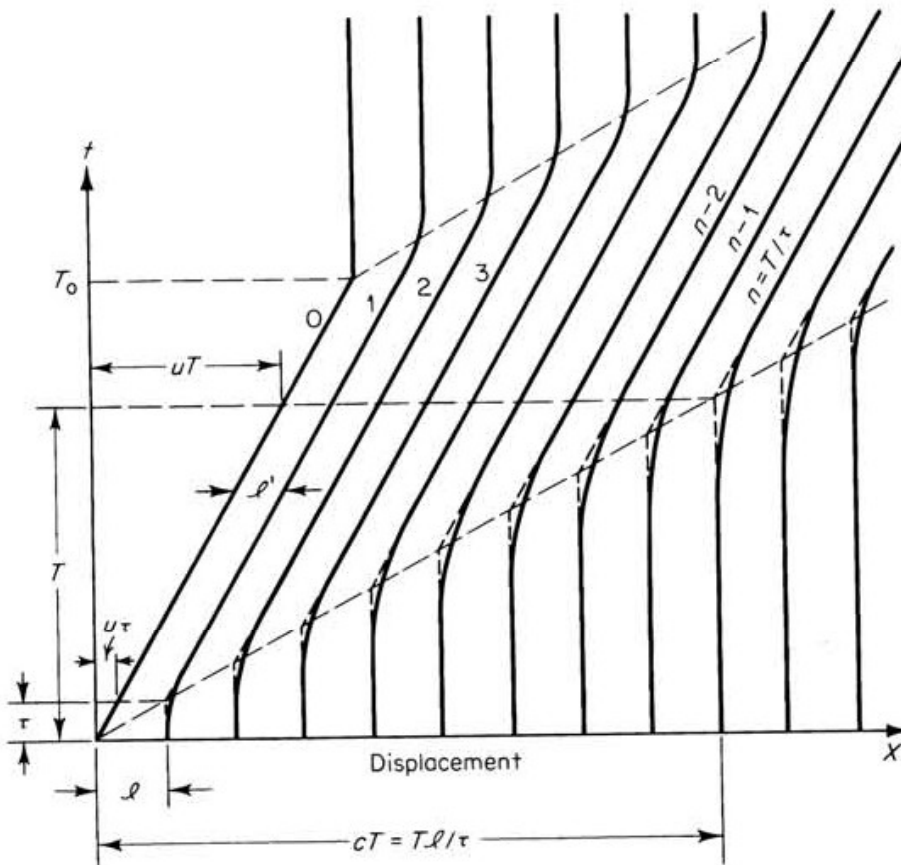


FIGURE 3.5

Wave motion in a linear array. Mass 0 is started into motion with velocity u at $t = 0$; effects of this disturbance are propagated with velocity c . Solid lines represent the exact motion of the masses; dotted lines are a simplified version of the motion.

second, or else (what is the same thing) the balance between the work done by the force pushing the spring end at 0 and the gain in energy of the system per second.

Setting the masses into motion requires a force f_1 on the left-hand end of the spring 0-1; this force is transmitted along the array by a compression of the springs (the moving masses are closer together than the stationary ones). If the stiffness constant of each spring is K , then the force f_1 , transmitted along

the array in order to accelerate the next mass, is equal to $K(l - l')$. A glance at Fig. 3.5 shows that f_1 also can be written as $K(u\tau)$, where $\tau = T/n = l/c$ is the time delay of the motion of one mass behind the preceding one in the wave motion. The impulse $f_1 T$ which the force imparts to the system in time T must be equal to the momentum nmu which the force has generated in the n masses set into motion in time T . Thus

$$nmu = K(u\tau)T = Ku \frac{l}{c} \frac{nl}{c}$$

or

$$m = K \left(\frac{l}{c} \right)^2 \quad \text{or} \quad c = l \sqrt{\frac{K}{m}} \quad (3.3.1)$$

which corresponds to the formula we arrived at earlier, by dimensional reasoning.

The same equation can be obtained from energy considerations. The work done by force f_1 on the left-hand end of the spring, from $t = 0$ to $t = T = n\tau = nl/c$, is f_1 times the distance uT the spring end has moved in time T . The energy gained by the system in time T is n times the kinetic energy $\frac{1}{2}mu^2$ gained by the n masses set into motion, plus n times the potential energy $\frac{1}{2}K(l - l')^2 = \frac{1}{2}K(ul/c)^2$ stored when each of the n springs is compressed by an amount $l - l'$. The equation is

$$f_1 u T = \frac{Kul}{c} u \frac{nl}{c} = \frac{1}{2}mu^2 + \frac{1}{2}nK \left(\frac{ul}{c} \right)^2$$

which again reduces to Eq. (3.3.1). Thus a force f_1 applied at one end of the array feeds both momentum and energy into the array, both of which are transmitted along the array with a velocity $c = l\sqrt{K/m}$, to accelerate and donate energy to the masses and springs at the front of the wave (or to be withdrawn from the array at its far end).

If, now, we stop moving the left-hand spring end at $t = T_0$, mass 1 will begin to decelerate as spring 0-1 is restretched; after mass 1 comes to rest, mass 2 will decelerate and come to rest a time $\tau = l/c$ later; and so on. Reapplication of the previous arguments shows that the velocity of the wave of stopping is equal to $c = l\sqrt{K/m}$, that of the wave of starting.

If, after $t = T_0$, no force is applied to the left-hand spring end, it will remain at rest, and no further momentum or energy will enter the system. The array still possesses internal momentum and energy, however. A certain number, $n' \simeq T_0 c / l = T_0 \sqrt{m/K}$, of the masses are moving with velocity u , though all the rest of the masses are at rest. Which of them are moving changes as time progresses. During a time $\tau = c/l$, the rearmost moving mass comes to rest, and the mass just ahead of the frontmost moving mass gets pushed into moving with velocity u . Each successive mass, as the wave reaches it and passes beyond, starts into motion, moves with velocity u

for a time $T_0 = n'\tau$, and then comes to rest after having moved a distance $uT_0 = n'l(u/c)$. Thus the wave carries $n'mu$ momentum and $n'mu^2$ energy (kinetic and potential) along with it.

The expression for the wave velocity c can be expressed in terms of properties per unit length of the array. For example, its mass per unit length is $\epsilon = m/l$. The stiffness per unit length is the reciprocal of the longitudinal compressibility κ , which is defined as the ratio between the fractional shortening $dl/l = (l - l')/l$ of any compressed portion of the spring system and the force $K(l - l')$ applied; $\kappa = (l - l')/K(l - l') = 1/Kl$. Thus the wave velocity can be written

$$c = \sqrt{\frac{1}{\kappa\epsilon}} \quad (3.3.2)$$

We have repeated several times that this elementary discussion has not been exact, that the motion of the individual masses is actually more complicated than the simple stop-start-stop assumed here and displayed by the dotted lines of Fig. 3.5, though the conclusions we have reached are generally true and the formulas are approximately correct. We must now analyze the motion in detail to find out exactly what happens.

Simple-harmonic wave motion

Returning to the system of Fig. 3.4, we denote by x_n the longitudinal displacement from equilibrium of mass n , and we denote by f_n the force exerted on both mass $n - 1$ and mass n by the spring between them. This force is zero if the spring has its equilibrium length l , but if it is longer or shorter than l (i.e., if $x_n - x_{n-1} \neq 0$), the force will be proportional to the change of length, the proportionality constant being the stiffness constant K of the spring. Therefore the equation for f_n in terms of the displacements x_n and x_{n-1} and the related equation for the acceleration of mass n in terms of forces f_n and f_{n+1} of the springs attached to it are

$$\begin{aligned} f_n &= K(x_{n-1} - x_n) = \frac{1}{4}m\omega_0^2(x_{n-1} - x_n) \\ m \frac{d^2x_n}{dt^2} &= f_n - f_{n+1} \end{aligned} \quad (3.3.3)$$

where we have introduced the constant $\omega_0^2 = 4K/m$ to simplify some of the later formulas. If all the x 's are kept zero except x_n , mass n will be a simple oscillator with natural frequency $(1/2\pi)\sqrt{2K/m} = \omega_0/\pi\sqrt{8}$.

These equations of motion can be put into a more symmetric form if we use the momenta $p_n = m(dx_n/dt)$ of the masses rather than their displacements.

$$\begin{aligned} \frac{df_{n+1}}{dt} &= \frac{1}{4}\omega_0^2(p_n - p_{n+1}) \\ \frac{dp_n}{dt} &= f_n - f_{n+1} \quad \omega_0 = 2\sqrt{\frac{K}{m}} \end{aligned} \quad (3.3.4)$$

We can now ask, as we have done before with other systems, whether this system can oscillate with simple-harmonic motion. To find out, we assume that both the f 's and the p 's have factors $e^{-i\omega t}$, so that the time derivative of either function is equal to $-i\omega$ times the function itself. In this case Eqs. (3.3.4) become

$$f_{n+1} = \frac{i\omega_0^2}{4\omega} (p_n - p_{n+1}) \quad -i\omega p_n = f_n - f_{n+1}$$

or

$$f_{n+1} = f_n + i\omega p_n \quad p_{n+1} = p_n \left(1 - \frac{4\omega^2}{\omega_0^2}\right) + \frac{4i\omega}{\omega_0^2} f_n$$

which expresses the force f_{n+1} and momentum p_{n+1} of one unit of the chain in terms of the force f_n and momentum p_n of the preceding unit. Evidently, these equations admit several solutions; we now ask whether one of them can represent wave motion in the direction of increasing n .

If the system can exhibit wave motion at frequency $\omega/2\pi$, the motion of p_{n+1} should lag behind that of p_n by a time delay which was called τ in Fig. 3.5. In other words, if $p_n = P_n e^{-i\omega t}$, then p_{n+1} should equal $P_n e^{-i\omega(t-\tau)}$, or p_{n+1}/p_n should equal $e^{i\omega\tau}$ (and similarly for f_{n+1}/f_n , where τ must be real if there is to be true wave motion at frequency $\omega/2\pi$ (if τ were a complex quantity, p_{n+1} would not only be delayed, but would also differ in magnitude from p_n , which does not occur in true wave motion). For the moment, setting $e^{i\omega\tau} = \gamma$ to simplify typesetting, if wave motion is to be present, Eqs. (3.3.4) must become

$$-i\omega\gamma f_n = \frac{1}{4}\omega_0^2(1 - \gamma)p_n \quad \text{and} \quad -i\omega p_n = (1 - \gamma)f_n$$

For this to be true, γ must satisfy the equation

$$\omega^2\gamma f_n = \frac{1}{4}\omega_0^2(1 - \gamma)^2 f_n \quad \text{or} \quad \gamma^2 - \left(2 - \frac{4\omega^2}{\omega_0^2}\right)\gamma + 1 = 0$$

so that

$$\gamma = 1 - \frac{2\omega^2}{\omega_0^2} \pm i\sqrt{1 - \left(1 - \frac{2\omega^2}{\omega_0^2}\right)^2} = \cos(\omega\tau) \pm i\sin(\omega\tau) = e^{\pm i\omega\tau} \quad (3.3.5)$$

where

$$\cos(\omega\tau) = 1 - \frac{2\omega^2}{\omega_0^2} \quad \sin\left(\frac{1}{2}\omega\tau\right) = \frac{\omega}{\omega_0}$$

as long as $\omega < \omega_0 = 2\sqrt{K/m}$.

The frequency $\omega_0/2\pi$ is $\sqrt{2}$ times the frequency with which any one of the masses in the array would oscillate if all the other masses were held fixed at their equilibrium positions. We see that simple-harmonic wave motion is possible only as long as *its frequency is less than $\omega_0/2\pi$* . Referring to Fig. 3.5,

we see that the velocity of propagation of such waves,

$$c = \frac{l}{\tau} = \frac{\omega l}{2 \sin^{-1}(\omega/\omega_0)} \rightarrow l \sqrt{\frac{K}{m}} \quad \omega \ll \omega_0 \quad (3.3.6)$$

depends on the value of ω , except in the limit of ω small. Only in this limit is the wave velocity equal to that of Eq. (3.3.1), the formula obtained by approximately balancing force and momentum. Therefore our previous discussion of Fig. 3.5 is valid only as long as the wave motion traversing the array can be analyzed into simple-harmonic components, all of which have frequencies considerably less than ω_0 . Any sudden changes in velocity (such as the sudden start-up shown by the dotted lines) would contain high-frequency components which would not propagate.

A system which transmits waves with a wave velocity which depends on the frequency of the wave is called a *dispersive* medium for waves, by analogy with the dispersion of light in glass (see Eq. 9.1.17). The linear array of masses and springs is thus a dispersive medium.

When the driving frequency $\omega/2\pi$ is larger than the basic resonance frequency $\omega_0/2\pi$, true wave motion cannot exist. Instead of successive time delay, the motions of the successive masses are all in phase (alternating in sign), but successively reduced in *amplitude*, by the factor

$$\gamma = - \left(\frac{2\omega^2}{\omega_0^2} - 1 \right) + \sqrt{\left(\frac{2\omega^2}{\omega_0^2} - 1 \right)^2 - 1} \rightarrow - \frac{\omega_0^2}{4\omega^2} \quad \omega \gg \omega_0$$

In other words, if the motion of mass 0 is given by $P_0 e^{-i\omega t}$, then, when $\omega > \omega_0$, the motion of mass n is $\gamma^n P_0 e^{-i\omega t}$ (γ real, negative, and of magnitude less than 1), which diminishes with n the more rapidly the larger ω is. As a transmission line, the array is a low-pass filter with a cutoff frequency at $\omega_0/2\pi$. Of course, the wave can go in either direction, as is implied by the $\pm$ sign in Eq. (3.3.5). The various forms for the expression for the displacement of the n th mass, for simple-harmonic motion, are thus

$$x_n = \begin{cases} A \exp \left(-i\omega t \pm 2in \sin^{-1} \frac{\omega}{\omega_0} \right) & \text{when } \omega \leq \omega_0 \\ A \exp \left(-i\omega t \mp i\pi n \mp 2n \cosh^{-1} \frac{\omega}{\omega_0} \right) & \text{when } \omega \geq \omega_0 \end{cases} \quad (3.3.7)$$

the first form representing true wave motion, the second representing the vibrations above the transmission cutoff frequency.

Sinusoidal wave motion

The formulas come out somewhat simpler when we ask under what circumstances the array can propagate sinusoidal waves, rather than asking for the configuration of the array when it is performing simple-harmonic

motion. A sinusoidal configuration means that, at any instant, the displacement of the n th mass would depend on n , according to the real part of the exponential $e^{i\alpha n}$, so that $x_n = x_0(t)e^{i\alpha n}$. Every time that n increases by the amount $2\pi/\alpha$, the motion is again in phase; so we can call l times this amount the *wavelength* $\lambda = l(2\pi/\alpha)$ of the sinusoidal wave. For this case Eqs. (3.3.3) become

$$m \frac{d^2 x_n}{dt^2} = f_n - f_{n+1} = \frac{1}{4} m \omega_0^2 (x_{n+1} - 2x_n + x_{n-1})$$

(3.3.8)

or

$$\frac{d^2 x_n}{dt^2} = \frac{1}{4} \omega_0^2 (e^{i\alpha} - 2 + e^{-i\alpha}) x_n = -\omega_0^2 \sin^2(\frac{1}{2}\alpha) x_n$$

where we have used the last of Eqs. (1.2.7) to take the last step.

But this is the equation for simple-harmonic oscillation, of frequency $\omega/2\pi = (\omega_0/2\pi) \sin(\frac{1}{2}\alpha)$, which is independent of n . Thus, when a sinusoidal wave of wavelength $\lambda = 2\pi l/\alpha$ traverses the periodic array, all parts of the array move with simple-harmonic motion, all with the same frequency $(\omega_0/2\pi) \sin(\frac{1}{2}\alpha)$.

$$x_n = A e^{i(\alpha n - \omega t)} \quad \omega = \omega_0 \sin \frac{\alpha}{2}$$

(3.3.9)

$$c = \lambda \cdot \text{frequency} = \omega_0 \frac{l}{\alpha} \sin \frac{\alpha}{2}$$

which is not an unexpected result.

For small values of α (long wavelength), the frequency of the simple-harmonic motion is approximately proportional to α and inversely proportional to λ ($\omega/2\pi \simeq \alpha \omega_0/4\pi = l\omega_0/2\lambda$). Therefore long-wavelength sinusoidal waves travel with a velocity $c \simeq \frac{1}{2} l \omega_0 = l \sqrt{K/m}$, which is independent of λ . As α is increased (or λ decreased), the frequency of the wave increases, but not proportionally to α (so c is no longer independent of λ), until finally, when $\alpha = \pi$ (or $\lambda = 2l$), $\sin(\alpha/2)$ and thus $\omega/2\pi$, reaches a maximum value ($\omega/2\pi \rightarrow \omega_0/2\pi$) which is $\sqrt{2}$ times the natural frequency of isolated motion of each mass. For this limiting wavelength, $\lambda = 2l$, we have $x_n = e^{i\pi} x_{n-1} = -x_{n-1} = -x_{n+1}$; each mass is moving in opposition to its neighbors. No shorter wavelength is possible for the system of Fig. 3.4, for if we make α larger than π (λ shorter than $2l$), the phase factor $e^{i\alpha}$ can be written $e^{-i(2\pi-\alpha)}$, which corresponds to an $\alpha' = -2\pi + \alpha$ of magnitude less than π but of negative sign. This can be interpreted as the phase factor for a wave traveling in the negative direction, of wavelength $2\pi l/\alpha' = 2\pi l/(2\pi - \alpha)$, longer than $2l$. Because the array is a discrete sequence of masses, spaced a finite distance l apart, wavelengths smaller than $2l$ have no meaning, and thus motion with frequency greater than the cutoff value $\omega_0/2\pi$ cannot be propagated as wave motion.

The only way we can make $(\omega_0/2\pi) \sin(\alpha/2)$ larger than $\omega_0/2\pi$ is to make α a complex quantity, $\alpha = \pi + 2i\beta$, so that

$$\begin{aligned}\omega &= \omega_0 \sin(\tfrac{1}{2}\pi + i\beta) = \omega_0[\sin(\tfrac{1}{2}\pi) \cos(i\beta) + \sin(i\beta) \cos(\tfrac{1}{2}\pi)] \\ &= \omega_0 \cosh \beta > \omega_0\end{aligned}$$

and

$$x_n = (-1)^n A \exp(-2n\beta - i\omega_0 t \cosh \beta)$$

Such motion is not wave motion; there is no phase lag, proportional to n , and the amplitude of motion decreases as n increases. This result is identical with Eqs. (3.3.7); only α (or β) is now the independent variable, not ω .

Wave impedance

The driving force necessary to send a wave along the linear array can be measured by removing the spring between masses 0 and 1 and determining the force f_1 which must be applied to mass 1 to produce the specified wave form in the array to the right of 1. If the applied force is simple-harmonic, of frequency $\omega/2\pi$, then Eqs. (3.3.7) or (3.3.9) show that the time delay τ or the phase lag α between successive masses is related to ω according to either Eq. (3.3.5) or Eqs. (3.3.9). The impedance with which the right-hand half of the array will resist a simple-harmonic driving force can be computed from Eq. (3.3.4) [according to Eq. (2.3.2), impedance is the ratio between force and velocity, for a driving-force frequency of $\omega/2\pi$].

$$\begin{aligned}Z &= \frac{f_1}{p_1/m} = m \frac{f_{n+1} e^{-in\alpha}}{p_{n+1} e^{-in\alpha}} = -im\omega \frac{f_{n+1}}{f_{n+1} - f_{n+2}} \\ &= im\omega \frac{f_{n+1}}{f_{n+1}(e^{i\alpha} - 1)} = \tfrac{1}{2}m\omega \frac{e^{-i\alpha/2}}{\sin(\alpha/2)} = \tfrac{1}{2}m\omega_0 e^{-i\alpha/2} \\ &= \tfrac{1}{2}m\omega_0 \cos \frac{\alpha}{2} - \tfrac{1}{2}im\omega = \tfrac{1}{2}m\sqrt{\omega_0^2 - \omega^2} - \tfrac{1}{2}im\omega \quad (3.3.10)\end{aligned}$$

At very low frequencies, $\omega \ll \omega_0$, this impedance is real, indicating that the load is purely resistive. The force is in phase with the velocity of mass 1; power is generated by the force, which is transmitted along the array with velocity $l\sqrt{K/m}$. As ω is increased, the resistive part of the impedance diminishes, and its imaginary part increases. As long as $\omega < \omega_0$, this imaginary part is equivalent to the reactance of a mass $\tfrac{1}{2}m$; the mass-spring chain acts as though it were a mass load equal to half of one of the masses, plus a frictional load of $\tfrac{1}{2}m\sqrt{\omega_0^2 - \omega^2}$. When $\omega = \omega_0$, the impedance is entirely the reactive load of the mass $\tfrac{1}{2}m$. For $\omega > \omega_0$, the impedance is entirely reactive, being $-\tfrac{1}{2}im(\omega + \sqrt{\omega^2 - \omega_0^2})$, the magnitude of which increases until, when $\omega \gg \omega_0$, it is just the impedance $-im\omega$ of mass 1. At high frequencies the springs are not stiff enough to transmit the motion, so that mass 1 is isolated.

Transient motion

The preceding pages have dealt with the steady-state motion of sinusoidal waves in the linear array of Fig. 3.4. The transient behavior, and the motion for other forms of waves, can be obtained by using the methods of the Laplace transform, as we have done several times before (examples of this will be given in the problems). Or it can be explored by taking advantage of an interesting parallel between the equations of motion (3.3.3) of the array and the set of recursion formulas (1.2.6), relating successive Bessel functions. The second set of recursion formulas, for the Bessel functions $J_m(\omega_0 t)$, is

$$\frac{d}{dt} J_m(\omega_0 t) = \frac{1}{2} \omega_0 J_{m-1}(\omega_0 t) - \frac{1}{2} \omega_0 J_{m+1}(\omega_0 t)$$

where m is any integer, positive or negative. Referring to Eqs. (3.3.3), we see that if we set

$$y_{2n} = \frac{dx_n}{dt} \quad \text{and} \quad y_{2n+1} = \frac{2}{m\omega_0} f_{n+1} = \frac{1}{2} \omega_0 (x_n - x_{n+1})$$

then the first of Eqs. (3.3.3) (differentiated once) corresponds to

$$\frac{d}{dt} f_{n+1} = \frac{1}{2} m \omega_0 \frac{d}{dt} y_{2n+1} = \frac{1}{4} m \omega_0^2 (y_{2n} - y_{2n+2})$$

and the second equation becomes

$$m \frac{d}{dt} y_{2n} = \frac{1}{2} m \omega_0 (y_{2n-1} - y_{2n+1})$$

Consequently, the quantities y_m must be proportional to the Bessel functions $J_m(\omega_0 t)$, where $m = 2n + \text{any integer}$. For example, if we set $x_n = A_p J_{2n-2p}(\omega_0 t)$, then

$$y_{2n} = \frac{dx_n}{dt} = \frac{1}{2} \omega_0 A_p [J_{2n-2p-1}(\omega_0 t) - J_{2n-2p+1}(\omega_0 t)]$$

and

$$y_{2n-1} = \frac{2f_n}{m\omega_0} = \frac{1}{2} \omega_0 A_p [J_{2n-2p-2}(\omega_0 t) - J_{2n-2p}(\omega_0 t)]$$

is a self-consistent solution of the equations of motion (3.3.3) for the array of Fig. 3.4.

Since these equations of motion are linear, a sum of solutions is also a solution. Therefore we can arrange a sum to satisfy initial conditions,

based on the fact that $J_n(0) = \delta_{0n}$ [Eq. (1.2.5)]. If, at $t = 0$, $x_n = a_n$ and $dx_n/dt = v_n$, then the subsequent motion of the system is given by the following equations:

$$x_n = \sum_{p=-\infty}^{\infty} \left[a_p J_{2n-2p}(\omega_0 t) + \left(\frac{2}{\omega_0} \sum_{m=p+1}^{\infty} v_m \right) J_{2n-2p-1}(\omega_0 t) \right]$$

$$\frac{d}{dt} x_n = \sum_{p=-\infty}^{\infty} [v_p J_{2n-2p}(\omega_0 t) + \frac{1}{2} \omega_0 (a_p - a_{p+1}) J_{2n-2p-1}(\omega_0 t)] \quad (3.3.11)$$

$$f_n = \frac{1}{2} m \omega_0 \sum_{p=-\infty}^{\infty} [\frac{1}{2} \omega_0 (a_{p-1} - a_p) J_{2n-2p}(\omega_0 t) + v_p J_{2n-2p-1}(\omega_0 t)]$$

Curves for x_n ($n = 0, 1, 2, 3, 4$) versus time are plotted in Fig. 3.6a for the case where mass 0 is given an initial impulse ($a_n = 0$, $v_n = \delta_{n0} u_0$), so that $x_n = \sum_{m=0}^{\infty} (2u_0/\omega_0) J_{2n+2m+1}(\omega_0 t)$. These curves exhibit the difference between the actual motion of a linear array and the simple approximation discussed earlier in this section. For a nondispersive wave medium, the curve for x_1 would be the same as for x_0 , except that it would be delayed a time $\tau = l/c = 2/\omega_0$ after that for x_0 , and so on; so $x_n(t)$ would equal $x_0(t - n\tau)$. The dashed lines of Fig. 3.6a correspond to the velocity $c = \frac{1}{2} l \omega_0 = l \sqrt{K/m}$. We see that in a general way each x_n curve duplicates the preceding one; it

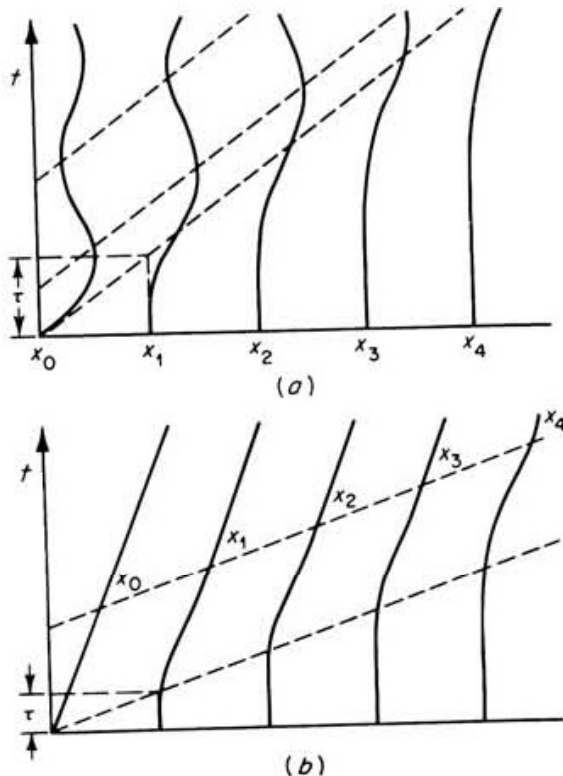


FIGURE 3.6
Wave motion in a linear array. Curves *a* are displacements of successive masses when mass 0 is given an initial impulse; curves *b* are for the situation of Fig. 3.5.

starts from rest at about $t = n\tau$, goes through a maximum, and then oscillates about a new equilibrium position, a distance u_0/ω_0 from its original position. Actually, each curve differs somewhat from the others; each successive x_n starts from zero a little earlier than $t = n\tau$, and its first maximum is broader and not so high, though the subsequent oscillations are quite similar to those for x_0 .

Such differences in the motion of successive parts of the linear array are, of course, due to the fact that the velocity of simple-harmonic waves is not the same for all frequencies; according to Eq. (3.3.6) it decreases as ω increases, going from $l\omega_0/2$, for the low-frequency components of the wave, down to $l\omega_0/\pi$ when $\omega = \omega_0$; for $\omega > \omega_0$ no wave motion is possible. The relatively low-frequency oscillations of the masses, after their first large displacement, propagate without much change from mass to mass. But the sharp change of velocity of x_0 at $t = 0$ (caused by the impulsive force applied then) does not propagate; the onset of the motion of each successive mass is more and more gradual (i.e., has less and less of the high frequencies). These are typical characteristics of wave transmission in dispersive media.

We can also use the Bessel function series to obtain the exact solution for the motion pictured in Fig. 3.5, where x_0 is moved with constant velocity after $t = 0$. Application of the equations following Eqs. (1.2.6) shows that the appropriate solution is

$$x_n = \frac{2u}{\omega_0} \sum_{m=0}^{\infty} (2m+1) J_{2n+2m+1}(\omega_0 t)$$

This solution is plotted in Fig. 3.6*b*. We see that the sharp change in velocity, at $t = 0$ for x_0 , does not propagate; each successive mass accelerates more smoothly, slightly overshoots the velocity u , and subsequently oscillates about the velocity u with diminishing amplitude. Here again the dispersive character of the array for wave motion is apparent.

Problems

- 1 Two oscillators, each of mass m and natural frequency ν_0 (i.e., if one oscillator is held at equilibrium, the other will oscillate with a frequency ν_0), are coupled so that moving one mass 1 cm from equilibrium produces a force on the other of C dynes. Show that if C is small compared with $4\pi^2 m \nu_0^2$ and if one oscillator is held 1 cm from equilibrium and the other at equilibrium and both are released at $t = 0$, the subsequent displacements of the masses will be

$$y_1 = \cos\left(\frac{Ct}{4\pi m \nu_0}\right) \cos(2\pi \nu_0 t) \quad y_2 = \sin\left(\frac{Ct}{4\pi m \nu_0}\right) \sin(2\pi \nu_0 t)$$

- 2 Suppose that each oscillator of Prob. 1 is acted on by a frictional force equal to R times its velocity. Show that the modulus of decay of the oscillations equals $2m/R$.
- 3 Discuss the forced vibrations of the coupled oscillators described in Prob. 2.

- 4 Three masses, each of m g, are equally spaced along a string of length $4a$. The string is under a tension T dynes. Show that the three allowed frequencies and the corresponding relations between the displacements for the normal modes are

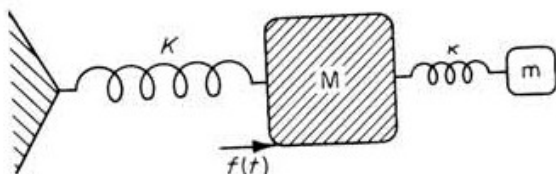
$$v = v_0 \sqrt{1 - \sqrt{\frac{1}{2}}} \quad \text{if } y_1 = y_3 = \frac{y_2}{\sqrt{2}}$$

$$v = v_0 \quad \text{if } y_1 = -y_2 \text{ and } y_3 = 0$$

$$v = v_0 \sqrt{1 + \sqrt{\frac{1}{2}}} \quad \text{if } y_1 = y_2 = \frac{-y_3}{\sqrt{2}}$$

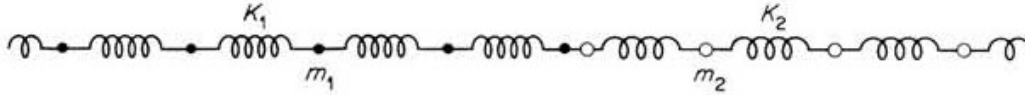
where $T/a = 2\pi^2 v_0^2 m$.

- 5 An arrangement sometimes used for vibration damping is shown schematically in the figure. A large mass M (a machine of some sort) is supported by a spring with stiffness constant K . A linear driving force $f(t) = F_0 e^{-i\omega t}$ is acting on M . By adding a second oscillator with mass m and stiffness constant κ as shown, the vibration amplitude of M can be eliminated. How are m and κ to be chosen to make this possible?

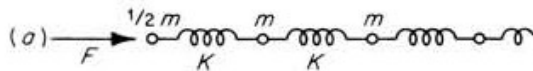
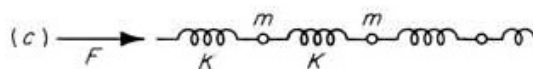


- 6 Four coupled harmonic oscillators have coupling constants U_{ij} , as defined in Eq. (3.2.8), related as follows: $U_{11} = U_{22} = U_{33} = U_{44} = \omega_0^2$, $U_{12} = U_{21} = U_{23} = U_{32} = U_{34} = U_{43} = U_{41} = U_{14} = \alpha\omega_0^2$, $U_{13} = U_{31} = U_{24} = U_{42} = 2\alpha\omega_0^2$. Show that the allowed natural frequencies of the system are $\omega_1^2 = \omega_0^2(1 - 4\alpha)$, $\omega_2^2 = \omega_0^2$, $\omega_3^2 = \omega_4^2 = \omega_0^2(1 + 2\alpha)$. At which frequency will all the masses be moving in phase with equal amplitude? What are the relative phases and amplitudes of motion of the masses when vibrating with the other frequencies? Compute the 16 unit components $E_i(n)$ of Eq. (3.2.23) for this system. Then calculate the motion of each mass if the first mass is struck a unit impulse at $t = 0$, the system being previously at rest.
- 7 Suppose the masses in the system of Fig. 3.4 move at right angles to the system axis, instead of longitudinally. The lattice is stretched to a tension $S = K(l - l_0)$, where l is the equilibrium spacing and l_0 the relaxed length of the spring, and the transverse displacement of each mass is much smaller than l . The average mass per unit length of the system being $\epsilon = m/l$, show that the speed of simple-harmonic transverse wave motion is $\sqrt{S/\epsilon}$, at sufficiently low frequencies. Prove that the wave speed of transverse waves is always smaller than the wave speed of longitudinal waves for this system.
- 8 With the system of Fig. 3.4, suppose there is a frictional force $R(dx_n/dt)$ acting on each mass. Consider a wave motion of the lattice in which the displacement of the n th mass is proportional to $\exp(ikx_n - i\omega t)$. Determine k as function of ω , and show that for small values of R the amplitude of oscillation decreases by a factor $\exp(4R/\omega_0 m)$ per lattice spacing [see Eq. (3.3.4) for definition of ω_0].

- 9 Two one-dimensional mass-spring systems are connected as shown in the figure, with $m_2 > m_1$ and $K_2 \neq K_1$. A simple-harmonic longitudinal wave of frequency $\omega/2\pi$ is traveling on the first lattice, from the left, and is partially reflected at the junction, so that the motion of the first lattice is a superposition of incident and reflected waves. In the low-frequency limit, determine the amplitudes of the transmitted and reflected waves, in terms of the amplitude of the incident wave. What is the reflected wave amplitude when the frequency of the incident wave exceeds the cutoff frequency of the second lattice?



- 10 A mass-spring lattice of the sort shown in Fig. 3.4 has constants $m = 1$, $K = 10^4$, and $l = 1$. (a) Determine the frequency range in which a harmonic wave can be transmitted without attenuation. (b) What is the phase velocity when the wavelength of the wave is much greater than l ? When the wavelength is $4l$? (c) If the lattice is driven at frequency $\sqrt{8K/m}$, determine the spatial attenuation of the amplitude of oscillation, and also the phase shift, per lattice spacing l .
- 11 Two longitudinal waves with slightly different frequencies, $\omega/2\pi$ and $(\omega + d\omega)/2\pi$, and corresponding wavenumbers, k and $k + dk$, are transmitted to the right along the mass-spring lattice of Fig. 3.4. Show that the resulting wave can be described as a wave of spatially varying amplitude, and that a point of constant displacement amplitude travels forward with a velocity $d\omega/dk$ (called the *group velocity* of the wave). What is the group velocity of the system, and what is its phase velocity ω/k , when the wavelength of the wave motion is four times the lattice spacing l ? Express your answers in terms of the value of the phase velocity in the limit of low frequencies.
- 12 A one-dimensional mass-spring lattice is driven at one end by a simple-harmonic force $F = F_0 \exp(-i\omega t)$ as shown in the figure. (a) Determine the velocity u of the first mass element (with mass $m/2$) in the lattice; show that, for $\omega \ll \sqrt{K/m}$, this is given by the simple formula $F = \epsilon cu$, where c is the phase velocity of the wave and $\epsilon = m/l$ is the mass per unit length of the lattice. Is this result consistent with Eq. (3.3.10)? (b) What is the average power transmitted to the lattice by the force when $\omega \ll \sqrt{K/m}$? When $\omega > 2\sqrt{K/m}$? (c) Suppose the first mass element is removed and the force is applied to the end of the spring as shown in the lower part of the figure. What, then, is the power delivered to the lattice when $\omega \ll \sqrt{K/m}$?



- 13 The linear array of Fig. 3.4 is made finite by fixing masses $n = 0$ and $n = N + 1$ at rest, so that the potential energy is

$$V = \frac{1}{2} m \omega_0^2 [x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + \cdots + (x_N - x_{N-1})^2 + x_N^2]$$

Show that the N allowed frequencies of longitudinal vibration of the system

are given by

$$\omega_v = \omega_0 \sin \frac{\pi v}{2(N+1)} \quad v = 1, 2, 3, \dots, N$$

and that, corresponding to the v th frequency, the unit solutions for the displacement amplitudes [Eq. (3.2.23)] are

$$E_n(v) = \sqrt{\frac{2}{N}} \sin \frac{\pi v n}{N+1}$$

Show that

$$\sum_{n=1}^N E_n(v) E_n(\mu) = \delta_{v\mu} \quad \sum_{v=1}^N E_n(v) E_m(v) = \delta_{nm}$$

and therefore that if at $t = 0$ the n th mass is at rest but displaced a distance x_n^0 from equilibrium, the subsequent motion of the system is given by the equations

$$x_n(t) = \sum_{v=1}^N E_n(v) \left[\sum_{m=1}^N x_m^0 E_m(v) \right] \cos(\omega_v t)$$

- 14** A finite circular array (with mass 1 connected to mass N) has a potential energy

$$V = \frac{1}{2} m \omega_0^2 [(x_2 - x_1)^2 + (x_3 - x_2)^2 + \dots + (x_N - x_1)^2]$$

Show that the possible motions of the array are given by

$$x_n = C_v \exp \left(\frac{2\pi i v n}{N} - i \omega_v t \right)$$

where

$$\omega_v = \omega_0 \sin \frac{\pi v}{N} \quad 0 \leq v < N$$

What is the motion corresponding to $v = 0$? What is the relation between motions for v and for $N - v$? Show that if the system is originally in equilibrium and, at $t = 0$, the m th mass is given a velocity v_m^0 , the subsequent motion is given by the equations

$$x_n(t) = \frac{1}{N} \left\{ v_m^0 t + \left[\sum_{v=0}^{N-1} \frac{v_m^0}{\omega_v} \exp \left(2\pi i v \frac{n-m}{N} \right) \right] \sin(\omega_v t) \right\}$$

- 15** For the circular array of Prob. 14, show that if mass j is held displaced an amount x_j^0 and the rest held at equilibrium, the whole being released at $t = 0$, the subsequent motion will be

$$x_n(t) = \frac{1}{N} \left[\sum_{v=0}^{N-1} x_j^0 \exp \left(2\pi i v \frac{n-j}{N} \right) \right] \cos(\omega_v t)$$

Utilize the equation [Eq. (1.2.10)]

$$\cos(z \sin \theta) = \sum_{m=-\infty}^{+\infty} J_{2m}(z) e^{-2im\theta}$$

to show that the motion of the system can also be expressed in terms of the series

$$x_n(t) = \sum_{k=-\infty}^{\infty} x_j^0 J_{2j-2n+2kN}(\omega_0 t)$$

For $N = 6$, $j = 6$, plot the displacements of masses 3 and 6 from $\omega_0 t = 0$ to $\omega_0 t = 10$.

CHAPTER

4

THE FLEXIBLE STRING

4.1 WAVES ON A STRING

Up to now we have been discussing systems which behave as though they were constructed of point masses and massless springs. A few systems which approximate this simplicity are encountered in acoustic practice. The mathematical models we have developed to describe their motion, however, are more important in practice than the systems they represent, for they exhibit in elementary form many of the properties displayed by more complex systems. Thus they have permitted us to introduce concepts of energy and impedance and resonance, for example, which will turn out to be useful aids in analyzing situations of greater intricacy.

Neglecting atomicity, systems have their mass distributed over their whole extent; every part has both density and elasticity. The vibrating string of a violin, for example, cannot be considered as having all its mass concentrated at the center of the string, or even concentrated at a finite number of points along the string; an essential property of the string is that its mass is spread *uniformly* along its length. Similarly, a loudspeaker diaphragm has a good portion of its mass spread out uniformly over its extent. In these cases, each portion of the system will vibrate with a somewhat different motion from that of any other portion. The position of just a few parts of the system will not suffice to describe its motion; the position of *every* point must be specified.

Of course, a rough approximation to the continuous string can be achieved by carrying the analysis of Sec. 3.3 to its limit. It might, for example, be considered to be a limiting case of the system portrayed in Fig. 3.1*c*, when the number of point masses is increased indefinitely and their mutual distance is correspondingly reduced to zero. The techniques developed in the preceding chapter allow this asymptotic solution to be worked out (indeed, it has been the subject of a problem), and this solution turns out to predict the right answers. But such a technique is really an awkward way of solving a problem which is essentially simple. What is needed is a new point of view, a new method of attack.

The new point of view can be summarized as follows: We must not concern ourselves with the motion of each of the infinite number of points on the string, considered as separate points; we must consider the *shape* of the string *as a whole*. At any instant the string will have a definite shape, which can be expressed mathematically by saying that y , the displacement from equilibrium of that part of the string a distance x cm from one end, is a function of x . The motion of the string at any instant will depend on the shape of the string, and the subsequent shape will depend on the motion; what we must do is to find the relation between the shape and the motion. In other words, the string's displacement y is a function of both x and t , and we must discover the relation between y 's dependence on x and its dependence on t . The same point of view will be necessary in studying the vibrations of diaphragms and of air, as we shall see later.

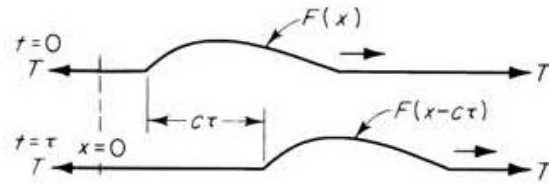
Wave motion

In this chapter we shall use one of the simplest continuous systems, the flexible string under tension, to illustrate this new point of view and how it can be used. We assume that the string has negligible transverse dimensions and that its mass is distributed uniformly, ϵ per unit length. We shall not be concerned here with longitudinal waves, analogous to the limiting form of the motions of the system of Fig. 3.4; we shall study only the transverse motion of the string. If the string is stretched under a tension T newtons, its equilibrium position (in the absence of gravity or other forces) will be straight-line. We can designate a point on the string by giving its distance x from some origin (perhaps one end of the string) while in its equilibrium position. The motions we shall study involve the displacement of the point x an amount y at right angles to the equilibrium line. We need only concern ourselves with displacements in one plane through the equilibrium line, for we can show that the general transverse motion is a linear combination of motions in two perpendicular planes (at least, when the displacements are small enough).

Transverse displacement y is thus a function of both x and t . Holding t constant and plotting y against x gives us a picture of the shape of the whole string at a given instant. Holding x constant and plotting y against t gives us a plot of the motion of a given point on the string as it moves in time. The basic question is, how are these two plots related? What is the relationship between the dependence of y on x and its dependence on t ? To answer this is to determine the nature of wave motion.

Even a casual observation of the behavior of ropes and strings indicates that transverse waves are propagated along them; a disturbance moves from one part to another with a velocity c . This, of course, has a bearing on the relationship between the dependence of y on x and its dependence on t . At $t = 0$ a string may have a shape given by $y = F(x)$ in Fig. 4.1. If this shape is to represent a wave traveling to the right, then a time τ later the displaced

FIGURE 4.1
Wave motion along a flexible string.



region should have moved to the right by a distance $c\tau$, where c is the wave velocity. If c is the same for all frequencies and shapes of wave (this was *not* the case for the waves of Sec. 3.3), the shape of the wave should be unchanged during its motion; at time $t = \tau$ the displacement should be $F(x - c\tau)$.

Thus, for “perfect” wave motion to the right, the relationship between dependence on x and on t should be especially simple. If the displacement at $t = 0$, as function of x , is $F(x)$, then the displacement at time t should be $F(x - ct)$, where c is the wave velocity, independent of the shape of F . In other words, t and x enter only in the combination $x - ct$, as, for instance, $\sin [k(x - ct)]$, or $A(x - ct)^3 \exp [-\beta^2(x - ct)^2]$, etc. For “perfect” wave motion to the left, the displacement should be $f(x + ct)$, independent of the shape of function f . If the equation of motion is linear, so that the sum of solutions is a solution, the general expression for “perfect” wave motion along the string will be

$$y(x, t) = F(x - ct) + f(x + ct) \quad (4.1.1)$$

where $F(z)$ and $f(z)$ are any continuous functions of variable z . Therefore many of our basic questions regarding wave motion on the string would be answered if we could show under what circumstances the displacement $y(x, t)$ of Eq. (4.1.1) is a solution of the equation of motion of the string.

The wave equation

The partial derivative $\partial y / \partial x = y'(x)$ represents the slope of the displacement curve, at point x on the string, at time t . The derivative $\partial y / \partial t = \dot{y}(x)$ represents the transverse velocity of point x at time t . If the shape of the string can be described by the function $F(x - ct)$, we shall have a particularly simple relationship between these partial derivatives, $\partial y / \partial t = -cF'(x - ct) = -c(\partial y / \partial x)$, where $F'(z)$ is the derivative of $F(z)$ with respect to its argument z . This says that, as the wave moves to the right, a portion having negative slope ($\partial y / \partial x < 0$) will move upward ($\partial y / \partial t > 0$), and vice versa.

For wave motion to the left, the relationship between the derivatives is $\partial y / \partial t = +c(\partial y / \partial x)$. For *both* wave motions, and thus for the motion represented by Eq. (4.1.1), the second-order equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (4.1.2)$$

holds. In fact, we can say that a solution of the form (4.1.1) implies Eq. (4.1.2) and also that an equation of motion of the form (4.1.2) has (4.1.1) as its general solution. If (4.1.1) represents "perfect" wave motion, Eq. (4.1.2) can appropriately be called the *wave equation*. We now must show when it is the equation of motion of the flexible string.

The left-hand term in Eq. (4.1.2) is the transverse acceleration of point x on the string at time t . This should be proportional to the net transverse force acting on the portion of the string near point x , if (4.1.2) is its equation of motion. Figure 4.2 shows the forces acting on the portion of string between the points which were at x and $x + dx$ when the string was in equilibrium. At equilibrium the string was under uniform tension T and had a mass ϵ per unit length. At time t , point x is displaced transversely an amount $y(x, t)$, and $x + dx$ is displaced an amount $y(x + dx, t)$. Figure 4.2 shows some of the complications which limit the validity of Eq. (4.1.2).

If y has a nonzero slope at x , either the length of the string between x and $x + dx$ is greater than dx or else the displacements $y(x)$ and $y(x + dx)$ are not exactly perpendicular to the equilibrium line. But if the string is stretched by the displacement, the tension in that part of the string must be greater than the value T , which it had in equilibrium. However, if $\partial y / \partial x$ is negligibly small compared with unity over the whole extent of the string, these complications can be disregarded; we can then neglect all quantities of second and higher order in $\partial y / \partial x$. To within the first order in this small quantity the displacement of the element of string will be perpendicular to the equilibrium line, and the tension in the element is equal to its equilibrium value T (see Chap. 14).

Also, to first order in $\partial y / \partial x$, the net horizontal force on the element of string is zero. The net vertical force is not zero to this order, however; it is equal to the difference in the vertical components of the tensions at the two ends, as shown in Fig. 4.2.

$$F_{\perp} = Ty'(x + dx) - Ty'(x) = T \frac{\partial^2 y}{\partial x^2} dx$$

If the string is perfectly flexible, so that no force is required to bend it into curved form, then $F_{\perp}$ is the only transverse force acting and the element's transverse acceleration must be proportional to $F_{\perp}$. In fact, since the mass

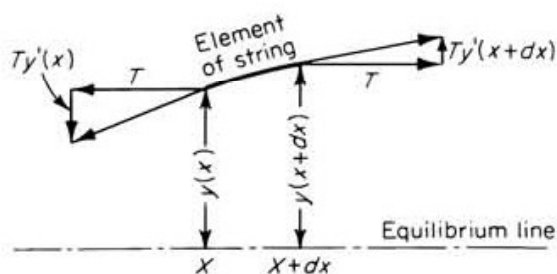


FIGURE 4.2
Forces acting on an elementary length of flexible string.

of the element is ϵdx , the equation of motion must be

$$\epsilon \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (4.1.3)$$

where $c^2 = T/\epsilon$ is independent of x and t to first order in $\partial y/\partial x$.

This is, of course, the wave equation (4.1.2). We have thus shown that if the string is perfectly flexible, is uniform in mass density and in tension along its length, and is nowhere much displaced from its equilibrium configuration, so that $(\partial y/\partial x)^2$ can be neglected compared with unity, the string's motion obeys the wave equation, and wave motion of the sort represented by (4.1.1) is possible, with a wave velocity $c = \sqrt{T/\epsilon}$ which is independent of waveshape and size. Under these circumstances "perfect" wave motion along the string is possible.

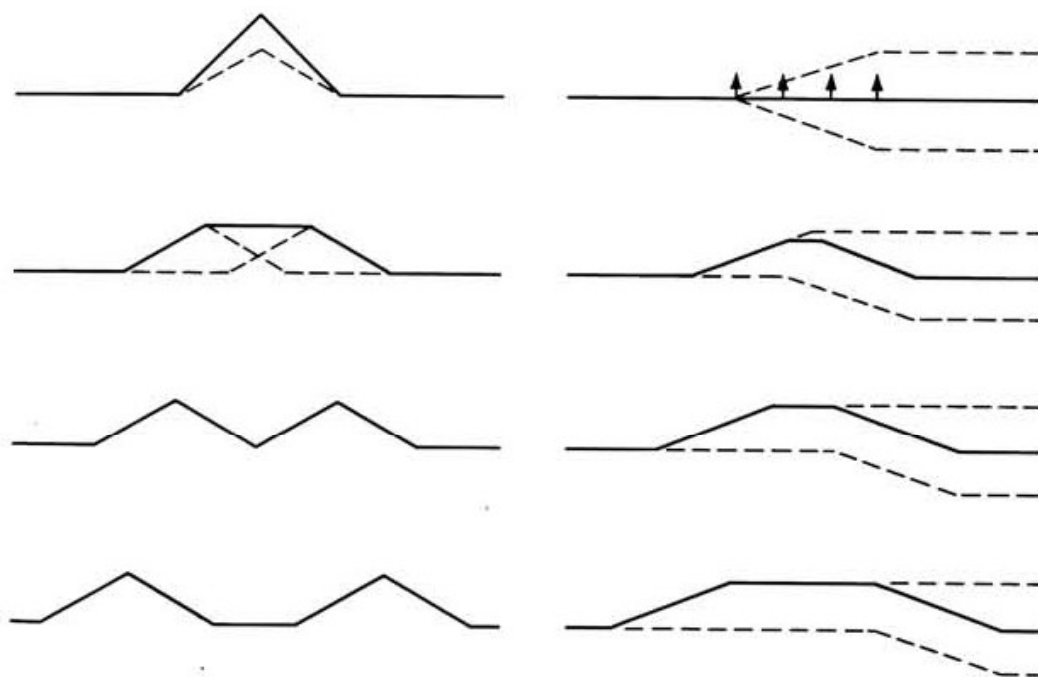


FIGURE 4.3
Motions of plucked and struck strings. The solid lines give the shapes of the strings at successive times; the dotted lines give the shapes of the two "partial waves" traveling in opposite directions whose sum is the actual shape of the string.

The forms of functions F and f in Eq. (4.1.1) can be determined from the initial transverse displacement and velocity of the string at $t = 0$. Since the equation of motion is a second-order equation, only the initial displacement and velocity are needed to determine the motion [see discussion of Eqs. (2.1.4)]. Suppose, at $t = 0$, the shape of the string is given by the function $y_0(x)$, and the initial transverse velocity of point x of the string is $v_0(x)$. Then we can express F and f in terms of y_0 and the function $S(x) = \int_0^x v_0(u) du$.

$$y(x,t) = \frac{1}{2}y_0(x-ct) + \frac{1}{2}y_0(x+ct) - \frac{1}{2c}S(x-ct) + \frac{1}{2c}S(x+ct) \quad (4.1.4)$$

as can be proved by computing y and $\partial y/\partial t$ for $t = 0$. In other words, the solution satisfying the initial conditions $y = y_0(x)$ and $\partial y/\partial t = v_0(x) = dS(x)/dx$ is that given in (4.1.1), with

$$F(z) = \frac{1}{2}y_0(z) - \frac{1}{2c}S(z) \quad f(z) = \frac{1}{2}y_0(z) + \frac{1}{2c}S(z)$$

Figure 4.3 shows the behavior of an infinite string subjected to different initial conditions. A string of infinite extent, a short length of which is pulled aside and let go at $t = 0$, returns to its initial equilibrium line; one given an impulse at $t = 0$ will come to rest along a line parallel to the original line.

Wave energy

When we come to compute the energy and momentum of wave motion, we must take into account the second-order terms we neglected in deriving Eq. (4.1.3), even when they are small. For example, in computing the potential energy gained by displacing a string from equilibrium, we must calculate the work required to move it from the shape $y(x) = 0$ to the shape $y(x)$, as shown in Fig. 4.4. The distance along such a shape, from $x = 0$ to $x = l$, would inevitably be greater than the distance l measured along the equilibrium line. Therefore, if the two end supports are kept a distance l apart, the string must increase its length by an amount

$$\int_0^l \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} dx - l \simeq \frac{1}{2} \int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx$$

when we neglect all terms of higher than second order in the expansion of the square root.

If the string is quite elastic, it has already been stretched quite a bit when

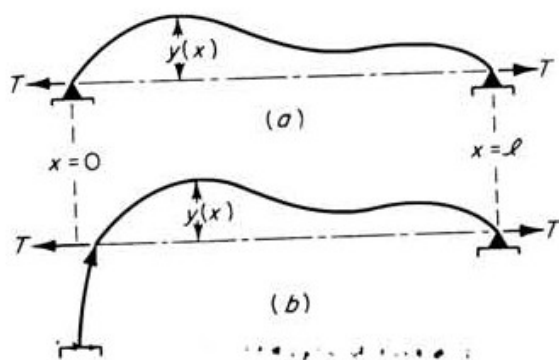


FIGURE 4.4
Displacing the string stretches the string or distorts the support.

it was put under tension T ; so the small increase in length involved in the displacement would not increase the tension by an appreciable amount. Consequently, the work required to stretch the string by displacing it from its equilibrium line to the shape $y(x)$ is equal to T times the change in length of the string, and this, to second order in $\partial y/\partial x$, is

$$V = \int_0^l T \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} dx - lT \simeq \frac{1}{2}T \int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx \quad (4.1.5)$$

If, on the other hand, the string does not elongate when placed under tension, one or both of the supports will be pulled inward when the string is displaced (assume it is the one at $x = 0$, as shown in Fig. 4.4*b*). Here again, if the tension is unchanged to the first order during the displacement, the potential energy is T times the displacement of the support, the identical formula for the potential energy of displacement given in Eq. (4.1.5), as long as $\partial y/\partial x \ll 1$. In the intermediate case, where the string stretches a bit and the supports move a bit, the expression for the potential energy of displacement is still given by Eq. (4.1.5), correct to the second order in the small quantity $\partial y/\partial x$.

One might be tempted to localize the potential energy and say that the element of string between x and $x + dx$ has the potential energy $\frac{1}{2}(\partial y/\partial x)^2 T dx$. But if the string does the stretching, this stretching is uniform along the string, not proportional to $(\partial y/\partial x)^2$ at each point; and if the support does the "stretching," the work is all done moving the end support. In any case it is more correct to say that the potential energy of the string as a whole is given by the integral of Eq. (4.1.5). If, later in this chapter, we find it convenient to talk about a "potential energy density" at x as being $V(x) = \frac{1}{2}T(\partial y/\partial x)^2$, we must keep in mind that the density has a certain degree of arbitrariness about it. We could just as well have said that $V(x)$ is equal to $\frac{1}{2}T(\partial y/\partial x)^2 + (\partial\Phi/\partial x)$, where $\Phi(x)$ is any continuous function of x which goes to zero at the two ends of the string. For really, the only measurable quantity is the integral of $V(x)$ over the whole length of the string, as given in (4.1.5), and $\Phi(x)$ has been chosen so that the integral of $\partial\Phi/\partial x$ over the string is zero.

To cast further light on the amount of energy and momentum carried by the wave, let us parallel the discussion given earlier, in connection with Fig. 3.5, for wave motion in the sequence of springs and masses. Suppose we start wave motion in a very long string, with one end at $x = 0$, by moving this support sideward with a velocity u . After a time τ the transverse displacement of the support is $u\tau$, the wave started by the displacement has reached a distance $c\tau$ along the string, and the shape of the string is that shown by the solid line of Fig. 4.5. The part of the string from $x = 0$ to $x = c\tau$ is moving upward with velocity u ; the part near $x = 0$ has been moving since $t = 0$; the part near $x = c\tau$ has just started to move; the part beyond $x = c\tau$ is still at rest.

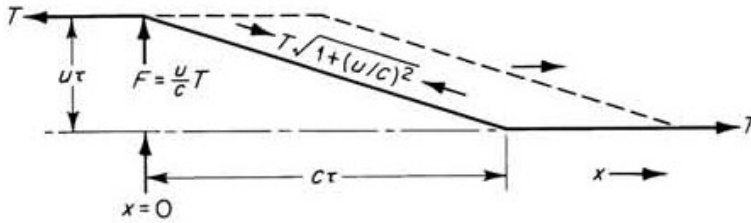


FIGURE 4.5
Initiation of wave motion
by transverse displacement
of support at $t = 0$ with
velocity u .

The transverse force necessary to keep the support moving sideward, with velocity u , is nearly equal to $T(u/c) = -Ty'(0)$ [where $y'(x) = \partial y(x)/\partial x$] as long as $y' \ll 1$. The work done on the string during time τ , in generating the wave motion, is the transverse force F times the distance $u\tau$, which equals $T(u/c)u\tau$. This must equal the kinetic energy $\frac{1}{2}\epsilon c\tau u^2$ of the length $c\tau$ of the string which is in motion at time τ , plus the potential energy of this part, which from Eq. (4.1.5) equals $\frac{1}{2}T\int (y')^2 dx = \frac{1}{2}T(u/c)^2 c\tau$. The resulting equation for the energy balance leads directly to a relation between tension, mass density, and wave velocity c .

$$T\left(\frac{u}{c}\right)u\tau = \frac{1}{2}\epsilon c\tau u^2 + \frac{1}{2}T\left(\frac{u}{c}\right)^2 c\tau$$

or

$$\frac{Tu^2\tau}{2c} = \frac{1}{2}\epsilon u^2 c\tau \quad \text{or} \quad c^2 = \frac{T}{\epsilon} \quad (4.1.6)$$

The velocity c required to balance energy input with energy flow along the string is of course the same as that given in Eq. (4.1.3).

The total energy density of the wave motion can thus be written as the sum of a kinetic energy density and a potential energy density.

$$H(x,t) = \frac{1}{2}\epsilon\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = W_{tt} \quad (4.1.7)$$

where we must keep in mind the reservations mentioned three paragraphs above. In addition, Eq. (4.1.6) shows that the energy flow along the string, across point x , during the wave motion, is $T(u/c)u = -T(\partial y/\partial x)(\partial y/\partial t)$, in watts. Finally, if the transverse force ceases at $t = \tau$, the sideward motion of the support stops, and the string, a short time later, will have the shape of the dotted line of Fig. 4.5. Energy no longer is fed into the string, but the energy it collected between $t = 0$ and $t = \tau$ continues to be propagated to the right.

Wave momentum

When we consider the momentum imparted to the string, we must be still more careful to keep track of quantities of the second order in $\partial y/\partial x$. For example, if the string elongates enough so the motion of the end at $x = 0$

is vertical, as shown in Fig. 4.5, then to second order in $\partial y/\partial x = u/c$, the tension in the moving part of the string must be $T\sqrt{1 + (u/c)^2}$. In this case the horizontal components of force at $x = c\tau$ exactly cancel and the vertical component, $T(u/c)$ to second order, serves to bring the next portion of the string into motion. On the other hand, if the string does not change length, the end $x = 0$ must move upward at an angle u/c to the vertical, as shown in Fig. 4.6. In this case, as can be seen from Fig. 4.6, the tension in the moving part of the string is $T\sqrt{1 - (u/c)^2}$, to second order, so that the unbalanced force at $t = c\tau$, with vertical component $T(u/c)$ and horizontal component $T(u/c)^2$ (to second order), starts the next part of the string up at the required angle u/c to the vertical.

These results do not invalidate our earlier derivation of Eq. (4.1.3) or (4.1.7); we there used values of the tension only to the first order in u/c , and the tension, in either case, is T throughout the string to first order.

The balance of forces at the end $x = 0$ also is affected by the degree of elasticity of the string. We first deal with the limiting case where the string does not stretch and the end support moves inward, as well as transversally, as shown in Fig. 4.6. In this case the motion of the end support is perpendicular to the string, at an angle u/c radians to the vertical, when u/c is small. The applied force therefore has a horizontal component $F(u/c) = T(u/c)^2$ to the second order in the small quantity $u/c = -(\partial y/\partial x)$. Consequently, the driving force produces both a vertical and a horizontal component of momentum in the string. The momentum transverse to the equilibrium line is the one usually considered; it amounts to ϵu times the length $c\tau$ of the string in motion at $t = \tau$ (to the second order in u/c). This must equal the transverse impulse, $F\tau = T(u\tau/c)$, of the driving force. Again equating the two gives rise to the equation $c^2 = T/\epsilon$ for the wave velocity.

The momentum in the longitudinal direction is not quite as obvious a matter. The horizontal impulse, for the inelastic string of Fig. 4.6, is $F(u/c)\tau = T\tau(u/c)^2$. This must impart a longitudinal momentum to the string. For a nonstretchable string the x component of momentum is $\epsilon(u/c)u$ per unit length, over the length $c\tau$ which is in motion. Again equating impulse to momentum leads to the equation $c^2 = T/\epsilon$; this time it indicates a *longitudinal momentum density* of $-\epsilon(\partial y/\partial t)(\partial y/\partial x)$ per unit length of string. This momentum would be "felt" by the support at the other end

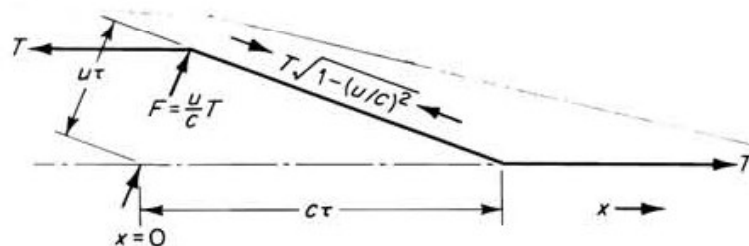


FIGURE 4.6
Balance of forces during
wave motion along an in-
elastic string.

and would produce a "radiation pressure" equal to c times the longitudinal momentum density, $-\epsilon c(\partial y/\partial t)(\partial y/\partial x) = \epsilon u^2 = T(u/c)^2 = F(u/c)$ for the case of Fig. 4.6.

If the string is elastic, as in Fig. 4.5, so that the motion of the support at $x = 0$ is exactly transverse, it is altered longitudinal stress which is carried, not longitudinal momentum; but the effect on the support at the far end is just the same (at least to second order in u/c). The tension in the moving part of the string is $T\sqrt{1 + (u/c)^2}$, greater than T . This stretches the string and moves it vertically upward and acts like a longitudinal momentum in its effects on the support at the other end [see discussion following Eq. (4.1.12)]. A longitudinal force equal to c times $-\epsilon(\partial y/\partial t)(\partial y/\partial x)$ will be produced. Thus, in either limit, or in the intermediate case of some string elasticity and some horizontal motion of the support at $x = 0$, the string acts as though it carries a longitudinal wave momentum of density $-\epsilon(\partial y/\partial t)(\partial y/\partial x)$ per unit length of string (as long as $\partial y/\partial x$ is small), which is carried along the string with the velocity c of the wave. We note that if the direction of the transverse force F and the motion of the string had been downward, this longitudinal momentum would still be to the right. The motion of both energy and longitudinal momentum is always in the direction of the wave.

Lagrange equation and stress-energy tensor

We can now recapitulate our findings for the stress, energy, and momentum of transverse waves in a perfectly flexible string, when $\partial y/\partial x$ is small throughout its length, and we can recast its equation of motion in the form of the Lagrange equation of classical dynamics. The kinetic energy of transverse motion is $\frac{1}{2}\epsilon(\partial y/\partial t)^2$ per unit length, and its potential energy density is $\frac{1}{2}T(\partial y/\partial x)^2$, to second order in $\partial y/\partial x$. If the system were a point mass m acted on by forces having a potential energy $V(x)$, we could define a Lagrange function $L = \frac{1}{2}m(dx/dt)^2 - V(x)$, and write the equation of motion as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad \dot{x} = \frac{dx}{dt}$$

In the case of the string, with its mass distributed along its length and with its dependent variable y a function of two independent variables, x and t , we must define a *Lagrange density* $L = \frac{1}{2}\epsilon(\partial y/\partial t)^2 - \frac{1}{2}T(\partial y/\partial x)^2$ and write the equation of motion, to be satisfied at every point of the string,

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{y}} + \frac{\partial}{\partial x} \frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \quad \dot{y} = \frac{\partial y}{\partial t} \quad y' = \frac{\partial y}{\partial x} \quad (4.1.8)$$

or

$$\epsilon \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = 0$$

which is the wave equation. For the derivation of the equation of motion from the Lagrange density, see the discussion preceding Eq. (5.2.6).

The transverse momentum density is $\partial L/\partial \dot{y} = \epsilon(\partial y/\partial t) = p(x, t)$. The transverse force by which the string to the left of point x acts on the part to its right is $\partial L/\partial y' = -T(\partial y/\partial x) = s(x, t)$. By analogy with particle dynamics we can define an energy density $H(x, t) = p\dot{y} - L$, but whereas H is stationary for the point mass, it may flow along the string. Thus we must define an energy flux W_{tx} as well as an energy density $W_{tt} = H$.

$$\begin{aligned} W_{tt}(x, t) &= \dot{y} \frac{\partial L}{\partial \dot{y}} - L = \frac{1}{2}\epsilon \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2}T \left(\frac{\partial y}{\partial x} \right)^2 \\ W_{tx}(x, t) &= \dot{y} \frac{\partial L}{\partial y'} = -T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \end{aligned} \quad (4.1.9)$$

From the first expression, again by analogy with point-mass dynamics, we can obtain a pair of Hamilton's equations, still another way of writing the equation of motion, which will be useful later.

$$\begin{aligned} H(p, s, y) &= \frac{p^2}{2\epsilon} + \frac{s^2}{2T} \\ \frac{\partial y}{\partial t} &= \frac{\partial H}{\partial p} \quad \frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} \frac{\partial H}{\partial y'} = -\frac{\partial H}{\partial y} \end{aligned} \quad (4.1.10)$$

which will again produce the wave equation.

The energy density W_{tt} and the energy flux W_{tx} must be related. If, for example, more energy flows across the point $x + dx$ than flows across x , then the energy contained in length dx of the string must diminish.

$$W_{tx}(x + dx) - W_{tx}(x) = -dx \frac{\partial W_{tx}}{\partial x} \quad \text{or} \quad \frac{\partial}{\partial x} W_{tx} + \frac{\partial}{\partial t} W_{tt} = 0 \quad (4.1.11)$$

which is the *equation of continuity* for energy. That the quantities given in Eqs. (4.1.9) indeed satisfy the equation of continuity can be quickly demonstrated, for

$$\begin{aligned} \frac{\partial}{\partial x} W_{tx} + \frac{\partial}{\partial t} W_{tt} &= -T \frac{\partial^2 y}{\partial x \partial t} \frac{\partial y}{\partial x} - T \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial x^2} + \epsilon \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} + T \frac{\partial y}{\partial x} \frac{\partial^2 y}{\partial t \partial x} \\ &= \frac{\partial y}{\partial t} \left(-T \frac{\partial^2 y}{\partial x^2} + \epsilon \frac{\partial^2 y}{\partial t^2} \right) = 0 \quad [\text{from Eq. (4.1.8)}] \end{aligned}$$

The quantities W_{tt} and W_{tx} are two elements in a tensor; its other

components are

$$\begin{aligned} W_{xt} &= y' \frac{\partial L}{\partial \dot{y}} = \epsilon \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} \\ W_{xx} &= y' \frac{\partial L}{\partial y'} - L = -\frac{1}{2} \epsilon \left(\frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 \end{aligned} \quad (4.1.12)$$

Reference to the special case of Fig. 4.6 indicates that $W_{xt} = -\epsilon u(u/c)$ is minus the longitudinal momentum density of the wave (distinguished from the transverse momentum $\epsilon \partial y / \partial t$, which is a first-order quantity). Another application of the equation of continuity would indicate that a time rate of change of momentum density should equal minus the space rate of change of stress in the string; net force must equal time rate of change of momentum. In the special case of Fig. 4.6, W_{xx} for the moving part of the string equals $-\frac{1}{2} \epsilon u^2 - \frac{1}{2} T(u/c)^2 = -T(u/c)^2$ and is zero for the rest of the string. However, Fig. 4.6 also shows that the net change in horizontal component of the stress (stress is minus the tension) caused by the string's displacement and motion is $-T[1 - (u/c)^2] + T = T(u/c)^2$ to the second order in u/c . Therefore W_{xt} is minus the longitudinal momentum density, and W_{xx} the net longitudinal stress in the string caused by the wave motion. The two satisfy the equation of continuity $\partial W_{xx} / \partial x + \partial W_{xt} / \partial t = 0$, as substitution from Eqs. (4.1.12) will show.

Thus we have obtained an expression, correct to the second order in $\partial y / \partial x$, for the Lagrange density $L(\dot{y}, y', y)$, from which we can obtain, by partial differentiation, the transverse momentum $p(x, t)$ and stress $s(x, t)$, the longitudinal momentum density W_{xt} and stress W_{xx} , and the energy density $W_{tt} = H$ and energy flux W_{tx} . The Lagrange equation (4.1.8) is the wave equation. Each of the pairs p and s , W_{xt} and W_{xx} , W_{tt} and W_{tx} satisfies an equation of continuity. We note again that the elements of the stress-energy tensor are not localized quantities which can be measured at each point of the string, but that the expressions given in Eqs. (4.1.9) and (4.1.12), when integrated over the whole length of the string, give correct values for the energy of the whole string and the energy and momentum input and output at the ends of the string, which are measurable.

4.2 FREE OSCILLATIONS

We have now demonstrated that the transverse motion of a flexible string obeys the wave equation (4.1.3) if the slope of its shape, $|\partial y / \partial x|$, is everywhere very small compared with unity and if the string is perfectly flexible. Neither limitation is exactly met in practice, but in a large number of cases of interest acoustically, they are conformed to so nearly that the actual motion corresponds fairly well to the solutions of Eq. (4.1.3). These

solutions, having the general form of (4.1.1), are noninterfering waves, traveling in either direction with a velocity $c = \sqrt{T/\epsilon}$, independent of the shape or size of the wave (within the limits mentioned). Of course, it is just the shape of the wave which moves along the string; the individual parts of the string just move up and down.

Initial conditions

For a string of infinite length, its initial transverse displacement and velocity at $t = 0$ being $y_0(x)$ and $v_0(x)$, respectively, the subsequent displacement of the string is given by Eqs. (4.1.4).

$$y(x, t) = \frac{1}{2c} [cy_0(x - ct) - S(x - ct)] + \frac{1}{2c} [cy_0(x + ct) + S(x + ct)] \quad (4.2.1)$$

where

$$S(z) = \int_0^z v_0(x) dx$$

The first bracket represents a wave going in the positive x direction; the second, a wave in the negative x direction. For this wave the elements of the stress-energy tensor are

$$\begin{aligned} W_{tt} &= H = \frac{1}{4} \{ [J(x - ct)]^2 + [K(x + ct)]^2 \} = -W_{xx} \\ W_{tx} &= \frac{c}{4} \{ [J(x - ct)]^2 - [K(x + ct)]^2 \} = -c^2 W_{xt} \\ J(z) &= \sqrt{T} y'_0(z) - \sqrt{\epsilon} v_0(z) \quad K(z) = \sqrt{T} y'_0(z) + \sqrt{\epsilon} v_0(z) \end{aligned} \quad (4.2.2)$$

The waves in the two directions each have their own energy, momentum, and stress, there being no cross-product terms. The energy and stress terms are both positive; these quantities are independent of wave direction. On the other hand, the momentum density and the energy flux are positive for the wave in the positive x direction, and negative for the wave in the negative x direction.

Boundary conditions

So far we have been treating the string as though it had an infinite extent; actually, it is fastened down somewhere, and this fastening affects the motion of the string. The fact that the string is fastened to a support is an example of a *boundary condition*. It is a requirement on the string at a given point in space which must be true *for all time*, as opposed to initial conditions, which fix the dependence of y and v on x *at a given time*. Boundary conditions are more important in determining the general behavior of the string, its allowed frequencies, etc., than are initial conditions.

If the support is rigid and the distance along the string is measured from it, the boundary condition is that y must be zero when $x = 0$, for all values of the time. If the support is springy, so that it is displaced sideward a distance

CF cm for a sideward force F dynes, the boundary condition is that y must always be equal to C times the component of the string's tension perpendicular to the equilibrium line: $y = CT(\partial y/\partial x)$ at $x = 0$. Many other sorts of boundary conditions are possible.

Reflection at a boundary

Let us take the simple case of the rigid support, requiring that $y = 0$ when $x = 0$, and see what effect this has on the motion of the string. The solutions $y = f(x - ct)$ or $y = f(x + ct)$ cannot be used in this case, for they will not always be zero when $x = 0$. However, the solution

$$y = -f(x - ct) + f(-x - ct)$$

will satisfy the boundary condition. At $x = 0$,

$$y = -f(-ct) + f(-ct) = 0$$

for all values of t . To see what motion of the string this expression corresponds to, let us suppose that $f(z)$ is a function that is large only when z is zero and drops off to zero on both sides of this maximum. Then, when $t = -10$, the function $f(-x - ct)$ will have a peak at $x = 10c$ representing a single pulse traveling leftward along the string. The function $f(x - ct)$ at this time would have its peak at $x = -10c$ if there were any string to the left

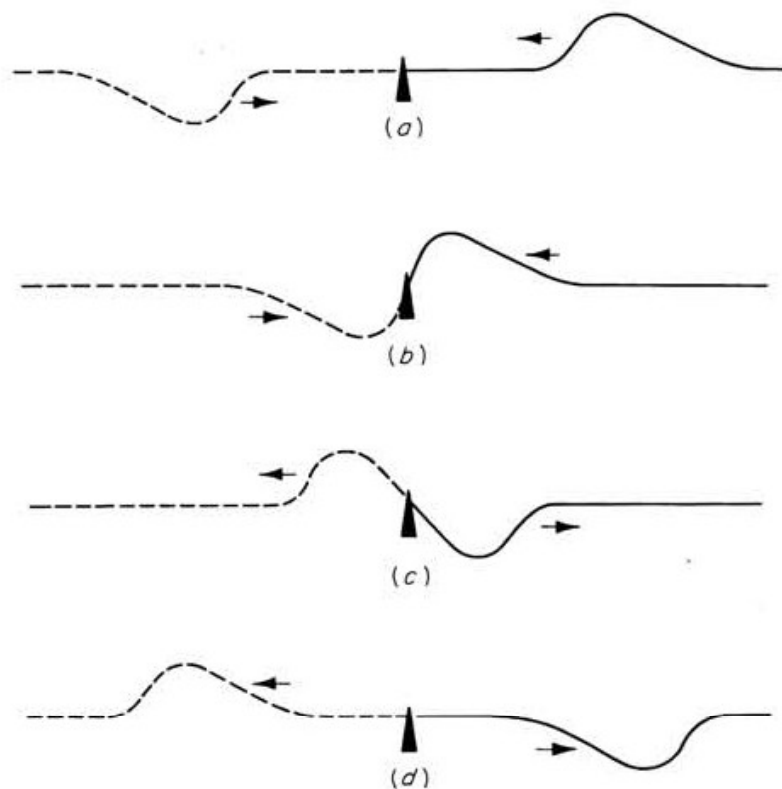


FIGURE 4.7

Reflection of a wave from the end support of a string. The solid lines show the shape of the string at successive instants of time; the dotted lines, the imaginary extension of the wave-form beyond the end of the string.

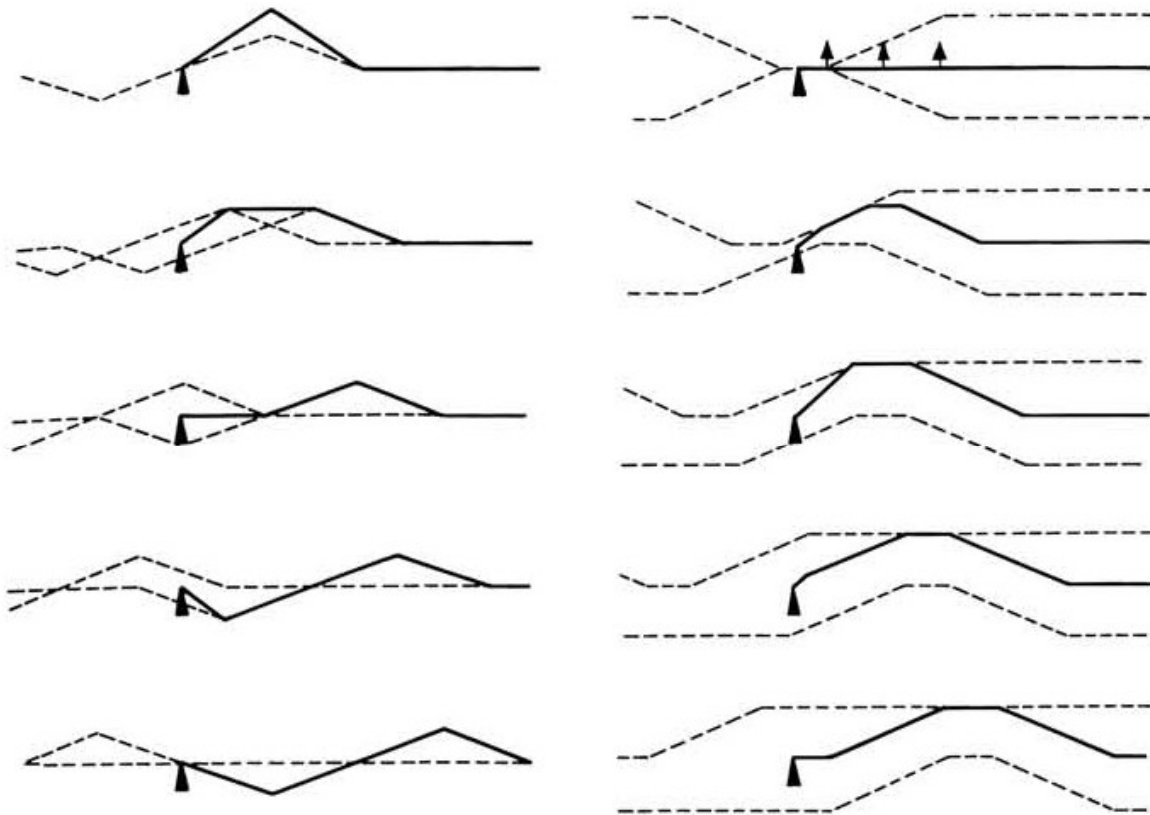


FIGURE 4.8
Motions of plucked and struck strings fixed at one end. Dotted lines show the traveling partial waves; their sum is the solid line, the actual shape of the string.

of the support, but since no string is there, the term $-f(x - ct)$ is not apparent in the shape of the string at $t = -10$. It has been represented in Fig. 4.7a by a dotted line to the left of $x = 0$.

As t increases, the wave in the actual string moves to the left, and the wave in the imagined extension of the string moves to the right, until they begin to coalesce at $x = 0$. During the coalescing the displacement of the point at $x = 0$ is always zero, for the effects of the two waves just cancel each other here. A little later the waves have passed by each other, the wave that had been on the imaginary part of the string now being on the actual string, and vice versa. The succession of events is pictured in Fig. 4.7. What has happened is that the pulse which had originally been traveling leftward is *reflected* at the point of support $x = 0$ and comes back headed toward the right, as a pulse of similar form but of opposite sign. The boundary condition at $x = 0$ has required this reflection, and the particularly simple sort of condition that we have imposed has required this very symmetric sort of reflection. Most other boundary conditions would require a greater difference between the original and the reflected wave. When a wave strikes it, a rigid support must pull up or down on the string by just the right amount to keep y zero, and in doing so it “generates” a reflected wave.

The expression for the motion of a string satisfying the boundary condition $y = 0$ at $x = 0$ and the initial conditions $y = y_0(x)$, $v = v_0(x)$ at $t = 0$ is

$$y = \frac{1}{2} \left[Y(x - ct) + Y(x + ct) - \frac{1}{c} H(x - ct) + \frac{1}{c} H(x + ct) \right] \quad (4.2.3)$$

where

$$Y(z) = \begin{cases} y_0(z) & z > 0 \\ -y_0(-z) & z < 0 \end{cases}$$

$$H(z) = \begin{cases} S(z) & z > 0 \\ S(-z) & z < 0 \end{cases} \quad S(z) = \int_0^z v_0(x) dx$$

These definitions of Y and H are necessary because y_0 and v_0 are defined only for positive values of x (where the string actually is), whereas the form of the partial waves used to build up the subsequent forms of the string must be given for all values of $z = x \pm ct$. The particular forms of Y and H are chosen so that they automatically satisfy the boundary conditions at $x = 0$ for all values of t . Two examples of the way in which the motion of the string can be built up by the use of these partial waves are given in Fig. 4.8.

Motion of the end support

When the end support at $x = 0$ is not perfectly rigid but moves transversally in response to transverse motion of the string, the problem becomes somewhat more complicated, though it can be worked out by means of the Laplace transform. If the end support yields to a force, it presumably yielded when the tension T was applied. But this has already been taken into account, and the support is, by definition, at $x = 0$ when the string is in equilibrium under tension T .

If the string is not in equilibrium, if it has a nonzero slope $y'(0)$ at the origin, the support will be subject to a transverse force and may move transversally. Suppose the support responds to this transverse force like a simple mass on a spring, like the systems discussed in Sec. 2.1. The transverse force is $Ty'(0)$, as long as $y'(0)$ is small, and the reaction of the support to it prescribes a relationship between the slope $y'(0)$, the displacement $y(0)$ of both string and support, its time derivatives and the effective mass m , frictional resistance R , and transverse stiffness K of the support.

$$Ty'(0, t) = m\ddot{y}(0, t) + R\dot{y}(0, t) + Ky(0, t) \quad (4.2.4)$$

where $y'(x, t) = \partial y / \partial x$, $\dot{y} = \partial y / \partial t$, and $\ddot{y} = \partial^2 y / \partial t^2$. This formula exhibits the stiffness, frictional, and inertive reaction of the support to transverse motion of the string end.

In accordance with Eq. (4.1.1), we assume that the displacement of the string is

$$y(x, t) = f(ct + x) + g(ct - x) \quad x \geq 0$$

where $f(x)$ is the shape of the left-bound wave, which is to be reflected from the support. Thus f is known; we must use Eq. (4.2.4) to determine the shape of $g(ct - x)$, the reflected wave. We assume that, at $t = 0$, the start of the calculation, the incoming wave has not yet reached the support, so that $f(x) = 0$ for $x < b$, where $b > 0$. Then, until $t = b/c$, the support will be at rest, and we can be sure that g is zero until that time.

Inserting the expression for $y(x, t)$ into Eq. (4.2.4), we get

$$mc^2 \frac{d^2 g}{dz^2} + (Rc + T) \frac{dg}{dz} + Kg = - \left[mc^2 \frac{d^2 f}{dz^2} + (Rc - T) \frac{df}{dz} + Kf \right] \quad (4.2.5)$$

for $g(z)$ and $f(z)$, where $z = ct$ in this equation [since Eq. (4.2.4) holds only for $x = 0$], but will be $ct - x$ when we come to compute the shape of the reflected wave $g(ct - x)$. Next we take Laplace transforms of both sides. The formulas of Table 2.1 show that if

$$F_L(s) = \int_0^\infty e^{-sz} f(z) dz$$

is the Laplace transform of the known function $f(z)$ (the variable is now z instead of t , but the computations are the same), the corresponding transform of the unknown function $g(z)$ is given by

$$[mc^2 s^2 + (Rc + T)s + K]G_L(s) = -F_L(s)[mc^2 s^2 + (Rc - T)s + K]$$

or

$$\begin{aligned} G_L(s) &= -F_L(s) + \frac{2TsF_L(s)}{mc^2 s^2 + (Rc + T)s + K} = -F_L(s) + \frac{2TF_L(s)}{cZ(ics)} \\ &= -F_L(s) + \frac{2\epsilon}{m} \frac{sF_L(s)}{[s + (k_r/c) - i(\omega_f/c)][s + (k_r/c) + i(\omega_f/c)]} \end{aligned} \quad (4.2.6)$$

where $k_r = (Rc + T)/2mc = (R + \epsilon c)/2m$ is a resistance parameter, like the k of Eq. (2.2.1) for the simple oscillator. In fact, k_r is the sum of the $k = R/2m$ of Eq. (2.2.1) for the transverse motion of the support, plus an additional term, $\epsilon c/2m$, produced by the attached string. We shall see later that ϵc is the "wave resistance" of the string; any transverse motion of the support produces a wave in the string, which carries energy away. Frequency $\omega_f/2\pi$ is related to the ν_f of Eq. (2.2.2); $\omega_f^2 = \omega_0^2 - k_r^2$, where $\omega_0^2 = K/m$; $\omega_0/2\pi$ is the resonance frequency of the support for transverse motion. Function $Z(\omega) = -i\omega m + (R + \epsilon c) + i(K/\omega)$ is the mechanical impedance of the support for transverse motion. Factor c in $\omega = ics$ is the ratio between wavelength and period.

The inverse transform of $G_L(s)$ thus has two terms. The first term, $-f(ct - x)$, is the wave which would be generated by a rigid boundary at $x = 0$, as shown in Fig. 4.7. The second term, the inverse transform of $2TF_L(s)/cZ(ics)$, is the wave motion generated by the motion of the support. It reflects the frequency response of the support, because of the impedance

term in the denominator; this response continues after the originating wave has been reflected.

An example may be in order. Suppose the original shape of the incident wave, going in the negative x direction, is $f(ct + x)$, where

$$f(z) = \begin{cases} 0 & 0 \leq z < l - a \\ \frac{b}{a}(z - l + a) & l - a < z < l \\ \frac{b}{a}(l + a - z) & l < z < l + a \\ 0 & l + a < z \end{cases}$$

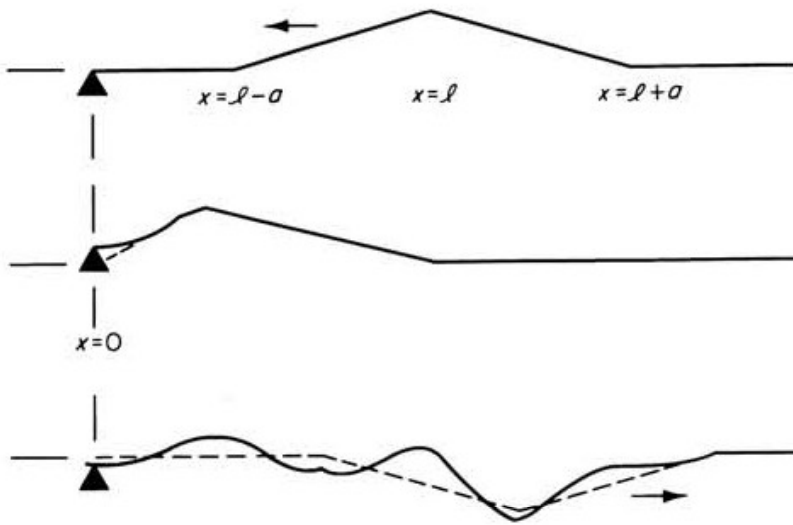


FIGURE 4.9

Reflection of wave from a nonrigid support. Dotted line is the shape if the support is rigid.

as shown in Fig. 4.9. The Laplace transform of $f(z)$ is then

$$F_L(s) = \frac{b}{s^2 a} (e^{s(a-l)} + e^{-s(a+l)} - 2e^{-sl})$$

and the difference between $G_L(s)$ and $-F_L(s)$ is

$$\begin{aligned} \frac{2TF_L(s)}{cZ(ics)} &= \frac{2Tb}{mc^2 a} \frac{e^{-s(l-a)} + e^{-s(l+a)} - 2e^{-sl}}{s[s + (k_r/c) - (i\omega_f/c)][s + (k_r/c) + (i\omega_f/c)]} \\ &= \frac{2Tb}{Ka} (e^{-s(l-a)} + e^{-s(l+a)} - 2e^{-sl}) \\ &\quad \times \left\{ \frac{1}{s} - \frac{\omega_0^2}{2i\omega_f(k_r - i\omega_f)[s + (k_r/c) - i(\omega_f/c)]} \right. \\ &\quad \left. + \frac{\omega_0^2}{2i\omega_f(k_r + i\omega_f)[s + (k_r/c) + i(\omega_f/c)]} \right\} \end{aligned}$$

Utilization of Table 2.1 shows that the inverse transform of this produces the solution

$$y(x,t) = f(ct + x) + g(ct - x)$$

where

$$g(z) = -f(z) + \frac{2Tb}{Ka} [Q(z - l + a) + Q(z - l - a) - 2Q(z - l)]$$

$$Q(w) = u(w) \left\{ 1 - e^{-k_r w/c} \frac{\cos [(\omega_f/c)w - \theta]}{\cos \theta} \right\} \rightarrow \begin{cases} -u(w) \left(\frac{\omega_f w}{c} \right)^2 & w \rightarrow 0 \\ 1 & w \rightarrow \infty \end{cases}$$

and $\cos \theta = \omega_f/\omega_0$, $\sin \theta = k_r/\omega_0$, $\omega_0^2 = K/m$, and $\omega_f^2 = \omega_0^2 - k_r^2$.

Figure 4.9 shows the shape of the string at $t = 0$ just after the incident wave strikes the support and after the wave has been reflected. The dotted lines show the shape the string would have if the support were rigid. When the incident wave strikes the support, it first is pulled upward and then oscillates with damped motion. This motion would be damped even if the inherent-friction term R were zero, because of the wave resistance ϵc of the attached string. Each of the three changes of slope of the wave generates a transient oscillation of the support, represented by the function Q of the formula. After the incident wave has gone, the support oscillation damps out rapidly. If k_r is larger than ω_0 , ω_f is imaginary, and the support returns exponentially to equilibrium without oscillation.

Strings of finite length

Actual strings are fastened at *both* ends, so that, really, two boundary conditions are imposed. For instance, the string can be fastened to rigid supports a distance l cm apart, so that y must always be zero both at $x = 0$ and at $x = l$. The most important effect of a second boundary condition of this sort is to require that the motion of the string be *periodic*. A pulse started at $x = 0$ travels to the other support at $x = l$ in a time l/c , is reflected, travels back to $x = 0$, and is again reflected. If the supports are rigid, the shape of the pulse after its second reflection is just the same as that of the original pulse, and the motion is periodic with a period equal to $2l/c$. The motion in this case is not, in general, harmonic, as we shall see, but it is always periodic. This periodicity of all motion of the string depends entirely on the fact that we have imposed a particular sort of boundary condition; if other conditions are imposed at $x = 0$ and $x = l$ (i.e., if the supports are not perfectly rigid), then it may not be true that every motion is periodic; in fact, it may never be periodic.

The quantitative manner of dealing with the two boundary conditions is by means of the partial waves f . When the string is only l cm long, we are free to give any shape to $f(z)$ for z larger than l or smaller than zero. "Free" is not the correct word, however, for we must choose that form of f which satisfies both boundary conditions. If we start out at $t = 0$ with a pulse of

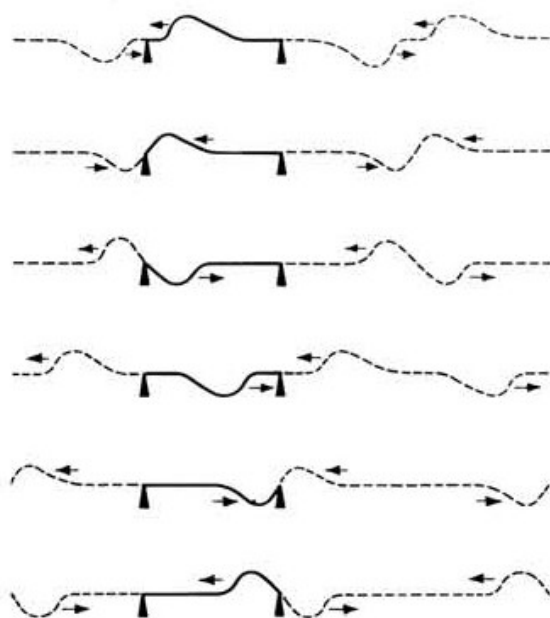


FIGURE 4.10

Periodic motion of a string fixed at both ends. Solid lines give the shape of the actual string at successive instants; dotted lines show the imaginary extension of the waveform beyond the ends of the string. The motion is made up of two partial waves going in opposite directions, each being periodic in x with period $2l$.

the form f traveling to the left, then, as before, we can satisfy the condition $y = 0$ at $x = 0$ by setting $y = -f(x - ct) + f(-x - ct)$. To have $y = 0$ at $x = l$, we must arrange the rest of the function f , beyond the limits of the actual string, so that $f(l - ct) = f(-l - ct)$ or, setting $z = -l - ct$, so that $f(z) = f(z + 2l)$ for all values of z . This means that the function $f(z)$, which must be defined for all values of z , must be *periodic* in z , repeating itself at intervals of $2l$ all along its length. An illustration of how this sort of partial wave can be used to determine the motion of the string is given in Fig. 4.10.

To satisfy the boundary conditions $y = 0$ at $x = 0$ and at $x = l$ and the initial conditions $y = y_0(x)$, $v = v_0(x)$ at $t = 0$, we build up a combination similar to that given in Eq. (4.2.3):

$$y = \frac{1}{2} \left[Y(x - ct) + Y(x + ct) - \frac{1}{c} H(x - ct) + \frac{1}{c} H(x + ct) \right] \quad (4.2.7)$$

where

$$Y(z) = \begin{cases} -y_0(-z) & -l < z < 0 \\ y_0(z) & 0 < z < l \\ -y_0(2l - z) & l < z < 2l \\ y_0(z - 2l) & 2l < z < 3l \\ \dots & \dots \end{cases}$$

$$H(z) = \begin{cases} S(-z) & -l < z < 0 \\ S(z) & 0 < z < l \\ S(2l - z) & l < z < 2l \\ S(z - 2l) & 2l < z < 3l \\ \dots & \dots \end{cases} \quad S(z) = \int_0^z v_0(x) dx$$

Two examples of the motion of such strings are given in Fig. 4.11. In Fig. 4.12 the displacement of a point on the string is plotted as a function of time, showing that the motion is periodic but not simple-harmonic.

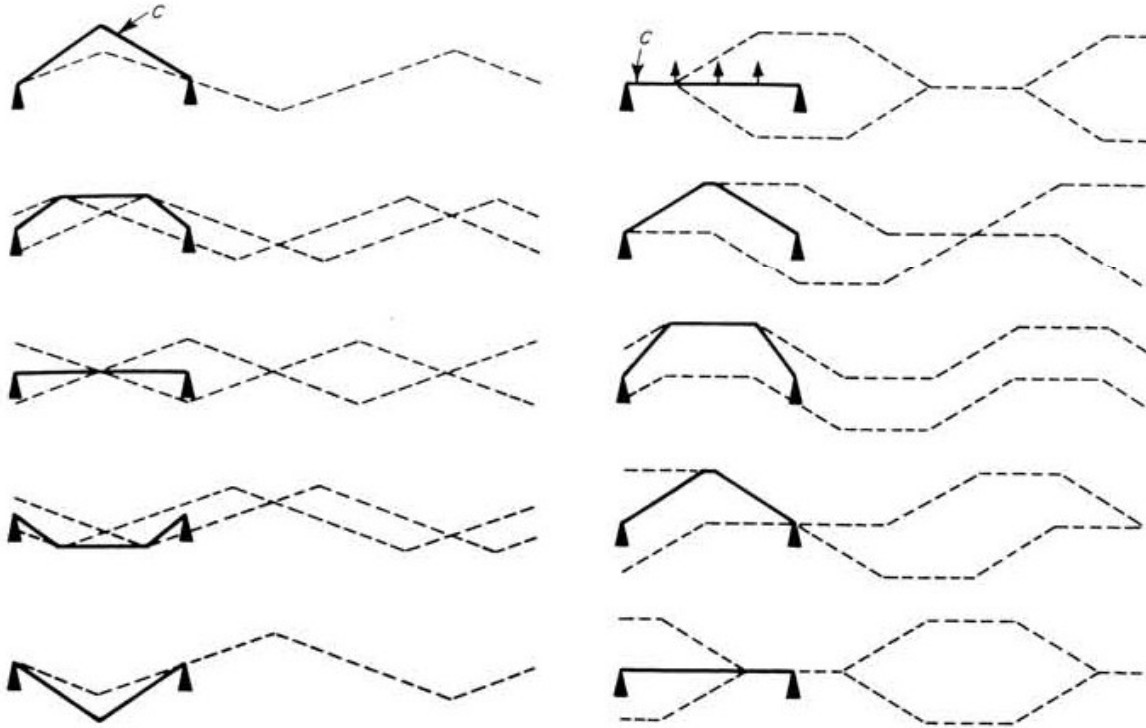


FIGURE 4.11

Motions of plucked and struck strings fixed at both ends. The solid lines show the successive shapes of the string during one half cycle. Shapes for the other half cycle are obtained by reversing the sign of the curves.

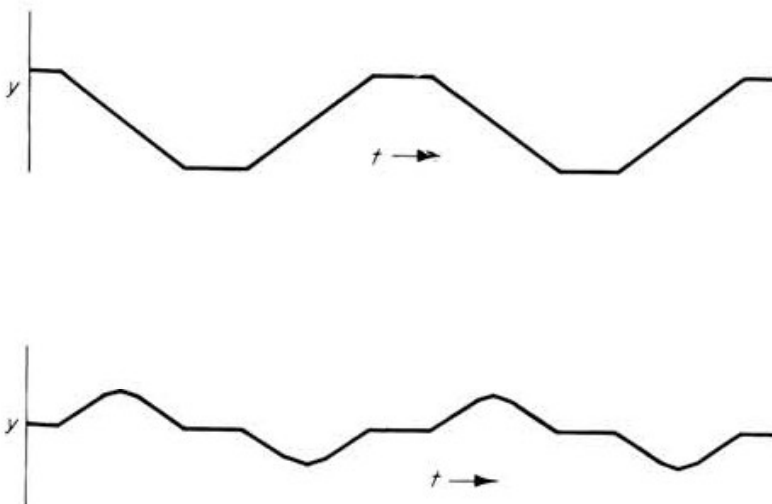


FIGURE 4.12

Displacements of the points marked c on the strings shown in Fig. 4.11, plotted as functions of the time.

4.3 SIMPLE-HARMONIC OSCILLATIONS

It has been seen in the last section that imposing boundary conditions limits the sorts of motion that a string can have, and that if the boundary conditions correspond to the fixing of both ends of the string to rigid supports, the motion is limited to *periodic* motion. The latter result is an unusual one, for we found in the last chapter that even as simple a system as a pair of coupled oscillators does not, in general, move with periodic motion. It is not unusual for a system to oscillate with simple-harmonic motion (which is a special type of periodic motion) when it is started off properly (we shall see that practically every vibrating system can do this); what is unusual in the string between rigid supports is that *every* motion is periodic, no matter how it is started.

Our problem in this section is to find the possible simple-harmonic oscillations of the string (the normal modes of vibration) and to see what the relation is between the frequencies of these vibrations that makes the resulting combined motion always periodic. The problem of determining the normal modes of vibration of a system is not just an academic exercise. For systems more complicated than that of the string between rigid supports, we have no method of graphical analysis similar to that of the last section, and the only feasible method of discussing the motion is to “take it apart” into its constituent simple-harmonic components. There is also a physiological reason for studying the problem, for the ear itself analyzes a sound into its simple-harmonic parts (if there are any). We distinguish between a note from a violin and a note from a bell, for instance, because of this analysis. If the frequencies present in a sound are all integral multiples of a fundamental frequency, as they are in a violin, the sound seems more musical than when the frequencies are not so simply related, as in the note from a bell.

Traveling and standing waves

We start our discussion with the wave equation (4.1.2), which, we showed, determines the motion of a perfectly flexible string as long as it is not displaced too far from equilibrium (as long as $|\partial y / \partial x| \ll 1$).

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad c^2 = \frac{T}{\epsilon} \quad (4.3.1)$$

The wave equation corresponds to a number of statements concerning the motion of a string. We saw in the last section that it implies that the wave motion travels with its shape unchanged, at a velocity c , independent of this shape. Since the derivative $\partial^2 y / \partial x^2$ is proportional to the curvature of the shape of the string at a given instant, Eq. (4.3.1) states that the *acceleration*

of any portion of the string is *directly proportional to the curvature* of that portion. If the curvature is downward, the acceleration is downward, and vice versa; and the greater the curvature, the faster the velocity changes.

If the string is infinite in extent, it can carry waves which travel exclusively in one direction. In that case, as was pointed out at the beginning of this chapter, if the time dependence of the wave is to be sinusoidal, its space dependence must also be sinusoidal. All simple-harmonic waves traveling in the positive x direction must have the form

$$y(x,t) = A \cos \left[\frac{\omega}{c} (x - ct) - \Phi \right]$$

or

$$y(x,t) = C \exp \left[\frac{i\omega}{c} (x - ct) \right] \quad (4.3.2)$$

if $C = Ae^{-i\Phi}$ and if physical meaning is attached only to the real part of the second expression. For a simple-harmonic wave in the negative x direction we substitute $-(x + ct)$ for $(x - ct)$ in these expressions. Incidentally, the reason we have chosen the time factor to be $e^{-i\omega t}$ rather than $e^{i\omega t}$ is that then the sign of the x part of the exponent, $e^{\pm i\omega x/c}$, indicates the direction of the wave.

For the wave of Eq. (4.3.2), the energy and momentum densities are [Eqs. (4.1.9) and (4.1.12)]

$$\text{Kinetic energy density} = U = \frac{1}{2} \epsilon \omega^2 A^2 \sin^2 \left[\frac{\omega}{c} (x - ct) - \Phi \right]$$

$$\text{Potential energy density} = V = \frac{1}{2} T \left(\frac{\omega}{c} \right)^2 A^2 \sin^2 \left[\frac{\omega}{c} (x - ct) - \Phi \right]$$

$$\text{Total energy density} = W_{tt} = \epsilon \omega^2 A^2 \sin^2 \left[\frac{\omega}{c} (x - ct) - \Phi \right] = H \quad (4.3.3)$$

$$\text{Energy flux} = W_{tx} = cH$$

$$\text{Longitudinal momentum density} = W_{xt} = \frac{H}{c}$$

$$\text{Longitudinal stress} = W_{xx} = \frac{H}{c^2}$$

The energy density is greatest where the string's slope and transverse velocity are greatest, each packet of energy spaced a half wavelength from its neighbor, each traveling with a velocity c . Consequently, the energy flux W_{tx} is equal to c times the energy density W_{tt} . The wavelength of these waves is, of

course, the distance between one wave peak and the next, a distance such that an increase of x by λ will increase $(\omega/c)(x - ct)$ by 2π , so that $(\omega/c)\lambda = 2\pi$, or $\lambda = 2\pi c/\omega$.

We could reach the same conclusions by asking what sort of shape the string will have when it vibrates with simple-harmonic motion, i.e., when its time dependence is through the factor $e^{-i\omega t}$. Setting $y(x, t) = Y(x)e^{-i\omega t}$ into the equation of motion (4.1.3) or (4.1.8), we obtain a familiar equation for $Y(x)$.

$$\frac{d^2 Y}{dx^2} + \left(\frac{\omega}{c}\right)^2 Y = 0 \quad c^2 = \frac{T}{\epsilon} \quad (4.3.4)$$

which is identical with Eq. (1.2.1). This is the equation for simple-harmonic dependence on x , with "angular frequency" $k = \omega/c$ and "period" $\lambda = 2\pi/k = 2\pi c/\omega$. The quantity k is called the *wavenumber* of the wave; its dimensions are inverse length. The quantity λ is the wavelength of the wave, the distance from crest to crest of a sinusoidal wave traveling in one direction.

The general solution of Eq. (4.3.4) can be written

$$Y(x) = C_+ e^{i\omega x/c} + C_- e^{-i\omega x/c}$$

so that

$$\begin{aligned} y(x, t) &= C_+ e^{i\omega(x-ct)/c} + C_- e^{-i\omega(x+ct)/c} \\ &= A_+ \cos \left[\frac{\omega}{c}(x - ct) - \Phi_+ \right] + A_- \cos \left[\frac{\omega}{c}(x + ct) + \Phi_- \right] \end{aligned} \quad (4.3.5)$$

representing two waves, of the same frequency and wavelength, traveling in opposite directions along the string. Since the wave equation is linear, neither wave has any effect on the other.

This mutual independence of the waves extends to expressions for their energy-momentum-stress terms, such as the total energy,

$$H = U + V = \omega^2(\theta_+^2 + 2\theta_+\theta_- + \theta_-^2)$$

where

$$\theta_+ = A_+ \sin \left[\frac{\omega}{c}(x - ct) - \Phi_+ \right] \quad \theta_- = A_- \sin \left[\frac{\omega}{c}(x + ct) + \Phi_- \right]$$

The mean value of the square terms is $\langle \theta_+^2 \rangle = \frac{1}{2}A_+^2$ and $\langle \theta_-^2 \rangle = \frac{1}{2}A_-^2$, neither of which is zero. But the cross terms can be written

$$\theta_+\theta_- = \frac{1}{2}A_+A_- \left[\cos(2\omega t + \Phi_+ + \Phi_-) - \cos\left(2\frac{\omega}{c}x - \Phi_+ + \Phi_-\right) \right]$$

When averaged over space and time, the average is zero. The energy flux has no cross term;

$$W_{tx} = -T \frac{\partial y}{\partial t} \frac{\partial y}{\partial x} = \epsilon c \omega^2 (\theta_+^2 - \theta_-^2)$$

so that even the instantaneous values of the flux are simply the differences between the two individual fluxes. Thus the average values of the stress-energy tensor are

$$\begin{aligned} H &= \langle W_{tt} \rangle = \frac{1}{2} \epsilon \omega^2 (A_+^2 + A_-^2) = \frac{1}{c^2} \langle W_{xx} \rangle \\ Y &= \langle W_{tx} \rangle = \frac{1}{2} \epsilon \omega^2 c (A_+^2 - A_-^2) = \frac{1}{c} \langle W_{xt} \rangle \end{aligned} \quad (4.3.6)$$

The energy and stress terms are the sum of the terms arising from each wave. The energy and momentum fluxes are the difference of the terms, since the two waves are flowing in opposite directions.

If the amplitudes of the two simple-harmonic waves are equal, there is no net flow of energy or momentum, and the combination is called a *standing wave*.

$$\begin{aligned} y(x, t) &= C_+ e^{i\omega(x-ct)/c} + C_- e^{-i\omega(x+ct)/c} \\ &= 2A \cos\left(\frac{\omega}{c}x + \frac{1}{2}\Phi_+ - \frac{1}{2}\Phi_-\right) \exp(-i\omega t + \frac{1}{2}\Phi_+ + \frac{1}{2}\Phi_-) \quad (\text{real part}) \\ &= 2A \cos\left(\frac{\omega}{c}x + \frac{1}{2}\Phi_+ - \frac{1}{2}\Phi_-\right) \cos(\omega t - \frac{1}{2}\Phi_+ - \frac{1}{2}\Phi_-) \end{aligned} \quad (4.3.7)$$

where $C_+ = Ae^{i\Phi_+}$ and $C_- = Ae^{i\Phi_-}$ have the same amplitude but different phases. In this case the shape of the wave does not move along the string; it simply oscillates in amplitude with simple-harmonic motion. At points where $\cos[(\omega/c)x + \frac{1}{2}\Phi_+ - \frac{1}{2}\Phi_-] = 0$, the two traveling waves always cancel each other and the string never moves. These points are called the *nodal points* of the wave motion. In the case that we are considering, where the density and tension are uniform, the nodal points are equally spaced along the string a distance $c/2\nu$ apart, two for each wavelength. Halfway between each pair of nodal points is the part of the string having the largest amplitude of motion, where the two traveling waves always add their effects. This portion of the wave is called a *loop*, or *antinode*.

We should ask how a standing wave gets established and is maintained, for if there is no motion at each node, there can be no flow of energy from one loop to its neighbors. The answer is that a standing wave is a steady-state situation. During the transient state, when energy is being distributed along the string, the nodes are not perfect (that is, y is not exactly zero there) and energy does pass from one loop to the next. Also, even for the steady-state situation, the nodes are only perfect when there is no friction. With zero friction, once a loop has acquired its energy, it can oscillate forever. If friction is present, the “nodes” are simply places of minimal (but not zero) amplitude of vibration; some energy flows from loop to loop.

Normal modes

So far, we have neglected boundary conditions. If we require that $y = 0$ when $x = 0$, the general form of (4.3.5) can no longer be used; the number of possible harmonic motions is limited. The expression for y that must be used is the standing-wave form (4.3.7) with the angles Φ so chosen that *a nodal point coincides with the point of support* $x = 0$:

$$y = A \sin \left(\frac{2\pi\nu}{c} x \right) \cos (2\pi\nu t - \Phi) \quad (4.3.8)$$

This agrees with the discussion in the previous section. For the simple boundary condition that we have used, the reflected wave has the same amplitude as the incident wave; and when the incident one is sinusoidal, the result is a set of standing waves. Any frequency is allowed, however.

When the second boundary condition $y = 0$ at $x = l$ is added, the number of possible simple-harmonic motions is still more severely limited. For now, of all the possible standing waves indicated in (4.3.8), *only those which have a nodal point at $x = l$ can be used*. Since the distance between nodal points depends on the frequency, the string fixed at both ends cannot vibrate with simple-harmonic motion of any frequency; only a discrete set of frequencies is allowed, the set that makes $\sin [(2\pi\nu/c)l]$ zero. The distance between nodal points must be l , or it must be $l/2$ or $l/3$, etc. The allowed frequencies are therefore $c/2l$, $2c/2l$, $3c/2l$, etc., and the different allowed simple-harmonic motions are all given by the expression

$$y = A_n \sin \left(\frac{\pi n x}{l} \right) \cos \left(\frac{\pi n c}{l} t - \Phi_n \right) \quad n = 1, 2, 3, 4, \dots \quad (4.3.9)$$

$$v_n = \frac{nc}{2l} = \frac{n}{2l} \sqrt{\frac{T}{\epsilon}}$$

The lowest allowed frequency $v_1 = c/2l$ is called the *fundamental frequency* of vibration of the string. It is the frequency of the general periodic motion of the string, as we showed in the last section. The higher frequencies are called *overtones*, the first overtone being v_2 , the second v_3 , and so on.

The equation for the allowed frequencies given in Eq. (4.3.9) expresses an extremely important property of the uniform flexible string stretched between rigid supports. It states that the frequencies of all the overtones of such a string are *integral multiples of the fundamental frequency*. Overtones bearing this simple relation to the fundamental are called *harmonics*, the fundamental frequency being called the first harmonic, the first overtone (twice the fundamental) being the second harmonic, and so on.

Very few vibrating systems have harmonic overtones, but these few are the bases of nearly all musical instruments. For when the overtones are harmonic, the sound seems particularly satisfying, or musical, to the ear.

Fourier series

To recapitulate: The string has an infinite number of possible frequencies of vibration; and if the supports are rigid, these frequencies have a particularly simple interrelation. If such a string is started in just the proper manner, it will vibrate with just one of these frequencies, but its general motion will be a combination of all of them:

$$y = A_1 \sin \frac{\pi x}{l} \cos \left(\frac{\pi c}{l} t - \Phi_1 \right) + A_2 \sin \frac{2\pi x}{l} \cos \left(\frac{2\pi c}{l} t - \Phi_2 \right) + \cdots$$

or symbolically,

$$\begin{aligned} y &= \sum_{n=1}^{\infty} A_n \sin \frac{\pi n x}{l} \cos \left(\frac{\pi n c}{l} t - \Phi_n \right) \\ &= \sum_{n=1}^{\infty} \sin \frac{\pi n x}{l} \left(B_n \cos \frac{\pi n c t}{l} + C_n \sin \frac{\pi n c t}{l} \right) \end{aligned} \quad (4.3.10)$$

where the symbol Σ indicates the summation over the number n , going from $n = 1$ to $n = \infty$. The value of A_n is called the *amplitude of the n th harmonic*.

Equation (4.3.10) is just another way of writing Eq. (4.2.7). The present form, however, shows clearly why all motion of the string must be periodic in character. Since all the overtones are harmonic, by the time the fundamental has finished one cycle, the second harmonic has finished just two cycles, the third harmonic just three cycles, and so on, so that during the second cycle of the fundamental the motion is an exact repetition of the first cycle. This is, of course, what we mean by periodic motion.

Equation (4.3.10) is in many ways more useful for writing the dependence of y on t and x than is Eq. (4.2.7). For it gives us a means of finding the relative intensities of the different harmonics of the sound given out by the string (corresponding to the analysis that the ear makes of the sound) and thus gives us a method of correlating the motion of the string with the tone quality of the resulting sound. We shall have to wait until farther along in the book to discuss the quantitative relations between the vibrations of bodies and the intensity of the resulting sounds, but it is obvious that the intensity of the n th harmonic in the sound depends on the value of the amplitude A_n . Once the values of all the A_n 's are determined, the future motion of the string and the quality of the sound which it will emit will both be determined.

The energy of vibration of this string is the integral of the energy density,

$$\begin{aligned} W_{tt} = H &= \frac{1}{2} \epsilon \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} T \left(\frac{\partial y}{\partial x} \right)^2 \\ &= \sum_{n=1}^{\infty} \left[\frac{1}{2} \epsilon \left(U_n^2 + \sum_{m \neq n} U_n U_m \right) + \frac{1}{2} T \left(S_n^2 + \sum_{m \neq n} S_n S_m \right) \right] \end{aligned}$$

over the length of the string, where

$$U_n = -\frac{\pi n c}{l} A_n \sin \frac{\pi n x}{l} \sin \left(\frac{\pi n c t}{l} - \Phi_n \right)$$

$$S_n = \frac{\pi n}{l} A_n \cos \frac{\pi n x}{l} \cos \left(\frac{\pi n c t}{l} - \Phi_n \right)$$

In the double sum which is the square of the derivative, we have written the squared terms ($m = n$) separately and left the other terms ($m \neq n$) together. The reason for this is that the terms for which $m \neq n$ all disappear when integrated over x [Eq. (1.3.9)],

$$\int_0^l \sin \frac{\pi n x}{l} \sin \frac{\pi m x}{l} dx = \begin{cases} \frac{1}{2}l & n = m \\ 0 & n \neq m \end{cases} \quad (4.3.11)$$

$$\int_0^l \cos \frac{\pi n x}{l} \cos \frac{\pi m x}{l} dx = \begin{cases} \frac{1}{2}l & n = m \\ 0 & n \neq m \end{cases}$$

leaving only the sum of the squared terms to constitute the energy.

Thus the total energy of vibration of a string oscillating transversally between two rigid supports a distance l apart is

$$E = \int_0^l H dx$$

$$= \sum_{n=1}^{\infty} \frac{1}{4} \epsilon \left(\frac{\pi c}{l} \right)^2 n^2 A_n^2 l \sin^2 \left(\frac{\pi n c t}{l} - \Phi_n \right) + \sum_{n=1}^{\infty} \frac{1}{4} T \left(\frac{\pi}{l} \right)^2 n^2 A_n^2 l \cos^2 \left(\frac{\pi n c t}{l} - \Phi_n \right)$$

$$= \frac{\pi^2 T}{4l} \sum_{n=1}^{\infty} n^2 A_n^2 \quad (4.3.12)$$

since

$$c^2 = \frac{T}{\epsilon}$$

There are no cross terms in this expression; each mode of oscillation of the string contributes its separate part to the total energy, without any interaction with other modes. We remind ourselves that these formulas represent the string's motion only when $|\partial y / \partial x| < \sum (\pi n / l) A_n \ll 1$ [which, incidentally, is also sufficient to ensure that series (4.3.10) and (4.3.12) converge].

Determining the series coefficients

The A_n 's and Φ_n 's, or the B_n 's and C_n 's, are an infinite number of arbitrary constants whose values are fixed by the initial conditions, corresponding to the infinite number of points along the string whose positions and velocities must all be specified at $t = 0$. Our analysis will not be complete until we devise a method for determining their values when the initial shape and velocity shape of the string are given.

The initial conditions must satisfy the equations, obtained from (4.3.10) by setting $t = 0$,

$$\begin{aligned} y(x,0) = y_0(x) &= \sum_{n=1}^{\infty} B_n \sin \frac{\pi n x}{l} \\ \left(\frac{\partial y}{\partial t} \right)_{t=0} = v_0(x) &= \frac{\pi c}{l} \sum_{n=1}^{\infty} n C_n \sin \frac{\pi n x}{l} \end{aligned} \quad (4.3.13)$$

Series like the right-hand sides of these equations are called *Fourier series* [Eq. (1.3.8)].

The initial shape and velocity shape of the string, the functions $y_0(x)$ and $v_0(x)$, are any continuous functions of x which go to zero at $x = 0$ and $x = l$ and are finite in value and slope in between. It is not immediately apparent that the Fourier series on the right-hand sides of the equations for y_0 and v_0 can represent any function of this sort. A proof is possible, however; it is given in any text on Fourier series. As physicists, of course, we know that the motion of the string is uniquely determined by its initial configuration and velocity, and since Eq. (4.3.10) represents the most general solution for the free vibration of a string of length l stretched between rigid supports, we perhaps are justified in assuming that the series can be made to fit any physically possible shape, as closely as one pleases, by adjusting the coefficients B_n , C_n appropriately. What is needed is a technique to determine their appropriate values.

The desired trick is embodied in Eqs. (4.3.11). To show this, we multiply both sides of the equations for y_0 and v_0 by $\sin(\pi m x/l)$ and integrate over x from 0 to l . The integrals on the right-hand side are all zero, except for the one for $n = m$, which is $(l/2)B_m$ or $(l/2)C_m$, respectively. For instance,

$$\begin{aligned} \int_0^l \sin \left(\frac{\pi m x}{l} \right) y_0(x) dx &= \sum_{n=1}^{\infty} B_n \int_0^l \sin \frac{\pi n x}{l} \sin \frac{\pi m x}{l} dx \\ &= \frac{l}{2} B_m \end{aligned}$$

so that

$$B_m = \frac{2}{l} \int_0^l \sin \left(\frac{\pi m x}{l} \right) y_0(x) dx$$

and similarly,

$$C_m = \frac{2}{\pi c m} \int_0^l \sin \left(\frac{\pi m x}{l} \right) v_0(x) dx \quad m = 1, 2, 3, 4, \dots \quad (4.3.14)$$

These equations provide a means of determining the values of the B_n 's and C_n 's in terms of the initial conditions.

A reference to Eq. (3.2.26) indicates that each of the standing waves in the series (4.3.13) is a normal mode of oscillation. When the string is vibrating with frequency $\omega_v = \pi v c/l$ (v an integer), its shape is given by

$y = A_v \sin(\pi v x/l)$; thus $\sin(\pi v x/l)$ is equivalent to the components $E_m(v)$ of Eq. (3.2.26), with the continuous variable x taking the place of the discrete index m (or n). The orthogonality equations following Eq. (3.2.23), which state that the eigenvectors $E(v)$ are mutually perpendicular (orthogonal), correspond to the orthogonality relations (4.3.11) for the continuous variable x . Series (4.3.10) is analogous to the series

$$x_m = \sum_{v=1}^N E_m(v) B_v e^{-i\omega_v t}$$

for the discrete set of N oscillators, and Eqs. (4.3.14), determining the normal-mode amplitudes B_m , C_m , are equivalent to the equations $B_v = \sum_{m=1}^N C_m E_m(v)$, which determine the normal-mode amplitudes B_v for the discrete set of oscillators. The string is equivalent to a system of an infinite number of coupled oscillators, with the particular oscillator specified by the continuous variable x rather than the discrete index m . The energy of the v th mode is, by Eq. (4.3.12), equal to $\pi^2 T v^2 A_v^2 / 4l$, independent of the amplitudes of the other modes.

Plucked string; struck string

A few examples will indicate how the method works. For instance, if we pull the center of the string out h cm and then let it go at $t = 0$, all the C_m 's will be zero, and

$$\begin{aligned} B_m &= \frac{2}{l} \left[\int_0^{l/2} \frac{2hx}{l} \sin \frac{\pi mx}{l} dx + \int_{l/2}^l \frac{2h}{l} (l-x) \sin \frac{\pi mx}{l} dx \right] \\ &= \frac{8h}{\pi^2 m^2} \sin \frac{\pi m}{2} \\ &= \begin{cases} 0 & \text{if } m \text{ is an even integer} \\ (-1)^{(m-1)/2} \frac{8h}{\pi^2 m^2} & \text{if } m \text{ is an odd integer} \end{cases} \end{aligned}$$

Therefore

$$y = \frac{8h}{\pi^2} \left[\sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{1}{9} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} \cdots \right] \quad (4.3.15)$$

Computing this series for y as a function of x and t gives the same values for the shape of the string at successive instants as are shown in the first sequence of Fig. 4.11. Figure 4.13 shows how the correct form is approached closer and closer, the more terms of the series are used.

At first sight the foregoing series appears to be simply a more awkward way of finding the shape of the string than the method used in the previous section. However, the series can tell us more about the string's motion than the results of the last section can. It tells us, for instance, that the second,

fourth, sixth, etc., harmonics will be absent from the sound given out by the string, for they are not present in the motion. It tells us that if, for example, the intensity of the sound emitted is proportional to the square of the amplitude of motion of the string, the fundamental frequency will be 81 times more intense than the third harmonic, 625 times more intense than the fifth harmonic, etc.

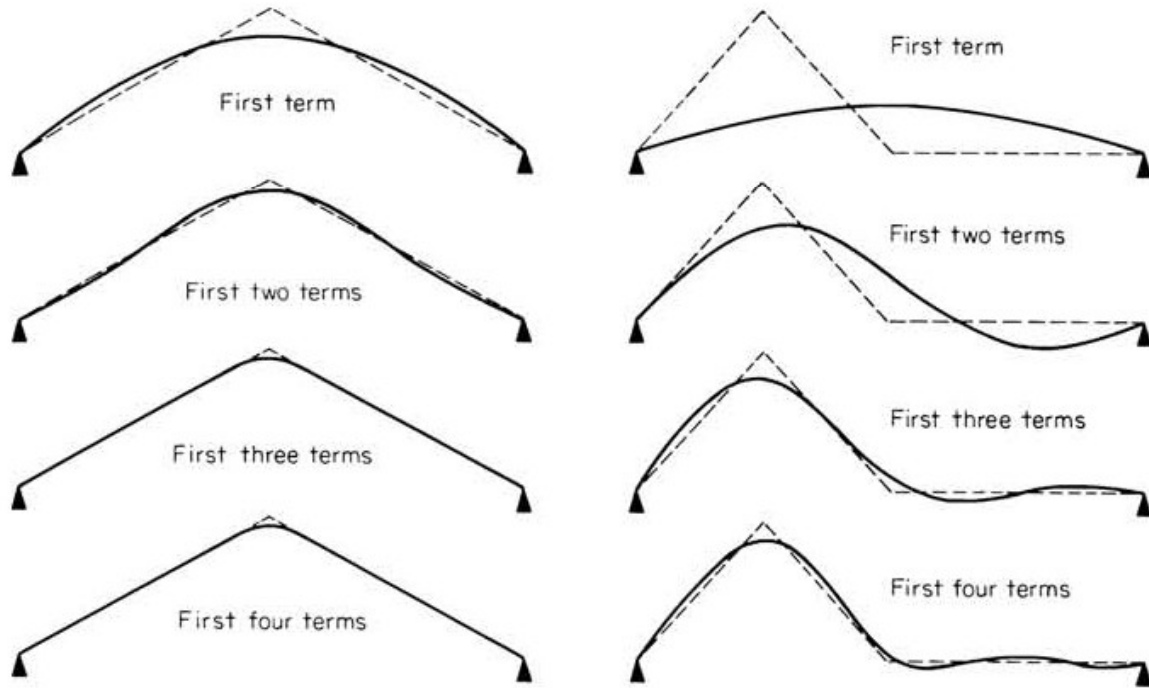


FIGURE 4.13
Fourier series representations of the initial form of the string given in Eq. (4.3.15) and the initial-velocity form given in Eq. (4.3.16). Successive solid curves show the effect of adding successive terms of the series; dotted curves show the actual form, given by the entire series.

The absent harmonics correspond to standing waves that have a nodal point at the center, the point pulled aside. This is an example of the general rule (which can be proved by computing the required integrals for B_m) that in the motion of any plucked string all those harmonics are absent which have a node at the point pulled aside.

If the string is struck, so that $y_0 = 0$ and

$$v_0(x) = \begin{cases} \frac{4\mu x}{l} & 0 < x < \frac{l}{4} \\ \frac{4\mu}{l} \left(\frac{l}{2} - x \right) & \frac{l}{4} < x < \frac{l}{2} \\ 0 & \frac{l}{2} < x < l \end{cases}$$

then all the B_m 's are zero, and

$$\begin{aligned}\frac{\pi cm}{l} C_m &= \frac{2}{l} \left(\int_0^{l/4} \frac{4\mu x}{l} \sin \frac{\pi mx}{l} dx + \int_{l/4}^{l/2} \frac{4\mu}{l} \left(\frac{l}{2} - x \right) \sin \frac{\pi mx}{l} dx \right) \\ &= \frac{8\mu}{\pi^2 m^2} \left(2 \sin \frac{\pi m}{4} - \sin \frac{\pi m}{2} \right)\end{aligned}$$

so that

$$\begin{aligned}y = \frac{8\mu l}{\pi^3 c} \left[(\sqrt{2} - 1) \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} + \frac{1}{4} \sin \frac{2\pi x}{l} \sin \frac{2\pi ct}{l} \right. \\ \left. + \frac{1 + \sqrt{2}}{27} \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l} \right. \\ \left. - \frac{1 + \sqrt{2}}{125} \sin \frac{5\pi x}{l} \sin \frac{5\pi ct}{l} \cdots \right] \quad (4.3.16)\end{aligned}$$

We note that the fourth, eighth, etc., harmonics, those having nodes at $x = l/4$, are absent in this case.

The energy of these free vibrations may be obtained from Eq. (4.3.12). For the plucked string, it is

$$E = \frac{16h^2 T}{\pi^2 l} \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots \right) = \frac{2h^2 T}{l}$$

since the sum in parentheses equals $\pi^2/8$. This is, of course, the same as the potential energy of the plucked string just before it has been released, $V_0 = \frac{1}{2} T \int (\partial y / \partial x)^2 dx = \frac{1}{2} T (2h/l)^2 l$. It is similarly possible to show that the energy of vibration of the struck string is equal to its initial kinetic energy,

$$E = \frac{1}{2} \epsilon \int_0^l \left(\frac{\partial y}{\partial t} \right)^2 dx = \epsilon \int_0^{l/4} \left(\frac{4\mu x}{l} \right)^2 dx = \frac{\epsilon l \mu^2}{12}$$

one-twelfth of the kinetic energy if all the mass of the string were concentrated at the point of greatest motion, $x = \frac{1}{4}l$. The total energy E remains independent of time because we have here neglected frictional forces.

Effect of friction

In the foregoing analysis we have neglected friction, although it is present in every vibrating string. To complete our discussion we should show, as with the simple oscillator, that the effect of friction is to damp out the free vibrations and to change slightly the allowed frequencies. To show that this is so is not difficult by the use of operational calculus, although it would be difficult by any other method.

The difficulty lies with the nature of the frictional term. The resistive force per unit length opposing the string's motion is due to the medium surrounding the string, the medium gaining the energy that the string loses. Part of the energy goes into heating the medium, the amount depending on the *viscosity* of the medium; and part goes into outgoing sound waves in the medium, the amount depending on the *radiation resistance* of the medium. The medium also adds an effective mass per unit length, which may not be negligible if the medium is a liquid. The important point, however, the one that is responsible for our difficulties, is that the effective resistance due to the medium (and also its added effective mass) *depends on the frequency* of the string's motion.

The equation of motion for the string when friction is included is

$$\epsilon \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - R(\omega) \frac{\partial y}{\partial t} \quad (4.3.17)$$

where R is the effective frictional resistance per unit length of string. To find the "normal modes" of the string involves a sort of circular process, since we cannot solve for the natural frequencies until we know the value of R , and we cannot know R unless we know the frequency. However, we can determine R as a function of the variable ω and note that, even if ω turns out to be a complex quantity, R is still computable. Usually, we can assume that the added mass of the medium is negligible compared with ϵ . For further discussion, see Eqs. (7.3.6) and (10.1.10).

A solution of Eq. (4.3.17) is the exponential $e^{i\beta x - i\omega t}$, with β and ω adjusted so that $\epsilon\omega^2 + i\omega R(\omega) = T\beta^2$. If the string is driven by a simple-harmonic force of frequency $\omega/2\pi$, then ω is real and known, and thus R is known, and the constant β can be computed (it turns out to be complex). For free vibrations of a string between rigid supports, however, β must be real and ω will then be complex. We set

$$y(x, t) = \sin\left(\frac{\pi n x}{l}\right) e^{-i\alpha_n t}$$

and solve the implicit equation $\alpha^2 + 2i\alpha k(\alpha) - (\pi n c/l)^2 = 0$ for α , where $k(\alpha) = (1/2\epsilon)R(\alpha)$. If $k(\alpha)$ is a slowly varying function of α , or if it is very small compared with $\pi n c/l$, we can solve this equation by guessing a value of α , using this to find a value of $k(\alpha)$, solving the equation to find a better value of α , and so on. In any case the value of α we finally find is related to n , c , l , etc., by the relationship

$$\alpha_n = -ik_n + \omega_n \quad (4.3.18)$$

where

$$\omega_n^2 = \left(\frac{\pi n c}{l}\right)^2 - k_n^2$$

and where

$$k_n = k(\alpha_n) = \frac{1}{2\epsilon} R(\alpha_n)$$

Therefore the possible motions of free vibration are given by the series

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{l}\right) e^{-k_n t} (B_n \cos \omega_n t + C_n \sin \omega_n t) \quad (4.3.19)$$

which differs from Eq. (4.3.10) in the presence of the exponential damping factor, and in the fact that the successive frequencies $\omega_n/2\pi$ are not integral multiples of the fundamental. The series coefficients B_n and C_n can be determined from the initial conditions by use of Eqs. (4.3.11), as were Eqs. (4.3.14) for the frictionless case.

4.4 FORCED MOTION

The forced motion of a flexible string exhibits some of the properties exhibited by the simple oscillator; it also displays new properties, arising from the system's spatial extension. The applied transverse force may be viewed as a combination of simple-harmonic components, in which case we can analyze the force into its frequency components by means of the Fourier transform, or else we can express it as a sequence of pulses and represent the motion as a sequence of responses to the impulses, as was done for the simple oscillator in Eq. (2.3.9).

Likewise, the transverse driving force can be concentrated at a single point x , or it can be spread out over a length of the string. Again we can analyze the spread-out force in terms of a Fourier integral (or series) in x , or we can consider the distribution to be a sum of point forces, with effects to be added. This latter method, the use of an analog of Eq. (2.3.9), will turn out to have a quite wide range of applicability.

String driven from one end

First let us examine the very simple case of a very long string, driven by the transverse motion of its support at the end $x = 0$. This support is supposed to maintain the longitudinal tension in the string at T dynes, but it can be moved transversally, as shown in Fig. 4.5, by an amount $y_0(t)$. This motion will generate a transverse wave along the string, and if the string is long enough, the waves reflected from the far end will not have time to return, so that, for a while, the wave motion will be all in the positive x direction. In this case the displacement of the string which fits the boundary condition $y(0, t) = y_0(t)$ is $y(x, t) = y_0[t - (x/c)]$, a solution of the wave equation (4.1.2), valid as long as $(1/c)y'_0[t - (x/c)]$ is everywhere small. The transverse velocity is $y'_0[t - (x/c)]$, where the prime denotes differentiation with respect to the argument.

Reference to Fig. 4.5 shows that the transverse force necessary to displace the end of the string is $-T(\partial y/\partial x)$ at $x = 0$, which is

$$\left(\frac{T}{c}\right)y'_0(t) = \frac{T}{c} \frac{\partial y}{\partial t} \quad \text{at} \quad x = 0$$

for waves entirely in the positive direction. There is thus a constant ratio between the transverse force $F_0(t)$ necessary to move the end of the string and the transverse velocity of the end,

$$\frac{F_0(t)}{y'_0(t)} = \frac{T}{c} = \epsilon c = \sqrt{\epsilon T} \quad (4.4.1)$$

for waves in the positive direction. In other words, if the end of the string is driven transversally by a force $F_0(t)$, then the wave produced in the string has the shape $y_0[t - (x/c)]$, where $(d/dt)y_0(t) = (c/T)F_0(t)$, or

$$\epsilon c y_0(t) = \int_0^t F_0(u) du.$$

A ratio between a force and a velocity is a mechanical impedance; in this case, being real and independent of frequency, it is a resistance. This ratio, ϵc , is called the *wave resistance* of the string; it measures the rate at which work must be done by the force to send a wave out along the string. The power transferred from driving force to string is

$$P = \frac{|F_0|^2}{\epsilon c} = \epsilon c \left| \frac{\partial y}{\partial t} \right|^2 \quad (4.4.2)$$

These simple results occur only when there is no wave present which is going in the negative x direction.

String driven along its length

Sometimes the transverse driving force is applied at points other than the end of the string. If it is applied along a length of the string, we must talk about the *force density* $f(x, t)$ in newtons per meter at point x at time t , tending to move the string sideward. Thus the element dx of string in Fig. 4.2 would have an additional force $f(x, t) dx$ acting on it, and the equation of motion (4.1.3) then becomes

$$\epsilon \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = f(x, t) \quad (4.4.3)$$

if the string has mass ϵ per unit length and is under tension T .

Let us first consider the consequent motion when the string is so long that no reflected waves return, from either end, to the region of application of the force, during the time period under consideration. If the force is concentrated on an infinitesimal length of the string, at $x = x_0$, so that

$f(x, t)$ can be represented as $F(x_0, t)\delta(x - x_0)$ (with F in newtons) in terms of the delta function of Eq. (1.3.24), we can consider the string as two separate strings, stretching in opposite directions, each ending at x_0 and each being driven transversally by half the force $F(x_0, t)$, since both strings are equivalent and both share the force. Thus, from our discussion of Eq. (4.4.1), there would be a wave going to the right, in the half-string $x > x_0$, of the form $Y\{t - [(x - x_0)/c]\}$, where $dY(t)/dt = (c/2T)F(x_0, t)$, and a wave going to the left, in the region $x < x_0$, of the form $Y\{t + [(x - x_0)/c]\}$.

Put another way, the displacement of the string acted on by a force $F(x_0, t)$ concentrated at point x_0 , in the absence of reflected waves, is given by the function

$$G(x, x_0; t) = \begin{cases} Y\left(x_0, t - \frac{x - x_0}{c}\right) & x > x_0 \\ Y\left(x_0, t + \frac{x - x_0}{c}\right) & x < x_0 \end{cases} \quad (4.4.4)$$

$$Y(x_0, z) = \frac{c}{2T} \int_0^z F(x_0, t) dt$$

To show that function $G(x, x_0; t)$ is a solution of

$$\epsilon \frac{\partial^2 G}{\partial t^2} - T \frac{\partial^2 G}{\partial x^2} = F(x_0, t) \delta(x - x_0) \quad (4.4.5)$$

we integrate both sides of this equation over x from $x_0 - \alpha$ to $x_0 + \alpha$ and then let α go to zero. The first term on the left integrates to $2\epsilon\alpha(\partial^2 G/\partial t^2) \rightarrow 0$; the second term integrates to

$$-T \left[\frac{\partial G}{\partial x} \right]_{x_0 - \alpha}^{x_0 + \alpha} = -\frac{c}{2} \left[-\frac{1}{c} F\left(x_0, t - \frac{\alpha}{c}\right) - \frac{1}{c} F\left(x_0, t + \frac{\alpha}{c}\right) \right] \rightarrow F(x_0, t)$$

and the term on the right, because of the properties of the delta function, also integrates to $F(x_0, t)$, thus proving that (4.4.4) is a solution of (4.4.5).

However, most forces are distributed along a length of string, not concentrated at a point. What is the solution of Eq. (4.4.3) if $f(x, t)$ is nonzero over the region between $x = a$ and $x = b$? Here the discussion preceding and following Eq. (2.3.9) may give us an idea. A distributed force function, nonzero between $x = a$ and $x = b$, can be represented as a sum of a whole series of point forces $f(x_0, t) \delta(x - x_0)$, distributed along the string from a to b ,

$$f(x, t) = \int_a^b f(x_0, t) \delta(x - x_0) dx_0 \quad (4.4.6)$$

according to Eq. (1.3.24). We should then expect that the solution of Eq. (4.4.3) for a distributed force would be the sum of the solutions of Eq. (4.4.5) for all the component point forces along the string. In other words, we

should expect that the solution of (4.4.3) is

$$y(x, t) = \int_a^b G(x, x_0; t) dx_0 \quad (4.4.7)$$

where G is the solution of Eq. (4.4.5) for a point force $f(x_0, t) \delta(x - x_0)$ at x_0 . We prove this by inserting (4.4.7) into Eq. (4.4.3)

$$\epsilon \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = \int_a^b \left(\epsilon \frac{\partial^2 G}{\partial t^2} - T \frac{\partial^2 G}{\partial x^2} \right) dx_0 = \int_a^b f(x_0, t) \delta(x - x_0) dx_0 = f(x, t)$$

and using (4.4.5) and (4.4.6).

Function $G(x, x_0; t)$ is called a *Green's function*. What we have shown is that a solution for a more general distributed force can be built up out of a sum of Green's functions.

An example may be desirable. Suppose that the transverse force is distributed uniformly along the length of the string from $x = 0$ to $x = b$ and that it is applied from $t = 0$ to $t = \tau$, so that

$$f(x, t) = \begin{cases} 0 & x < 0 \quad \text{or} \quad t < 0 \\ A & 0 < x < b \quad \text{and} \quad 0 < t < \tau \\ 0 & x > b \quad \text{or} \quad t > \tau \end{cases} \quad (4.4.8)$$

The point force to use in Eq. (4.4.5), to obtain the required Green's function, is thus one which is $A \delta(x - x_0)$ when $0 < t < \tau$ and which is zero for $t < 0$ and $t > \tau$. The Green's function is then

$$G(x, x_0; t) = \frac{A}{2T} \begin{cases} 0 & ct < |x - x_0| \\ ct - |x - x_0| & |x - x_0| < ct < |x - x_0| + c\tau \\ c\tau & |x - x_0| + c\tau < ct \end{cases}$$

which is zero until a time $|x - x_0|/c$, when the wave first reaches the point x from the source point x_0 , then rises linearly until the end of the wave produced by the force arrives at time $t = \tau + (|x - x_0|/c)$. This wave is pictured in the left-hand sequence of Fig. 4.14.

The solution for the force function of Eq. (4.4.8) is then obtained by integrating G over x_0 from 0 to b . The results are simple enough in principle, though rather complicated to write out because of the sequence of limits involved. If $c\tau > b$, then

$$y(x, t) = \begin{cases} 0 & x > ct + b; \quad ct < x - b \\ A(ct + b - x)^2 & ct < x < ct + b; \quad x - b < ct < x \\ A[b^2 + 2b(ct - x)] & ct - c\tau + b < x < ct; \quad x < ct < x - b + c\tau \\ A[2bc\tau - (x - ct + c\tau)^2] & ct - c\tau < x < ct - c\tau + b; \quad x - b + c\tau < ct < x + c\tau \\ A2bc\tau & ct - c\tau > x; \quad x + c\tau < ct \end{cases}$$

for the range $x > b$, and

$$y(x,t) = \begin{cases} 0 & ct < 0 \\ A(ct)^2 & 0 < ct < b-x \\ A[(ct+b-x)^2 - 2(b-x)^2] & b-x < ct < x \\ A[2b(ct+x) - b^2 - 2x^2] & x < ct < c\tau \\ A[2b(ct+x) - b^2 - 2x^2 - 2(ct-c\tau)^2] & c\tau < ct < c\tau + b-x \\ A[2bc\tau - (x-ct+c\tau)^2] & c\tau + b-x < ct < c\tau + x \\ A \cdot 2bc\tau & c\tau + x < ct \end{cases}$$

for $\frac{1}{2}b < x < b$, the shape for $x < \frac{1}{2}b$ being the mirror image of the shape for $x > \frac{1}{2}b$. This wave motion is pictured in the right-hand sequence of Fig. 4.14.

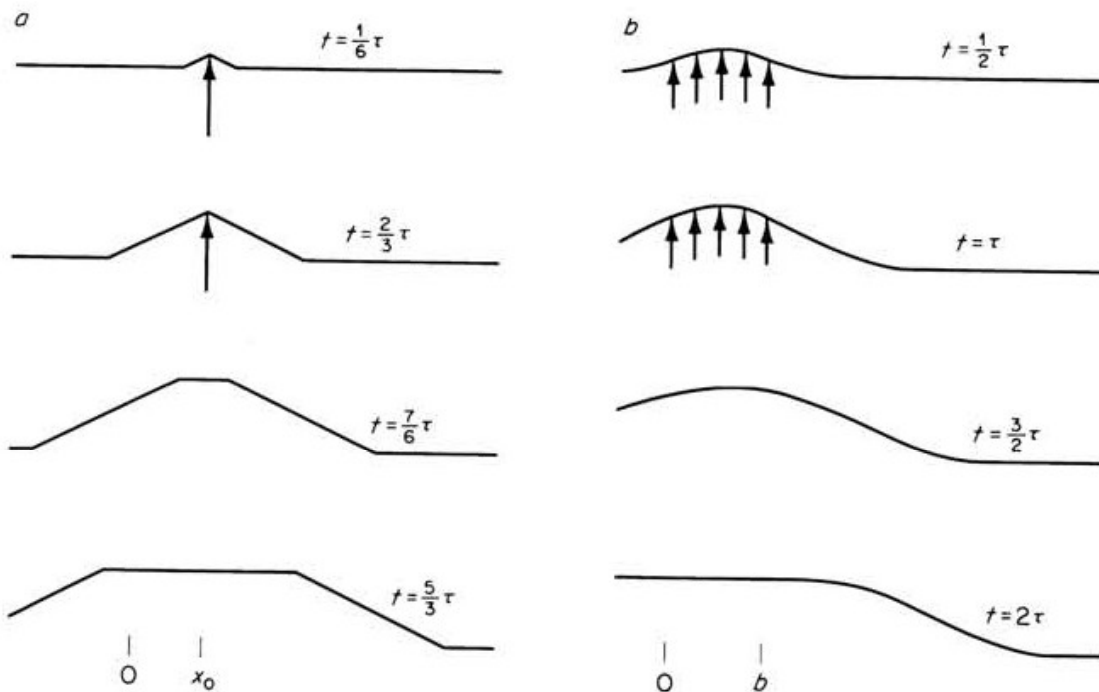


FIGURE 4.14

Infinite string driven (a) by a force concentrated at $x = x_0$ and (b) by a force distributed from 0 to b along the string.

A step-function force, with discontinuous beginning and end in both time and space, thus produces a rounded wave, made up of pieces of parabolas and straight lines, because of the double integration, once to go from f to G and again to integrate G over x_0 .

Simple-harmonic driving force

If the driving force is simple-harmonic, of frequency $\omega/2\pi$, so that $f(x,t) = F(x)e^{-i\omega t}$, we can represent the steady-state shape of the string with the same time factor, $y(x,t) = Y(x)e^{-i\omega t}$. Each point on the string has its

amplitude of motion, $Y(x)$, which does not change with time; if $Y(x)$ is real for all x , then all points on the string vibrate in phase, reaching maximum or going through zero simultaneously.

Again we start with the Green's function, the solution for a driving force concentrated at point x_0 on the string. In fact, we start with the solution for a concentrated force of magnitude equal to the tension in the string, $f(x, t) = T \delta(x - x_0) e^{-i\omega t}$, with corresponding steady-state shape $y(x, t) = g(x | x_0) e^{-i\omega t}$, where g is a solution of the Green's function equation,

$$\frac{d^2 g}{dx^2} + \left(\frac{\omega}{c}\right)^2 g = -\delta(x - x_0) \quad (4.4.9)$$

analogous to Eq. (4.4.5), but with the time dependence removed and the inhomogeneous term normalized to unit amplitude.

If the string is very long, so that no reflected waves return to complicate the motion, the wave on the positive side of x_0 must be a constant times $e^{i(\omega x/c) - i\omega t}$, corresponding to a wave of frequency $\omega/2\pi$ and wavelength $2\pi c/\omega$, traveling in the positive x direction. The wave in the region $x < x_0$ must be represented by a constant times $e^{-i(\omega x/c) - i\omega t}$, a wave in the negative x direction. Since the string must be continuous in value across $x = x_0$, the constants must be adjusted so that we can write $g(x | x_0) = A e^{i(\omega/c) |x - x_0|}$, where $|x - x_0| = x - x_0$ when $x > x_0$ and equals $x_0 - x$ when $x < x_0$. Thus g has a discontinuity in *slope* at $x = x_0$. The magnitude of this discontinuity, and therefore the value of A , is adjusted to satisfy Eq. (4.4.9). According to the discussion following Eq. (4.3.1), a distributed force corresponds to a curvature of the string; therefore a concentrated force would require an infinite curvature at the point of application—a discontinuity of slope there.

To find the magnitude of the discontinuity in slope at x_0 we proceed as before to integrate Eq. (4.4.9) over x from $x_0 - \alpha$ to $x_0 + \alpha$, and then let α go to zero.

$$\int_{x_0 - \alpha}^{x_0 + \alpha} \frac{d^2 g}{dx^2} dx + \left(\frac{\omega}{c}\right)^2 \int_{x_0 - \alpha}^{x_0 + \alpha} g dx = - \int_{x_0 - \alpha}^{x_0 + \alpha} \delta(x - x_0) dx$$

or

$$\left[\frac{dg}{dx} \right]_{x_0 - \alpha}^{x_0 + \alpha} + \left(\frac{\omega}{c}\right)^2 2\alpha g = -1$$

or

$$\lim_{\alpha \rightarrow 0} \left[\left(\frac{dg}{dx} \right)_{x_0 + \alpha} - \left(\frac{dg}{dx} \right)_{x_0 - \alpha} \right] = -1 \quad (4.4.10)$$

The discontinuity in slope at $x = x_0$, for a solution of the Green's function equation, must be minus unity.

For an infinite string, with no reflected waves, the slope of the solution $g = A e^{i(\omega/c) |x - x_0|}$ at $x_0 + \alpha$ is $i(\omega/c) A e^{i(\omega/c)\alpha}$, and that at $x_0 - \alpha$ is

$-i(\omega/c)e^{i(\omega/c)x}$; so Eq. (4.4.10) becomes $-2i(\omega/c)A = 1$, or

$$g(x | x_0) = i \frac{c}{2\omega} e^{i(\omega/c) |x-x_0|} \quad (4.4.11)$$

This is the Green's function for the infinite string (outgoing waves only). Point x is the point at which g is measured; so it can be called the *observation point*. Point x_0 is the point of application of the driving force which generates the wave; so it can be called the *source point*. We note that g is symmetric in x and x_0 ; its value is not changed if x and x_0 are interchanged. This general property of Green's functions, that their value is not changed when source and observation points are interchanged, is called the *principle of reciprocity*. We shall discuss its range of validity and its utility in wave calculations several times in this book.

By extension of Eq. (4.4.7) we should expect that a solution of the inhomogeneous equation for a general distributed force on a string,

$$\epsilon \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = F(x)e^{-i\omega t} \quad (4.4.12)$$

would, in the case when no reflected waves are present, be

$$\begin{aligned} y(x, t) &= \frac{1}{T} \int_{-\infty}^{\infty} F(x_0) g(x | x_0) dx_0 e^{-i\omega t} \\ &= \frac{i}{2\epsilon c \omega} \int_{-\infty}^{\infty} F(x_0) e^{i(\omega/c) |x-x_0|} dx_0 e^{-i\omega t} \end{aligned} \quad (4.4.13)$$

This can be proved by inserting the integral into Eq. (4.4.12) and using (4.4.9), as was done with Eq. (4.4.7).

As an example, suppose the string were acted on by a force $f(x, t) = A \cos(\pi x/a) e^{-i\omega t}$ between $x = \frac{1}{2}a$ and $x = -\frac{1}{2}a$ and $f(x, t) = 0$ for $x < -\frac{1}{2}a$ and $x > \frac{1}{2}a$. The integral solution works out to be

$$\begin{aligned} y(x, t) &= i \frac{cA}{2\omega T} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \cos\left(\frac{\pi x_0}{a}\right) e^{i(\omega/c) |x-x_0|} dx_0 \\ &= \begin{cases} \frac{iaA}{\pi\omega\epsilon c} \frac{\cos(\omega a/2c)}{1 - (\omega a/\pi c)^2} e^{i(\omega/c)x - i\omega t} & x > \frac{1}{2}a \\ \frac{iaA/\pi\omega\epsilon c}{1 - (\omega a/\pi c)^2} \left[e^{i\omega a/2c} \cos \frac{\omega x}{c} \right. \\ \quad \left. - i \frac{\omega a}{\pi c} \cos \frac{\pi x}{a} \right] e^{-i\omega t} & -\frac{1}{2}a < x < \frac{1}{2}a \\ \frac{iaA}{\pi\omega\epsilon c} \frac{\cos(\omega a/2c)}{1 - (\omega a/\pi c)^2} e^{-i(\omega/c)x - i\omega t} & x < -\frac{1}{2}a \end{cases} \end{aligned}$$

representing simple-harmonic waves going outward from the region $-\frac{1}{2}a < x < \frac{1}{2}a$, where the force is applied, with a mixed oscillation in the region of application of the force. The formulas look as though they become infinite at $\omega = \pi c/a$, but in each range of x the numerator of the function is also zero at $\omega = \pi c/a$; so $y(x, t)$ is finite for all values of ω except for $\omega = 0$. For a steady force $\omega = 0$, the region of application is pushed steadily outward with a constant velocity $aA/\pi\epsilon c$, so that the steady-state amplitude is infinite. Since the string must be infinite in length to have no reflected waves and for (4.4.11) to be valid, it is possible to have an arbitrarily large displacement of the region near $x = 0$.

Effect of wave reflections

Strings held under tension, however, have to terminate in supports which maintain the tension, and usually these supports produce reflected waves. If the driving force acts for a long enough time, these reflected waves return to the region of application of the force, and the simple outgoing-wave solutions of Eqs. (4.4.11) and (4.4.13) will not suffice. The Green's function for a wave of finite length must satisfy Eq. (4.4.9) if the point driving force is simple-harmonic. It must also satisfy the boundary conditions at the two ends of the string.

In other words, the Green's function for a string must be a solution of the homogeneous equation $(d^2g/dx^2) + (\omega/c)^2g = 0$ for all appropriate values of x , *except* at the point $x = x_0$, where it must have a discontinuity of slope of -1 . It must also satisfy the boundary conditions at the ends of the string, imposed by the nature of the supports.

If the string supports are rigid and a distance l apart, $g(x | x_0)$ must be zero at $x = 0$ and $x = l$. A solution of the homogeneous equation which is zero at $x = 0$ is $D \sin(\omega x/c)$, and one which is zero at $x = l$ is

$$E \sin [\omega(l - x)/c].$$

For g to be continuous in value (though not in slope) at $x = x_0$, we must have $D \sin(\omega x_0/c) = E \sin[\omega(l - x_0)/c]$. This can be done by setting

$$G(x | x_0) = \begin{cases} A \sin \frac{\omega(l - x_0)}{c} \sin \frac{\omega x}{c} & 0 < x < x_0 < l \\ A \sin \frac{\omega(l - x)}{c} \sin \frac{\omega x_0}{c} & 0 < x_0 < x < l \end{cases}$$

where A must be adjusted so that the discontinuity in slope at $x = x_0$ is -1 . The slope, for $x > x_0$, is $-(A\omega/c) \cos[\omega(l - x)/c] \sin(\omega x_0/c)$, and that for $x < x_0$ is $(A\omega/c) \sin[\omega(l - x_0)/c] \cos(\omega x/c)$; so the difference between them at $x = x_0$ is

$$-\left(\frac{A\omega}{c}\right) \sin \left[\frac{\omega}{c}(l - x_0) + \frac{\omega}{c}x_0 \right] = -\frac{A\omega}{c} \sin \frac{\omega l}{c}$$

which must equal -1 . Therefore the Green's function for the flexible string of length l between rigid supports, representing the steady-state response to a simple-harmonic force of amplitude T concentrated at the point x_0 , is

$$G(x | x_0) = \frac{c/\omega}{\sin(\omega l/c)} \begin{cases} \sin \frac{\omega(l-x_0)}{c} \sin \frac{\omega x}{c} & 0 < x < x_0 < l \\ \sin \frac{\omega x_0}{c} \sin \frac{\omega(l-x)}{c} & 0 < x_0 < x < l \end{cases}$$

$$\xrightarrow{(\omega/c) \rightarrow 0} \begin{cases} (l-x_0) \frac{x}{l} & x < x_0 \\ \frac{x_0}{l} (l-x) & x_0 < x \end{cases} \quad (4.4.14)$$

which is the solution of Eq. (4.4.9), satisfying the boundary conditions $g = 0$ at $x = 0$ and $x = l$.

A comparison of this Green's function with that of Eq. (4.4.11), for the infinite string, shows that the presence of wave reflections from the rigid-end supports introduces a factor $\sin(\omega l/c)$ in the denominator of the amplitude factor. Since this term goes to zero whenever $\omega l/c = n\pi$ ($n = 0, 1, 2, \dots$) or whenever $n\lambda \equiv n(2\pi c/\omega) = 2l$, the Green's function (and therefore any solution built up out of Green's functions for different x_0 's) exhibits a sequence of resonance peaks at the natural frequencies of free vibration $\nu_n = n(c/2l)$ of the string, as given in Eq. (4.3.9). Just as with the simple oscillator, whenever the driving force has frequency equal to a frequency of free vibration of the system, resonance occurs, and in the absence of frictional forces, the amplitude of motion becomes infinite. Since the string has an infinity of such frequencies, there are an infinite number of different resonant frequencies, the fundamental ($n = 1$) and all the harmonics ($n > 1$). At resonance the reflections from the string ends return just in time to reinforce the driving force, and the amplitude builds up without limit, unless friction and nonlinear effects are taken into account.

The factors $\sin(\omega x_0/c)$ and $\sin[\omega(l-x_0)/c]$ express the dependence of the solution on the location of the source point. If the distance x_0 of this point of application of the force from the left-hand end of the string is an integral multiple of half wavelengths $\pi c/\omega$, then a node in the standing wave, set up to the left of the source point, occurs exactly at x_0 . In this case the portion of the string to the right of the force, which is not in resonance, is motionless. Conversely, if $l - x_0$ is an integral number of half wavelengths, the region between 0 and x_0 is at rest, as is shown in Fig. 4.15 for $\gamma = 1.5$.

In either case the point of application of the force does not move. This is strictly true only when friction is zero; for the case of small friction, the motion of the point of application would be small compared with the

amplitude of the resonating region, and it would take a long time for the steady-state condition to be built up. In the cases where the whole string resonates, on the other hand, the source point x_0 is not motionless, but oscillates with the string, with infinite amplitude (if there is no friction).

The one exception to both rules is when there is an integral number of half wavelengths between 0 and x_0 and also between x_0 and l (for example,

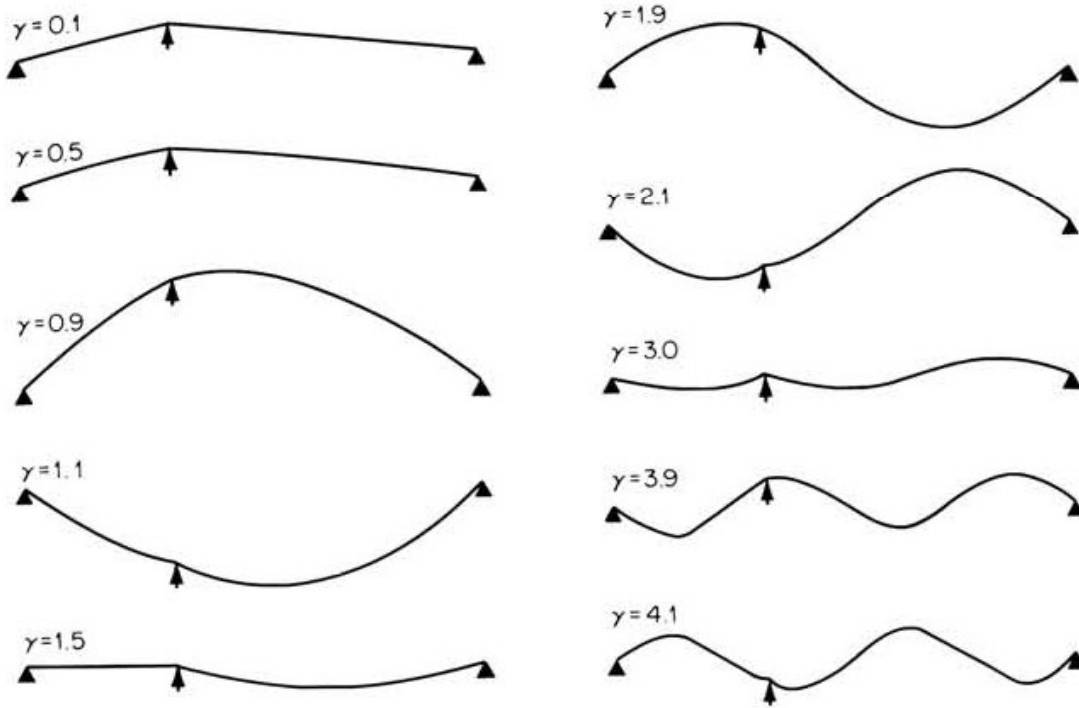


FIGURE 4.15

Forced vibration of a string driven at the point $x_0 = \frac{1}{3}l$. Curves give amplitude of vibration of the string for various values of $\gamma = \omega l/\pi c = 2vl/c = 2l/\lambda$.

when $nx_0 = ml$, n and m integers, and when $n\pi c/\omega = l$). In this case the limiting expression is

$$G(x | x_0) = -\frac{l}{\pi n} \sin\left(\frac{\pi n}{l} |x - x_0|\right) \quad \text{when } l = \frac{\pi n c}{\omega} \quad x_0 = \frac{\pi m c}{\omega}$$

where again the source point is at rest (see Fig. 4.15 for $\gamma = 3$). Figure 4.15 shows the shapes of function G , for $x_0 = \frac{1}{3}l$, for different driving frequencies $\omega/2\pi = \gamma c/2l$. The discontinuity in slope at x_0 is apparent at all frequencies.

Examination of Eq. (4.4.14) indicates that this Green's function, like that of Eq. (4.4.11), is symmetric in x and x_0 and is thus subject to the principle of reciprocity.

As before, the steady-state response, to a distributed force $F(x)e^{-i\omega t}$, is the combination

$$y(x, t) = \frac{1}{T} \int_0^l F(x_0) G(x | x_0) dx_0 e^{-i\omega t} \quad (4.4.15)$$

This is the solution of Eq. (4.4.12), which goes to zero at $x = 0$ and $x = l$.

As an example, we compute the motion of the string when the driving force is distributed uniformly between $x = a$ and $x = b$.

$$F(x) = \begin{cases} 0 & 0 < x < a \\ F_0 & a < x < b \\ 0 & b < x < l \end{cases} \quad 0 < a < b < l$$

Inserting this equation and expression (4.4.14) for g into Eq. (4.4.15), we find

$$y(x,t) = \begin{cases} \frac{F_0}{T} \frac{\sin(\omega x/c)}{\sin(\omega l/c)} \left\{ \cos \left[\frac{\omega}{c}(l-b) \right] - \cos \left[\frac{\omega}{c}(l-a) \right] \right\} & 0 < x < a \\ \frac{F_0/T}{\sin(\omega l/c)} \left\{ \sin \frac{\omega x}{c} \cos \left[\frac{\omega}{c}(l-b) \right] \right. & \\ \quad \left. + \sin \left[\frac{\omega}{c}(l-x) \right] \cos \frac{\omega a}{c} - \sin \frac{\omega l}{c} \right\} & a < x < b \\ \frac{F_0}{T} \frac{\sin[\omega(l-x)/c]}{\sin(\omega l/c)} \left\{ \cos \frac{\omega a}{c} - \cos \frac{\omega b}{c} \right\} & b < x < l \end{cases} \quad (4.4.16)$$

times the factor $e^{-i\omega t}$, which of course exhibits the resonance behavior discussed earlier.

Wave impedance

When the string is part of a larger mechanical system, the contribution of the string to the system's behavior can usefully be measured in terms of the mechanical impedance of the string at the point of contact of the coupling force. Early in this section we mentioned that for a very long string, driven from one end, the ratio between transverse driving force and transverse velocity of the end of the string is $(T/c) = \epsilon c = \sqrt{\epsilon T}$, the wave resistance of an infinite string [Eq. (4.4.1)]. For an infinite string driven at point x_0 by transverse force $F(t)$, Eq. (4.4.4) indicates that the transverse velocity of point x on the string at time t is $(c/2T)F[t - (1/c)|x - x_0|]$. Consequently, the impedance at x_0 , the ratio between applied force and the velocity of the point of contact, is $2T/c = 2\epsilon c$, twice the impedance of an infinite string, driven at one end. Since the application of the force to a point x_0 on an infinite string is equivalent to driving two infinite strings, end to end, from the ends of each, the factor 2 is not surprising. We note that these impedances are real and are independent of the driving frequency. The energy supplied to the string by the force continually radiates away, the power lost by the force being $F(t)v(x_0,t) = F^2(t)/2\epsilon c$.

For a string of length l , driven at the end $x = 0$ and held rigidly at the end $x = l$, the situation is quite different. Here the reflected wave does return to alter the nature of the string reaction at $x = 0$. Since no energy is

lost in reflection from the rigid support at $x = l$, when steady-state conditions are reached, the force is no longer feeding energy into the string; so the impedance must be reactive rather than resistive. Furthermore, since the phase relation of the reflected wave at $x = 0$ depends on the frequency of the driving force, the impedance must depend on the frequency.

A string of length l , with its end at $x = l$ held rigidly, must have a displacement $y(x, t) = A \sin [(\omega/c)(l - x)]e^{-i\omega t}$ for it to oscillate with frequency $\omega/2\pi$. Its slope at $x = 0$ would be $-(\omega/c) A \cos (\omega l/c)e^{-i\omega t}$, and its velocity there would be $v(0, t) = -i\omega A \sin (\omega l/c)e^{-i\omega t}$. The driving force would be $-T$ times the slope, $f(t) = (T\omega/c)A \cos (\omega l/c)e^{-i\omega t}$. The ratio between this force and the velocity at $x = 0$ is the impedance,

$$Z_m(\omega) = \frac{f(t)}{v(0, t)} = i \frac{T}{c} \cot \frac{\omega l}{c} = \frac{i\epsilon c}{\tan (\omega l/c)} \quad (4.4.17)$$

which is imaginary (reactive) and a function of ω , as predicted.

For frequencies very much smaller than $c/2\pi l$, the function $\tan (\omega l/c)$ is approximately equal to $\omega l/c$; so the impedance is $i(T/\omega l)$. Comparison with Eq. (2.3.4) shows that at these low frequencies the string end is stiffness-controlled for transverse oscillation, with a stiffness constant T/l , which checks.

As the driving frequency is increased, the impedance $Z(\omega)$ decreases in magnitude, from $i\infty$ to 0 when $\omega l/c = \frac{1}{2}\pi$ or $\lambda \equiv 2\pi c/\omega = 4l$. This is the condition of resonance; at this frequency the shape of the string is such that its slope is zero at $x = 0$; so it cannot oppose any reaction to the driving force. At a still higher frequency, when $\omega = \pi c/l$, the cotangent is again infinity; no amount of transverse force of this frequency will move the end $x = 0$. A node of the standing wave occurs at $x = 0$; so motion of $x = 0$ at this frequency is impossible (as long as there is no friction or the support at $x = l$ does not move). When the driving frequency is further increased, the impedance alternates between positive-imaginary and negative-imaginary values, being zero (resonance) whenever $\omega = (\pi c/l)(n + \frac{1}{2})$ and $\pm i\infty$ (driven end a standing-wave node) whenever $\omega = (\pi c/l)m$, where n and m are integers.

Use of the Laplace transform

Finally, we can utilize the Laplace-transform technique of Table 2.1 and Eq. (2.3.16) to calculate the behavior of the string when driven by a transient force applied at the end $x = 0$. If both force $f(t)$ and velocity $v(0, t)$ are zero for $t < 0$, we can separate the dependence of both force and velocity into their frequency components by taking Laplace transforms.

$$\begin{aligned} P_L(s) &= \int_0^\infty e^{-st} f(t) dt \\ V_L(s) &= \int_0^\infty e^{-st} v(0, t) dt \\ &= \int_0^\infty e^{-st} \frac{d}{dt} y(0, t) dt = s Y_L(s) \end{aligned}$$

where the usual relationship, $\omega = is$, holds between s and the frequency parameter ω of a Fourier transform. The relationship between the components P_L and V_L at a given frequency ω is, by definition, $P_L = Z(\omega)V_L$, where $Z(\omega) = Z(is)$ is the impedance. Therefore we find the velocity $v(0, t)$ or the displacement $y(0, t)$ of the string end by taking the inverse Laplace transforms of

$$V_L(s) = \frac{P_L(s)}{Z(is)} \quad Y_L(s) = \frac{P_L(s)}{sZ(is)} \quad (4.4.18)$$

For a string of length l , driven at one end and held rigidly at the other, the transforms of the velocity and displacement of the end $x = 0$, when acted on by a force having Laplace transform $P_L(s)$, are

$$V_L(s) = \frac{cP_L(s)}{iT} \frac{\sin(isl/c)}{\cos(isl/c)} = \frac{cP_L(s)}{T} \frac{\sinh(sl/c)}{\cosh(sl/c)} = sY_L(s) \quad (4.4.19)$$

where f , v , and y were all zero before $t = 0$. If we know the transform of $f(t)$, Table (2.1) (or a more complete table) will enable us to compute $v(0, t)$ or $y(0, t)$, the corresponding velocity and displacement of the end of the string.

For example, if the transient force $f(t)$ is a unit impulse $\delta(t)$ at $t = 0$, its Laplace transform is $P_L(s) = 1$, and we can use formula (20) in Table 2.1 to find $y(0, t)$. The transform $F_L(s)$ of this formula is $(c/sT) \sinh(sl/c) = (c/2sT)(e^{sl/c} - e^{-sl/c})$, which has an inverse transform $(c/2T)\{u[t + (l/c)] - u[t - (l/c)]\}$, where u is the step function of Eq. (1.3.22). Therefore

$$\begin{aligned} y(0, t) &= \frac{c}{T} \sum_{n=0}^{\infty} (-1)^n \left\{ u\left[t - 2n\frac{l}{c}\right] - u\left[t - 2(n+1)\frac{l}{c}\right] \right\} \\ &= \frac{c}{T} u(t) + \frac{2c}{T} \sum_{n=1}^{\infty} (-1)^n u\left(t - 2n\frac{l}{c}\right) \end{aligned} \quad (4.4.20)$$

which is periodic function with period $4l/c$. From $t = 0$ to $t = 2l/c$, y is equal to c/T ; from $t = 2l/c$ to $t = 4l/c$, y is equal to $-(c/T)$, after which it repeats the square-wave motion indefinitely. The unit impulsive force pushes the end at $x = 0$ an amount T/c transversally; this motion is transmitted down the string and reflects back, to arrive at $x = 0$ after a time $2l/c$. By that time there is no transverse force acting on that end; so the slope $\partial y/\partial x$ must remain zero there. This requires a reflected wave of negative sign, which goes down and back, after which it changes back to positive and starts a new cycle.

The solution (4.4.20) is for an impulsive force $\delta(t)$. If the end $x = 0$ is acted on by a force $f(t)$, which differs from zero only between $t = 0$ and $t = b$, then an extension of Eq. (2.3.9) (or a further application of Table 2.1) produces the following solution for the transverse displacement of the end $x = 0$:

$$y(0, t) = \frac{c}{T} Y(t) + \frac{2c}{T} \sum_{n=1}^{\infty} (-1)^n Y\left(t - 2n\frac{l}{c}\right) \quad (4.4.21)$$

where

$$Y(t) = \int_0^t f(u) du$$

For example, the lower curve of Fig. 4.16 displays the motion of the driven end of a string acted on by a force which occurs in the interval between $t = 0$

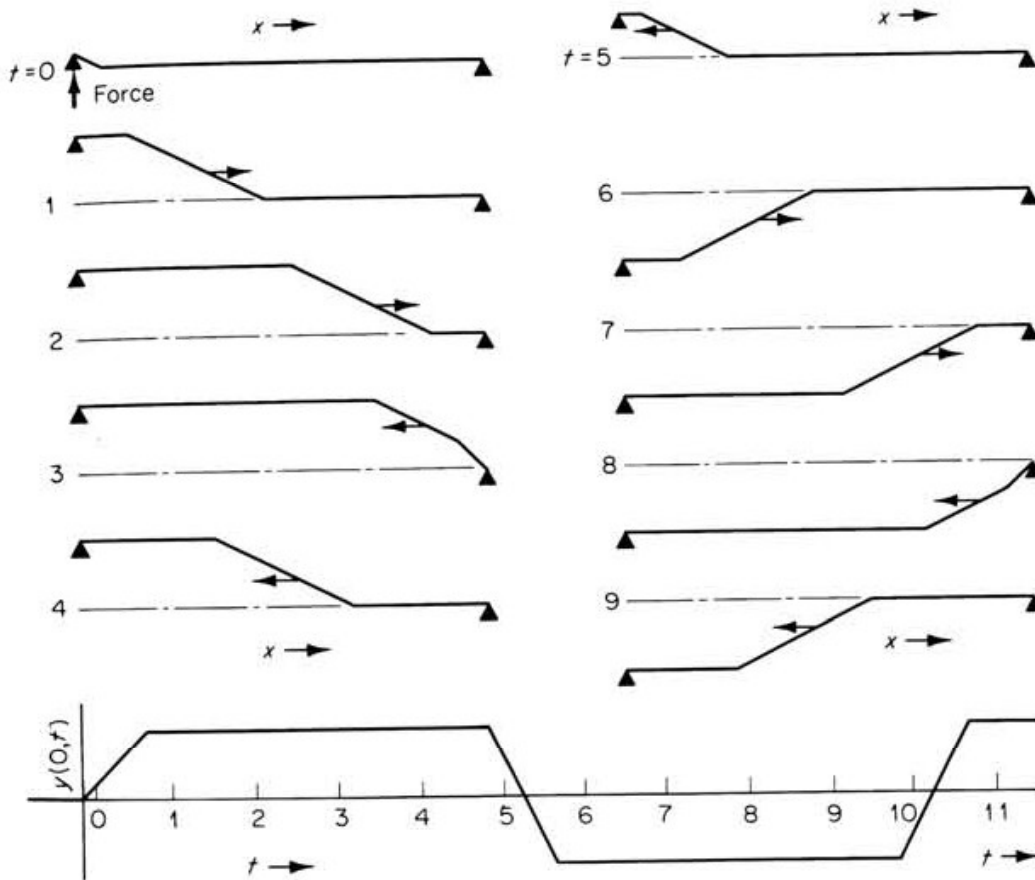


FIGURE 4.16

Motion of string fixed at $x = l$, driven by transverse force applied at $x = 0$, between $t = -\frac{1}{5}$ and $t = \frac{2}{5}$. Upper curves are successive shapes of string; lower curve is displacement of point of application $x = 0$. Period of free vibration is 10; c is thus $l/5$.

and $t = l/3c$. The upper curves show the successive waveforms at the times indicated on the lower curve. The wave reflects from the fixed end at $x = l$ without change of sign; from the "open" end $x = 0$ with change of sign. The motion is periodic after the force has ceased acting.

Uniform friction

When the frictional resistance of the medium surrounding the string is appreciable, the solutions of this section must be modified. In accordance with Eq. (4.3.17), the equation of motion of a string, driven by a force

$F(x)e^{-i\omega t}$ and acted on by a frictional force, proportional to its transverse velocity, along its length, is

$$\frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t} - c^2 \frac{\partial^2 y}{\partial x^2} = \frac{F}{\epsilon} e^{-i\omega t}$$

where $k = R(\omega)/2\epsilon$. If the force has amplitude T and is concentrated at the point x_0 , the solution will be a Green's function $g(x | x_0)e^{-i\omega t}$, where g satisfies an inhomogeneous equation like (4.4.9).

$$\frac{d^2 g}{dx^2} + \left(\frac{\omega}{c}\right)^2 \left(1 + \frac{2ik}{\omega}\right) g = -\delta(x - x_0) \quad (4.4.22)$$

In contrast to the free-vibration solutions of Eq. (4.3.16), where the frequency turns out to be complex, for forced motion (where ω must be real), the space dependence is complex.

For a string of length l , between rigid supports, the Green's function is an obvious modification of Eq. (4.4.14).

$$G(x | x_0) = \begin{cases} \frac{1}{\beta} \frac{\sin(\beta x)}{\sin(\beta l)} \sin[\beta(l - x_0)] & 0 < x < x_0 < l \\ \frac{1}{\beta} \frac{\sin[\beta(l - x)]}{\sin(\beta l)} \sin(\beta x_0) & 0 < x_0 < x < l \end{cases} \quad (4.4.23)$$

where

$$\beta = \frac{\omega}{c} \sqrt{1 + \frac{2ik}{\omega}} \simeq \frac{\omega}{c} + i \frac{k}{c} \quad k \ll \omega$$

When $k \ll \omega$, the resonance denominator becomes

$$\begin{aligned} \sin(\beta l) &\simeq \sin \frac{\omega l}{c} \cosh \frac{kl}{c} + i \cos \frac{\omega l}{c} \sinh \frac{kl}{c} \\ &\simeq \sin \frac{\omega l}{c} + i \frac{kl}{c} \cos \frac{\omega l}{c} \end{aligned}$$

Consequently, at the resonance frequencies $\nu_n = \omega_n/2\pi = nc/2l$, this denominator does not go to zero; it just becomes quite small and imaginary. The corresponding amplitude of motion becomes large but not infinite, and the motion is out of phase with the force.

The fact that the x dependence of g is also complex means that the motion of the string is not all in phase; the parts farther away from x_0 lag farther behind than the parts nearby. Figure 4.17 shows sequences for three different driving frequencies. They should be compared with the amplitudes given in Fig. 4.15 for $\gamma = 1.5, 1.9$, and 3.0 , for the case of no friction, when the motion is all in phase.

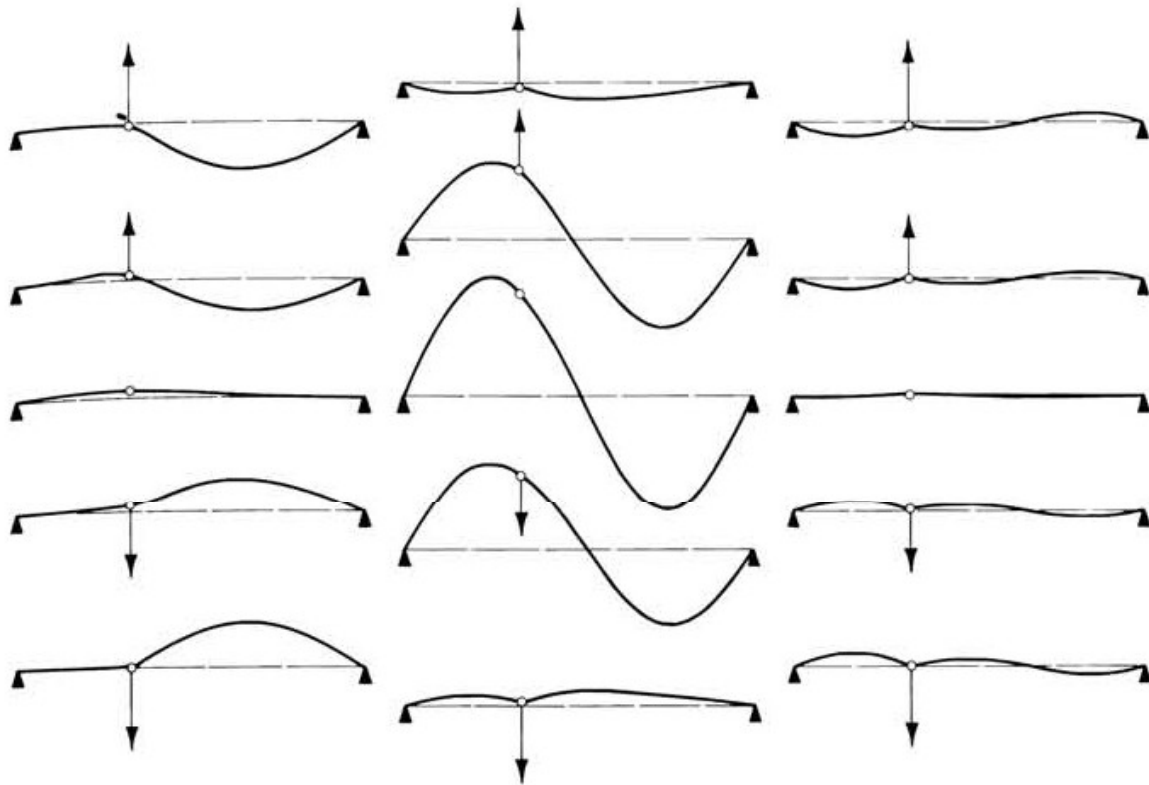


FIGURE 4.17

Sequences showing the successive shapes of a damped string, driven by a force of frequency ν , during a half cycle of the steady-state motion. The force is concentrated at the point marked by the circle, and the arrows show its successive magnitudes and directions. The three sequences are for the driving frequency ν , equal to $\frac{1}{2}$, two and three times the fundamental frequency of the corresponding undamped string.

4.5 THE EFFECT OF INHOMOGENEITIES

A wave in a uniform flexible string travels with constant velocity and unchanged shape, at least as long as the wave amplitude is not too great. Any irregularity in the mass density of the string, however, will distort this perfect wave motion. If the wave is traveling along a very long string, the irregularity will reflect part of the wave and will distort the rest as it is transmitted beyond the irregularity. If the motion is a standing wave, the irregularity will change the values of the frequencies of free vibration of the string. In this section we shall discuss various methods for computing these effects.

It is often useful to consider the behavior of an inhomogeneous region of a string as a simple example of the interaction between two vibrating systems, the uniform string being one system and the inhomogeneity another. As such, we should expect the combination to have some of the properties of coupled oscillators. Later we shall deal with the coupled motions of a wide variety of acoustic systems; in fact, most acoustic problems of interest involve the interaction of two or more systems. A string vibrating in vacuum between two rigid supports is a very simple—but a quite idealized—subject for study.

When we add the interaction between the string's motion and the air, and when we also consider the effects of the motion on supports which are not completely rigid, the problem becomes both very much more difficult to solve and also of much greater practical interest.

Thus the effect of inhomogeneities on the wave motion of a string deserves study because it is one of the simplest examples of the interaction between an extended system and a concentrated one.

Wave reflection from a point inhomogeneity

We start by considering an infinite string, under tension T and with uniform mass ϵ per unit length, having attached to it a simple oscillator, which can move transversally with the string, at point $x = 0$. The string's displacement $y(0, t)$ equals the displacement of the oscillator, and the shape of the string, near $x = 0$, must be such as to move the oscillator along with the string. As we saw in the previous section, the force required to produce the motion must be caused by a discontinuity in the string's slope at $x = 0$.

$$F = \lim_{\Delta \rightarrow 0} \left\{ T \left[\left(\frac{\partial y}{\partial x} \right)_{x=\Delta} - \left(\frac{\partial y}{\partial x} \right)_{x=-\Delta} \right] \right\} \quad (4.5.1)$$

The relationship between this coupling force and the displacement $y(0, t)$ of both string and oscillator at $x = 0$ will, of course, depend on the nature of the oscillator. The discontinuous change in slope required by Eq. (4.5.1) will distort the waveform as it passes through $x = 0$.

To see what happens we consider the simple case of a steplike wave $u(ct - x)$ approaching the origin from the left. Of course, a perfect step-function waveform would not satisfy the requirement that $|\partial y / \partial x|$ be small for the wave equation to be valid; but the step function can be considered to be the limiting form of an actual wave with steep wavefront. Such a wave, when it reaches the origin, would act like an impulsive force, of total impulse T . What this impulse does to the attached oscillator depends on the nature of the oscillator.

For example, if the "oscillator" at $x = 0$ is simply an extra mass m , attached to the string at $x = 0$, the impulse will give the mass an initial upward velocity T/m ; but at $t = 0$ the initial displacement $y(0, 0)$ will still be zero. Thus, to the steep-fronted wave, the extra mass will behave as though there were an initially rigid support at $x = 0$. As was shown in Sec. 4.2 (Fig. 4.7), a wave is reflected from a rigid support with change of sign, so the initial effect of the mass will be to start a wave of cancellation $-u(ct + x)$ back toward $-\infty$, canceling the $+1$ of the incident wave $u(ct - x)$. But the mass at $x = 0$ does not stay at rest; immediately after $t = 0$ it begins to rise, with initial velocity T/m , bringing the string along with it. The "delayed reaction" of the mass produces a wave in both directions from $x = 0$; the generation of such a wave requires energy, which must come

from the kinetic energy of the mass. Thus the mass will slow down, eventually coming to rest a unit distance from its initial position. The sequence is shown in the left-hand sequence of Fig. 4.18. The reflected wave has a steep wavefront, followed by a "wake" produced by the motion of the mass m . The transmitted wave only registers the motion of the mass.

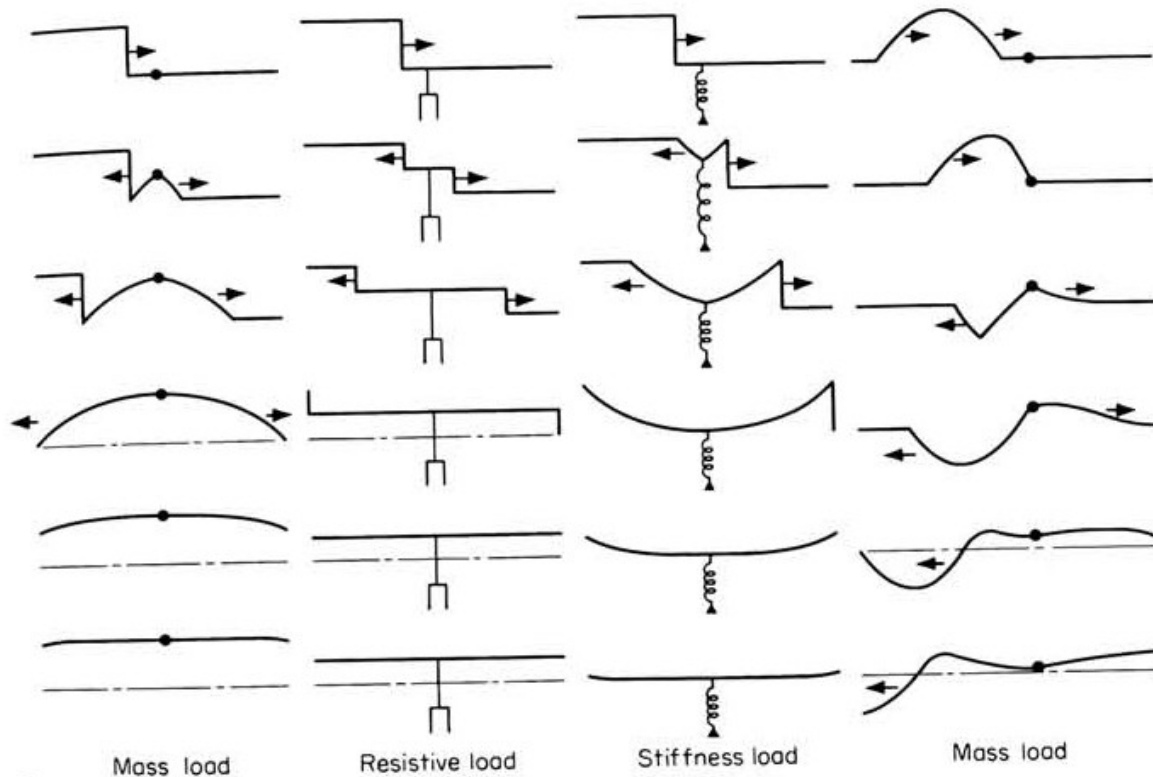


FIGURE 4.18
Reflection and transmission of a wave at a point load on a string. First three time sequences are for an incident step-function wave. Last sequence is for half a sine wave.

If the load at $x = 0$ is purely resistive, with no effective mass, the impulsive force of the wavefront will jerk the load upward by an amount T/R , where R is the load's mechanical resistance. As soon as the wavefront has passed, the frictional load will stop moving. Therefore both reflected and transmitted waves are steep-fronted, as shown in the second sequence of Fig. 4.18.

Finally, if the load is a massless spring attached to the string at $x = 0$, the impulsive force will jerk the spring end up so that $y(0, t)$, just after $t = 0$, is unity and the wavefront will pass unchanged into the region $x > 0$. However, now that the spring is displaced from equilibrium, it will exert a downward force on the string, creating a wave which spreads out in both directions from the origin. The sequence of events is shown in the third sequence of Fig. 4.18; in this case the steep wavefront stays with the transmitted wave, not the reflected one.

To work out the events quantitatively, we assume that the wave on the string for $x < 0$ has both an incident and a reflected wave, whereas the wave in the region $x > 0$ has only a transmitted part.

$$y(x,t) = \begin{cases} I(ct - x) + Q(ct + x) & x < 0 \\ P(ct - x) & x > 0 \end{cases} \quad (4.5.2)$$

where we can assume that $I(z)$, $Q(z)$, and $P(z)$ are all zero for $z < 0$, without serious loss of generality. In order that the string be continuous at $x = 0$ we must have

$$I(z) + Q(z) = P(z) \quad z = ct \quad (4.5.3)$$

and also

$$F(t) = T[I'(ct) - Q'(ct) - P'(ct)]$$

where $I'(z) = (d/dz)I(z)$ represents the force on the extra load at $x = 0$, as specified by Eq. (4.5.1).

If the load is purely inertial, a mass m , then the force F will cause an acceleration

$$m \frac{d^2}{dt^2} y(0,t) = mc^2 P''(ct) = F = \epsilon c^2 [I'(ct) - Q'(ct) - P'(ct)]$$

where $P''(z)$ is the second derivative of $P(z)$ with respect to its argument. This equation and the first of Eqs. (4.5.3), between them, enable us to eliminate the unknown function Q and obtain an equation for the transmitted wave P in terms of the known incident wave $I(z)$. We integrate the last equation, obtaining $I(z) - Q(z) - P(z) = (m/\epsilon)P'(z)$, and thence obtain

$$\frac{m}{2\epsilon} P'(z) + P(z) = I(z)$$

which has for solution

$$P(z) = \frac{2\epsilon}{m} e^{-(2\epsilon/m)z} \int_0^z I(u) e^{(2\epsilon/m)u} du \quad z > 0 \quad (4.5.4)$$

and the reflected wave is $Q(z) = P(z) - I(z)$ from Eq. (4.5.3).

The transmitted wave $P(ct - x)$ is thus not $I(ct - x)$, the incident wave, but a delayed response of the mass to the incident wave, expressible in terms of the integral involving exponentials, of Eq. (4.5.4). This is illustrated by the first sequence of Fig. 4.18; the reflected and transmitted waves are exponentials. The last sequence of Fig. 4.18 is the reaction of the string-mass combination to an incident wave of the form

$$I(z) = \begin{cases} 0 & z < 0 \\ \sin \frac{\pi z}{a} & 0 < z < a \\ 0 & a < z \end{cases}$$

The mass is initially impelled upward, but after the incident wave has gone by, the reflected and transmitted waves pull the mass back down again, then slow it up, so that it eventually comes to rest at $y = 0$.

The waves produced by resistive or springy loads can be worked out in a very similar manner; they will be left for a problem.

Simple-harmonic waves

When a sinusoidal wave is incident on the point $x = 0$, to which is attached a simple oscillator, the Green's function of Eq. (4.4.11) can be used to calculate the reflected and transmitted waves. The force which the string must exert on the oscillator to make it move with the string is, by Newton's third law, equal and opposite to the reactive force exerted by the oscillator on the string at $x = 0$. If the string's motion is sinusoidal, $y(0, t) = Y(0)e^{-i\omega t}$, then the reactive force, which is equal to the oscillator's velocity times its impedance $Z(\omega)$ [Eq. (2.3.3)],

$$F(t) = -Z(\omega) \frac{d}{dt} y(0, t) = i\omega Z(\omega) Y(0) e^{-i\omega t} \quad (4.5.5)$$

must also vary sinusoidally with time. Thus the equation of motion of both string and oscillator is given by a modification of Eq. (4.4.12).

$$\epsilon \frac{\partial^2 y}{\partial t^2} - T \frac{\partial^2 y}{\partial x^2} = i\omega Z(\omega) \delta(x) y(x, t)$$

or, since y depends on time only through the factor $e^{-i\omega t}$ so that $y(x, t) = Y(x)e^{-i\omega t}$, the equation for $Y(x)$ is

$$\frac{d^2 Y}{dx^2} + \omega^2 \frac{\epsilon}{T} Y(x) = -i\omega \frac{Z}{T} Y(x) \delta(x) \quad (4.5.6)$$

The displacement of the oscillator is $Y(0)e^{-i\omega t}$, equal to the displacement of the string at $x = 0$. The reactive force is concentrated at $x = 0$ (which accounts for the delta function) and is proportional to the displacement $Y(0)$ itself. Comparison with Eqs. (4.4.9) and (4.4.11) indicates that this reactive force must generate a wave

$$Y_g(x) = i \frac{c}{2\omega} i\omega \frac{Z}{T} Y(0) e^{i\omega |x|/c}$$

or

$$y_g(x, t) = -\frac{Z}{2\epsilon c} Y(0) \exp \left[\frac{i\omega}{c} (|x| - ct) \right]$$

which spreads out equally in both directions from the point of application $x = 0$ of the force. In the present case the force is produced by the motion, so that if y_g were the whole solution and $Y(0)$ on the right-hand side were equal to $Y_g(0)$, then $Y(0)$ would have to be zero. With no external driving force, no motion could result.

The driving force results from the incident wave

$$y_i(x, t) = Ae^{i\omega(x-ct)/c} \quad Y_i(x) = Ae^{i\omega x/c}$$

which is produced at the far end of the string, $x = -\infty$, by some external force. We note that $Y_g(x)$ is only a particular solution to the equation of motion (4.5.6); to it we can add any amount of a solution of the homogeneous equation $(d^2Y/dx^2) + (\omega/c)^2 Y = 0$ [see also the discussion preceding Eq. (4.4.14)]. For example, the function $Y(x) = Y_i(x) + Y_g(x)$, or

$$Y(x) = Ae^{i\omega x/c} - \frac{Z}{2\epsilon c} Y(0)e^{i\omega |x|/c} = \begin{cases} Ae^{i\omega x/c} - \frac{Z}{2\epsilon c} Y(0)e^{-i\omega x/c} & x < 0 \\ \left[A - \frac{Z}{2\epsilon c} Y(0) \right] e^{i\omega x/c} & x > 0 \end{cases} \quad (4.5.7)$$

is a solution of Eq. (4.5.6), which corresponds to the physical situation we have described. The first term in the expression for $Y(x)$ is the incident wave, generated at $x = -\infty$ and going clear to $x = +\infty$ without change of amplitude or phase. The second term is the wave going out from $x = 0$, caused by the reaction of the oscillator attached to the string there. Thus, in the region $x < 0$, there is an incident and a reflected wave, each going in opposite directions; in region $x > 0$ the incident and generated waves combine to produce a transmitted wave, of amplitude $A - (Z/2\epsilon c)Y(0)$, going in the positive direction.

But we still have not solved the problem, for the unknown Y appears on both sides of Eq. (4.5.7). The reason is that the equation of motion (4.5.6) is a homogeneous equation, not an inhomogeneous one like Eq. (4.4.5). The force term contains the unknown Y , since the reaction force is proportional to the displacement itself, though we assumed it to be known when we wrote down the particular solution Y_g . An inhomogeneous term emerges only when we introduce the incident wave Y_i into the equation for the solution. This term enables us to solve for the magnitude of Y by setting $x = 0$ in Eq. (4.5.7).

$$Y(0) = A - \frac{Z}{2\epsilon c} Y(0) \quad \text{or} \quad Y(0) = \frac{2\epsilon c}{Z + 2\epsilon c} A$$

or

$$Y(x) = Ae^{i\omega x/c} - \frac{AZe^{i\omega |x|/c}}{Z + 2\epsilon c} = \begin{cases} Ae^{i\omega x/c} - \frac{AZe^{-i\omega x/c}}{Z + 2\epsilon c} & x < 0 \\ \frac{2\epsilon c Ae^{i\omega x/c}}{Z + 2\epsilon c} & x > 0 \end{cases} \quad (4.5.8)$$

$$y(x, t) = Y(x)e^{-i\omega t}$$

This gives the amplitudes of the reflected and transmitted waves in terms of the amplitude A of the incident wave. The proportionality factors involve the impedance Z with which the oscillator opposes the motion of the string, and the impedance $2\epsilon c$ with which the string opposes the motion of the oscillator [see discussion of Eq. (4.4.1)]. If the impedance of the load Z is much larger than $2\epsilon c$, the reflected wave is large and the transmitted wave is small compared with the incident wave. The reverse is true if $|Z|$ is small compared with $2\epsilon c$.

Energy flow and resonance effects

The mechanical impedance of a simple oscillator, $-i\omega m + R + i(K/\omega)$, has a real part, the mechanical resistance term R , and an imaginary part, the reactance $-iX = -i[\omega m - (K/\omega)]$, caused by the oscillator's mass and stiffness. Unless X is zero, both reflected and transmitted waves are out of phase with the incident wave. Unless R is zero, the energy carried away by the reflected and transmitted waves is less than the energy introduced by the incident wave.

According to Eq. (4.4.2), the mean incident power Υ_i transmitted by the incident wave from $-\infty$ to 0, the mean power Υ_r reflected back to $-\infty$, and the mean power Υ_t transmitted from $x = 0$ on to $+\infty$ are

$$\begin{aligned}\Upsilon_i &= \frac{1}{2}\epsilon c\omega^2 |A|^2 & \Upsilon_r &= \frac{1}{2}\epsilon c\omega^2 |A|^2 \frac{R^2 + X^2}{(R + 2\epsilon c)^2 + X^2} \\ \Upsilon_t &= \frac{1}{2}\epsilon c\omega^2 |A|^2 \frac{4(\epsilon c)^2}{(R + 2\epsilon c)^2 + X^2}\end{aligned}\quad (4.5.9)$$

The sum of Υ_t and Υ_r is the power which leaves the region $x = 0$, whereas Υ_i is the power fed into the region by the incident wave. The difference,

$$\begin{aligned}\Upsilon_i - \Upsilon_t - \Upsilon_r &= \frac{\frac{1}{2}\epsilon c\omega^2 |A|^2}{(R + 2\epsilon c)^2 + X^2} [(R + 2\epsilon c)^2 + X^2 - R^2 - X^2 - 4(\epsilon c)^2] \\ &= \frac{1}{2}R\omega^2 |Y(0)|^2 = \frac{1}{2}R \left| \left(\frac{\partial y}{\partial t} \right)_{x=0} \right|^2\end{aligned}\quad (4.5.10)$$

is the amount absorbed in moving the oscillator against its resistance R .

For a simple oscillator the reactance $iX = -i\omega m + i(K/\omega)$ is zero at the resonance frequency $\omega_0/2\pi$, where $\omega_0^2 = K/m$. Near this frequency the transmitted wave carries away most of the incident energy, if $R < 2\epsilon c$; far from this frequency the reflected wave carries more. The effect is more pronounced the larger the Q of the oscillator, $\sqrt{Km}/R$ [Eq. (2.2.4)]. If this quantity is large, the incident wave usually finds the region $x = 0$ hard to penetrate; for all frequencies except those very near $\omega_0/2\pi$, the transmitted power Υ_t is much smaller than the incident power Υ_i or the reflected power Υ_r . At resonance, however, X is zero, and (if $R < \sqrt{Km}$ and $R < 2\epsilon m$)

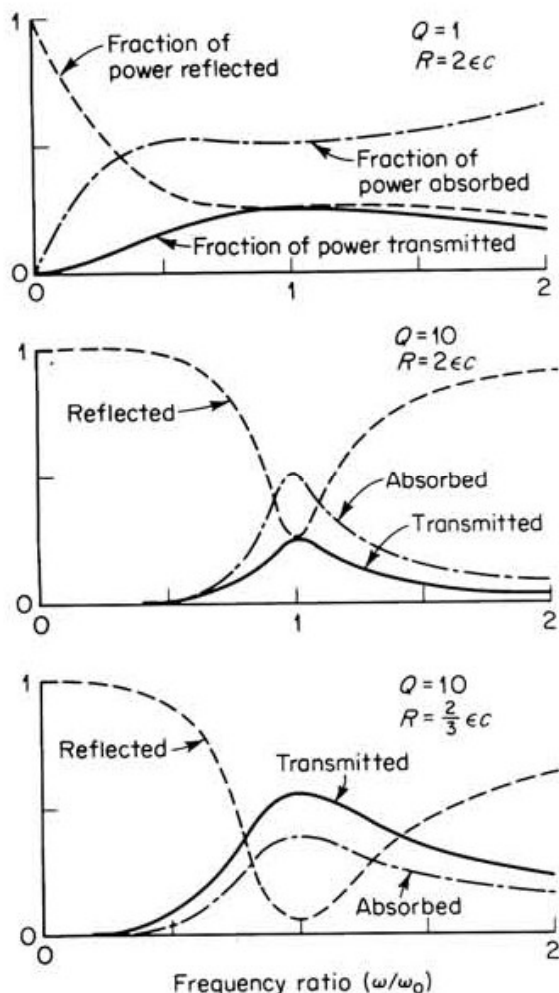


FIGURE 4.19

Ratio of power reflected, transmitted, and absorbed to incident power, by a simple oscillator attached to a very long string, for simple-harmonic waves of frequency $\omega/2\pi$, for different values of $Q = \sqrt{Km}/R$.

the point $x = 0$ is nearly transparent to the incident wave; Y_r is small compared with Y_i and Y_t . Figure 4.19 shows examples of this; the transmitted and reflected fractions of the incident power are there plotted as a function of ω/ω_0 . For the cases of $Q = 10$, the effects of the resonance are quite noticeable; for $Q = 1$ ($R = \sqrt{Km}$), the resistance has broadened out the resonance peak and damped out the string reflection at high frequencies. In all cases, however, for frequencies sufficiently above resonance so that $|X| \rightarrow \omega m$ is larger than $R + 2\epsilon c$, the transmitted fraction falls off inversely with ω^2 . In this frequency range the inertia of the point load effectively opposes motion of the string and reflects most of the incident energy.

Extended range of inhomogeneity

Now suppose that the oscillator is attached to an extended range of length of the string, rather than at the point $x = 0$. In other words, suppose that over the length of string from $x = -\frac{1}{2}a$ to $x = \frac{1}{2}a$, there is an additional load on the string, which opposes its displacement by a reaction force which is proportional to the string's displacement at x . For a simple-harmonic wave of frequency $\omega/2\pi$, the ratio between reaction-force density and displacement could be written as $-i\omega z(x)$, where $z(x)$ is the *impedance density*

of the load in mechanical ohms per unit length, at x . Setting $y(x, t) = Y(x)e^{-i\omega t}$ as before, the equation for the amplitude-phase function $Y(x)$ is

$$\frac{d^2 Y}{dx^2} + \left(\frac{\omega}{c}\right)^2 Y = -i \frac{\omega}{T} z(x) Y(x) \quad (4.5.11)$$

which is to be compared with Eq. (4.5.6). Here the load spreads out over a range of x (z differs from zero only in the range $-\frac{1}{2}a < x < +\frac{1}{2}a$) rather than being concentrated at $x = 0$. In a few cases, where z is a simple mathematical function (but not very physical), this equation can be solved exactly; in most cases of practical interest we must be satisfied with an approximate solution.

To solve the equation we use the Green's function technique of Eqs. (4.4.13) and (4.4.15), assuming for the moment that the right-hand term in (4.5.11) is a known force density. For example, if the string is long enough so that reflections from the ends may be neglected, we can use the Green's function of Eqs. (4.4.11) and (4.4.13) and transform Eq. (4.5.11) into an integral equation, similar to Eq. (4.5.7) for a load concentrated at $x = 0$. We multiply Eq. (4.5.11) (expressed in terms of x_0 instead of x) by $g(x_0 | x)$ of Eq. (4.4.11) and subtract from this $Y(x_0)$ times Eq. (4.4.9) for $g(x_0 | x)$, with x_0 and x interchanged, giving

$$\begin{aligned} g(x_0 | x) \frac{d^2}{dx_0^2} Y(x_0) - Y(x_0) \frac{d^2}{dx_0^2} g(x_0 | x) \\ = \delta(x_0 - x) Y(x_0) - \frac{i\omega}{T} \frac{ic}{2\omega} e^{i\omega |x_0 - x|/c} z(x_0) Y(x_0) \end{aligned} \quad (4.5.12)$$

Now integrate both sides of this equation with respect to x_0 from $-d$ to $+b$, where $-\frac{1}{2}a > -d < x < +b > \frac{1}{2}a$, so that the range of integration includes the point $x_0 = x$ and also the range $-\frac{1}{2}a < x_0 < \frac{1}{2}a$, for which the load impedance $z(x_0)$ differs from zero. On the right-hand side, the integral of the first term is equal to $Y(x)$; the second term becomes an integral of $[z(x_0)/2\epsilon c] e^{i\omega |x_0 - x|/c} Y(x_0)$ over x_0 from $-\frac{1}{2}a$ to $\frac{1}{2}a$, the range for which $z(x_0)$ differs from zero. On the left-hand side the terms can be integrated directly; so the integral becomes the difference between an expression evaluated at b and one evaluated at $-d$.

$$\begin{aligned} \left[g(x_0 | x) \frac{d}{dx_0} Y(x_0) - Y(x_0) \frac{d}{dx_0} g(x_0 | x) \right]_{-d}^b \\ = \frac{ic}{2\omega} \left[e^{i\omega |x - x_0|/c} \frac{d}{dx_0} Y(x_0) - Y(x_0) \frac{d}{dx_0} e^{i\omega |x - x_0|/c} \right]_{-d}^b \end{aligned}$$

At this point we have to specify something about the general properties of the solution $Y(x)$, in order to evaluate these terms. Points b and $-d$

are outside both the point x and the range of x_0 upon which the load falls. If we are investigating the effect of the load on an incident wave, we are assuming that the solution $Y(x_0)$, to the left of the load, has an incident wave $Ae^{i\omega x_0/c}$ plus a reflected wave $Qe^{-i\omega x_0/c}$ (where A is known but Q has, as yet, to be determined); and we are assuming that $Y(x_0)$, to the right of the load, is a transmitted wave $Pe^{i\omega x_0/c}$ (where P is as yet unknown).

This degree of specification of the solution is sufficient to enable us to evaluate the integral of the left-hand side of (4.5.12). Since $b > x$ and $-d < x$, at $x_0 = b$ the quantity $|x - x_0|$ equals $x_0 - x$ and at $x_0 = -d$ it equals $x - x_0$. Therefore the integral becomes

$$\begin{aligned} \frac{ic}{2\omega} \left\{ \left[e^{i\omega(x_0-x)/c} \frac{d}{dx_0} (Pe^{i\omega x_0/c}) - Pe^{i\omega x_0/c} \frac{d}{dx_0} e^{i\omega(x_0-x)/c} \right]_{x_0=b} \right. \\ \left. - \left[e^{i\omega(x-x_0)/c} \frac{d}{dx_0} (Ae^{i\omega x_0/c} + Qe^{-i\omega x_0/c}) \right. \right. \\ \left. \left. - (Ae^{i\omega x_0/c} + Qe^{-i\omega x_0/c}) \frac{d}{dx_0} e^{i\omega(x-x_0)/c} \right]_{x_0=-d} \right\} = Ae^{i\omega x/c} \end{aligned}$$

which is just the expression for the incident wave. Therefore the integral of Eq. (4.5.12), for a long string on which an incident wave $Ae^{i\omega x/c}$ impinges on a load between $-\frac{1}{2}a$ and $\frac{1}{2}a$, becomes the integral equation

$$Y(x) = Ae^{i\omega x/c} - \frac{1}{2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{i\omega|x-x_0|/c} \zeta(x_0) Y(x_0) dx_0 \quad (4.5.13)$$

where $\zeta = z/\epsilon c$, the ratio between the impedance density of the load and the radiation resistance of the string, may be called the *impedance ratio* of the load. It has the dimensions of inverse length.

This is an inhomogeneous integral equation for the waveform $Y(x)$. It contains more specifications about the solution than does the equation of motion (4.5.11), for it specifies the nature of the solution at large distances from the load; in particular, it includes the fact that an incident wave $Ae^{i\omega x/c}$ is coming in from $-\infty$. A solution of Eq. (4.5.13) is a solution of Eq. (4.5.11), but not all solutions of (4.5.11) are solutions of (4.5.13). The integral equation is a natural extension of Eq. (4.5.7) for a point load; it states that the solution $Y(x)$ is a sum of an incident wave plus a term representing the distortion of the wave by the added load. Each loaded point x_0 generates a wave $e^{i\omega|x-x_0|/c}$, radiating in both directions, of amplitude proportional to the amplitude $Y(x_0)$ of the string at x_0 and also proportional to the ratio $\zeta(x_0) = z(x_0)/\epsilon c$ between the transverse mechanical impedance density at x_0 and the wave impedance of the string.

Unfortunately, exact solutions of integral equations like (4.5.13) are no easier to obtain than are exact solutions of differential equations, about which we talked in Sec. 1.2. There are several advantages to the integral-equation

formulation, however. In the first place, it contains in it boundary conditions, as well as dynamic specifications, as discussed in the preceding paragraph. Because of this, one can derive general properties of the solution without exact knowledge of its form. And finally, the integral equation provides a more natural starting point for approximate solutions than does the equivalent combination of differential equation and boundary conditions.

For example, if we know the true form of $Y(x)$ in the range $-\frac{1}{2}a < x < \frac{1}{2}a$, where the load is attached, we can determine its form everywhere else in terms of the function

$$\varphi(\eta) = \begin{cases} 0 & \eta < -\frac{a}{2c} \\ \frac{\pi c \zeta(c\eta) Y(c\eta)}{A} = \pi \frac{z(c\eta)}{\epsilon} \frac{Y(c\eta)}{A} & -\frac{a}{2c} < \eta < \frac{a}{2c} \\ 0 & \frac{a}{2c} < \eta \end{cases}$$

The Fourier transform of this quantity with respect to the variable η , as defined in Eqs. (1.3.16), is

$$\Phi(\omega) = \frac{1}{2\pi} \int_{-a/2c}^{a/2c} e^{i\omega\eta} \varphi(\eta) d\eta = \frac{1}{2A} \int_{-a/2}^{a/2} e^{i\omega x_0/c} \zeta(x_0) Y(x_0) dx_0 \quad (4.5.14)$$

Conversely, if $\Phi(\omega)$, as a function of ω , can be determined, its inverse transform will determine the value of $Y(x)$ in the range $-\frac{1}{2}a < x < \frac{1}{2}a$.

The function $\Phi(\omega)$ is intimately related to the amplitude and phase of the waves reflected and transmitted from the loaded portion of the string, for an incident wave of frequency $\omega/2\pi$. This can be seen if we make $x < -\frac{1}{2}a$ or $x > \frac{1}{2}a$ in Eq. (4.5.13).

$$\begin{aligned} Y(x) &= Ae^{i\omega x/c} - \frac{1}{2}e^{-i\omega x/c} \int_{-a/2}^{a/2} e^{i\omega x_0/c} \zeta(x_0) Y(x_0) dx_0 \\ &= Ae^{i\omega x/c} - A\Phi(\omega)e^{-i\omega x/c} \quad x < -\frac{1}{2}a \end{aligned} \quad (4.5.15)$$

$$\text{Also} \quad Y(x) = A[1 - \Phi(-\omega)]e^{i\omega x/c} \quad x > \frac{1}{2}a$$

expresses the transmitted wave in terms of the transform of φ for the value $-\omega$. Thus a measurement of the ratios (amplitude and phase angle) between the incident and reflected waves and between the incident and transmitted waves, for the complete range of frequencies of the incident wave, will serve to measure the transform $\Phi(\omega)$ for ω from $-\infty$ to $+\infty$. The inverse transform will then give us $Y(x)$ in the range $-\frac{1}{2}a < x < \frac{1}{2}a$, and Eq. (4.5.15) provides the form of $Y(x)$ in the unloaded part of the string. Contrariwise, if we can compute the form of $Y(x)$ in the range $-\frac{1}{2}a < x < \frac{1}{2}a$, we can compute it for all other ranges of x and can thus calculate the reflection and transmission of the wave.

Successive approximations

If a , the length of string over which the load is applied, is very small compared with the wavelength $2\pi c/\omega$, then the term $e^{i\omega|x-x_0|/c}Y(x_0)$ in the integrand of (4.5.13) can be replaced by $e^{i\omega|x|/c}Y(0)$, and Eq. (4.5.13) reduces to Eq. (4.5.7) for a load applied at point $x = 0$, with Z equal to the integral of $z(x)$ over x from $-\frac{1}{2}a$ to $\frac{1}{2}a$. For larger values of a the form of $Y(x)$ in the range $-\frac{1}{2}a < x < \frac{1}{2}a$ is usually computed by means of successive approximations.

We can, for example, assume that the load does not change the form of the string very much, and insert $Y = Y_0(x) = Ae^{i\omega x/c}$ into the integral of Eq. (4.5.13), giving a first approximation.

$$\begin{aligned} Y_1(x) &= A \left[e^{i\omega x/c} - \frac{1}{2} \int_{-a/2}^{a/2} e^{i\omega|x-x_0|/c} \zeta(x_0) e^{i\omega x_0/c} dx_0 \right] \\ &= A \left\{ e^{i\omega x/c} \left[1 - \frac{1}{2} \int_{-a/2}^x \zeta(x_0) dx_0 \right] \right. \\ &\quad \left. - \frac{1}{2} e^{-i\omega x/c} \int_x^{a/2} e^{2i\omega x_0/c} \zeta(x_0) dx_0 \right\} \quad -\frac{1}{2}a < x < \frac{1}{2}a \quad (4.5.16) \end{aligned}$$

For x close to $-\frac{1}{2}a$, the positive exponential wave is nearly that of the incident wave, $Ae^{i\omega x/c}$; the negative exponential part is nearly that of the reflected wave. As x increases from $-\frac{1}{2}a$ to $\frac{1}{2}a$, the wave in the positive direction differs more and more from the incident wave, and the negatively directed wave decreases in amplitude, until, at $x = \frac{1}{2}a$, the wave is all in the positive direction. The reflected and transmitted amplitude functions of Eqs. (4.5.15) are, to this approximation,

$$\begin{aligned} \Phi_1(\omega) &= \frac{1}{2} \int_{-a/2}^{a/2} e^{2i\omega x_0/c} \zeta(x_0) dx_0 = \pi \Xi \left(\frac{2\omega}{c} \right) \\ \Phi_1(-\omega) &= \frac{1}{2} \int_{-a/2}^{a/2} \zeta(x_0) dx_0 = \pi \Xi(0) \end{aligned} \quad (4.5.17)$$

where

$$\Xi(w) = \left(\frac{1}{2\pi} \right) \int_{-a/2}^{a/2} \zeta(x_0) e^{iw x_0} dx_0$$

is the Fourier transform of the impedance ratio $\zeta(x)$. This approximation is usually called the *Born approximation*.

Better approximations can be obtained by reinserting $Y_1(x)$ into the integral of Eq. (4.5.13). In fact, one can compute a sequence of approximations, $Y_n(x)$, where

$$Y_{n+1}(x) = Ae^{i\omega x/c} - \frac{1}{2} \int_{-a/2}^{a/2} e^{i\omega|x-x_0|/c} \zeta(x_0) Y_n(x_0) dx_0 \quad (4.5.18)$$

which converges on the correct answer as n increases. The computations become progressively more tedious to carry out analytically, but can often be carried out numerically on a digital computer.

A variational technique

If we are more interested in calculating the transmission and reflection factors $\Phi(-\omega)$ and $\Phi(\omega)$ than in determining the exact shape of the wave in the loaded region $-\frac{1}{2}a < x < \frac{1}{2}a$, we can be satisfied with a process which leads directly to improved approximations for these factors. To explain the process, which will be of considerable use in solving many difficult problems, we consider an integral equation which is in some ways more general than Eq. (4.5.13) and in some ways simpler:

$$M(x) = \int_a^b K(x | x_0) Y(x_0) dx_0 \quad (\text{equality holds for } a < x < b) \quad (4.5.19)$$

where $M(x)$ is a known function of x ; $K(x | x_0)$ is symmetric [that is, $K(x_0 | x) = K(x | x_0)$]; and both M and K are real functions. The unknown function is $Y(x)$.

The first step is to multiply both sides of the equation by $Y(x)$ and then integrate over x from a to b , obtaining

$$0 = \int_a^b M(x) Y(x) dx - \int_a^b Y(x) dx \int_a^b K(x | x_0) Y(x_0) dx_0 \quad (4.5.20)$$

It usually happens that the integral $\int M Y dx$ has some physical importance in its own right. For example, in Eq. (4.5.13), $(1/2A)A\zeta(x)e^{i\omega x/c}$ is equivalent to $M(x)$; so $\int M Y dx$ is the transmission factor $\Phi(\omega)$, the quantity of interest in the present case. Thus we give this quantity a special symbol,

$$V(Y) = \int_a^b M(x) Y(x) dx \quad (4.5.21)$$

Adding the two equations and substituting for the solution $Y(x)$ the general function $u(x)$, we define the functional

$$V(u) = 2 \int_a^b M(x) u(x) dx - \int_a^b u(x) dx \int_a^b K(x | x_0) u(x_0) dx_0 \quad (4.5.22)$$

where $u(x)$ is any function integrable between a and b . Thus $V(u)$ is defined so that, when $u = Y$, the solution of Eq. (4.5.19), then $V(u)$ will have the correct value of the integral $\int M Y dx$.

This does not seem to be a very productive equation; a functional $V(u)$ is equal to a set of integrals involving the undetermined function u . Of course, if we knew the exact form of $Y(x)$ between a and b , we could use the equation to calculate $V(Y)$, but we know that already; also, Eq. (4.5.21) is an easier formula to use to compute $V(Y)$ (if we know Y). However, the right-hand side of Eq. (4.5.22) has several very important properties. Its first term is linear in the function u , and the second term, quadratic in u , is also symmetric in the u 's. In other words, if we had multiplied Eq. (4.5.19) by $v(x)$ and integrated, the integral $\int v dx \int K(x | x_0) u dx_0$ would remain the same if u and v were interchanged (this is because K is symmetric in x and x_0). Because of this symmetry, Eq. (4.5.22) can be made the basis of a variational principle, similar to Hamilton's principle.

To see what is meant by this statement, we note that the value of $V(u)$ is changed if the shape of $u(x)$ is changed. A small change, say, changing u to $u + \lambda v(x)$, where λ is a small quantity and v an integrable function, will usually produce a change in V proportional to λ . What we propose to show is that if, and only if, $u(x)$ is equal to $Y(x)$, the solution of Eq. (4.5.19), then the change in V is not proportional to λ , but is proportional to λ^2 . Variation in V produced by small changes in shape of $u + \lambda v$ around the function $Y(x)$ is *stationary* to this variation, to first order in λ , if u is Y . We cannot say that V is necessarily maximum or minimum for this choice of shape of u ; all we can say is that, if V is considered as a function of the shape-varying parameter λ , then $dV/d\lambda$ will be zero when $\lambda = 0$ if $u = Y$.

To prove these statements, suppose we set $u(x) = y(x) + \lambda v(x)$. Then

$$V(u) = V(y) + 2\lambda \int_a^b v(x) dx \left[M(x) - \int_a^b K(x | x_0) y(x_0) dx_0 \right] \\ - \lambda^2 \int_a^b v(x) dx \int_a^b K(x | x_0) v(x_0) dx_0$$

In order that $dV/d\lambda$ be zero at $\lambda = 0$, the quantity in square brackets must be zero; in other words, y must be the solution of Eq. (4.5.19). In many cases, of course, it is difficult to find the exact form of solution $Y(x)$; indeed, if we could find it easily, we should not need the assistance of functional $V(u)$. Often we are reduced to devising a form for u , containing parameters, which is too simple to be the exact solution for any set of values of the parameters; we should then like to know for what values of the parameters u comes as close as possible to Y . And more important, we want to know how close to $V(Y)$ is the corresponding value of $V(u)$ for these "best values" of the parameters. Equation (4.5.22) is eminently successful in attaining these more modest aims.

Suppose we choose a form for the trial function u , equipped with parameter λ , which we fear will never exactly equal Y for any value of λ but which we hope will not differ much from Y for some value of λ . In other words, we hope that u will have the form $u(\lambda, x) = Y(x) + \epsilon w(x) + \lambda v(x)$, where we do not know the exact form of function w but we hope that constant ϵ is small. Inserting this form into the integrals for $V(u)$, we have

$$V(u) = V(Y) - \epsilon^2 \int w \int K w - 2\lambda \epsilon \int v \int K w - \lambda^2 \int v \int K v$$

where the meaning of the shorthand symbols for the integrals should be obvious. If we knew the shape of w and the magnitude of ϵ , constituting the difference between $u(0, x)$ and Y , we could obtain the exact solution. But we do not know $\epsilon w(x)$; so all we can do is to vary the explicit parameter λ until $dV/d\lambda = 0$, to obtain a "best" value for λ and a corresponding "best"

shape for u , as well as the stationary value for V .

$$\lambda_s = -\epsilon \frac{\int w \int Kv}{\int v \int Kv} \quad u_s = Y(x) + \epsilon \left[w(x) - v(x) \frac{\int w \int Kv}{\int v \int Kv} \right]$$

$$V(u_s) = V(Y) - \epsilon^2 \left[\int w \int Kw - \frac{\left(\int w \int Kv \right)^2}{\int v \int Kv} \right]$$

Note that $V(u_s)$ may be maximum or minimum for V (or neither), depending on the relationship between w , v , and K .

We see that, although u_s differs from the correct solution Y by a term of the first order in ϵ , it actually is closer to Y than is $Y + \epsilon w$, and it may be quite close if v happens to have a shape like w in the region where K is large. We also see that the stationary value of the variational expression $V(u_s)$ is very much closer to the correct value $V(Y)$ if ϵ is small than is the value $V(Y + \epsilon w)$, which one would obtain by arbitrarily setting $\lambda = 0$. It thus is obvious that the success of the variational procedure is critically dependent on the user's physical intuition regarding the shape of the unknown function $Y(x)$ in the range of x from a to b , and particularly in the range where K and M are large. If one can devise a parametrized form $u(\lambda, x)$ which is fairly close in form to Y , then setting $dV/d\lambda$ equal to zero will produce a good approximation for u and a very good (second-order) value for V .

Finally, we should note that the variational calculation has concerned itself entirely with the shape of Y in the range $a < x < b$; once this is known, Eq. (4.5.19) can usually be extended to enable us to compute Y elsewhere [as will be shown for Eq. (4.5.13)].

Solutions and adjoint solutions

The procedure just described must be generalized somewhat to handle Eq. (4.5.13), since both terms on the right are complex quantities. To symmetrize the physical situation we should also consider the case when a wave is incident on the inhomogeneity, from $-\frac{1}{2}a$ to $+\frac{1}{2}a$, from the right, with a reflected wave going back to $x \rightarrow \infty$ and a transmitted wave going to $-\infty$. The solution corresponding to this inverse situation will be written $Y^-(x)$ and will be called the *adjoint* solution. The equation it must satisfy is like Eq. (4.5.13) but with a different inhomogeneous term,

$$Y^-(x) = Ae^{-i\omega x/c} - \frac{1}{2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{i\omega |x-x_0|/c} \zeta(x_0) Y^-(x_0) dx_0 \quad (4.5.23)$$

with reflection and transmission factors $\Phi^-(\pm\omega)$ similar in form to those of Eq. (4.5.14).

The variational form for both these functions can be found by multiplying Eq. (4.5.13) by $\zeta(x)Y^-(x)$ and integrating; by adding the equation for $2A^2\Phi(-\omega)$ to this, we obtain the analogue to Eq. (4.5.22).

$$\begin{aligned}
 V(u, u^-) = & A \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta(x)u(x)e^{-i\omega x/c} dx + A \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta(x)u^-(x)e^{i\omega x/c} dx \\
 & - \frac{1}{2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta(x)u^-(x) dx \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{i\omega |x-x_0|/c} \zeta(x_0)u(x_0) dx_0 \\
 & - \int_{-\frac{1}{2}a}^{\frac{1}{2}a} u^-(x)\zeta(x)u(x) dx \quad (4.5.24)
 \end{aligned}$$

If we regard u and u^- as separate but related functions, we see that a requirement that the change in V , because of a variation in u^- , be stationary (i.e., be second order in the variation) is equivalent to requiring that u satisfy Eq. (4.5.13); whereas the requirement that V be stationary to variation of the form of u is equivalent to stating that u^- be a solution of Eq. (4.5.23). If we set $u(\lambda, x)$ as a trial function with a shape we hope is close to that of Y , and also devise an adjoint $u^-(\lambda, x)$ for every value of λ , then the value λ_s , for which $dV/d\lambda = 0$, is the "best" value of λ , the corresponding value of V should be quite close to the exact value $2A^2\Phi(-\omega)$, and the corresponding functions $u(\lambda_s, x)$ and $u^-(\lambda_s, x)$ should be fairly close to the correct forms $Y(x)$ and $Y^-(x)$.

For example, we could guess that the general form of the solution in the region $-\frac{1}{2}a < x < +\frac{1}{2}a$ resembles the function $\chi(x)$, but not be sure about its amplitude. In this case we could assume the trial function $u(x)$ to be $\lambda\chi(x)$, and work out that the adjoint function is $\lambda\chi^*(x)$ (because of the symmetry of the problem, u^- should be the complex conjugate of u in the range $-\frac{1}{2}a < x < +\frac{1}{2}a$), λ , the amplitude, being the variational parameter. Then

$$V = 2\lambda AC - \lambda^2(D + G)$$

where

$$\begin{aligned}
 D &= \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \chi(x)\zeta(x)\chi^*(x) dx \\
 C &= \frac{1}{2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta(x)\chi(x)e^{-i\omega x/c} dx + \frac{1}{2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta(x)\chi^*(x)e^{i\omega x/c} dx
 \end{aligned}$$

and

$$\begin{aligned}
 G &= \frac{1}{2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta\chi dx \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{i\omega |x-x_0|/c} \zeta\chi^* dx_0 \\
 &= \frac{1}{2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta\chi^* dx \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{i\omega |x-x_0|/c} \zeta\chi dx_0
 \end{aligned}$$

Setting $dV/d\lambda = 0$ results in a value for λ and a "best" approximation for Y ,

Y^- , $\Phi(\omega)$, and $\Phi(-\omega)$ for this trial function.

$$\begin{aligned}\lambda_s &= \frac{AC}{D+G} & Y &\simeq \lambda_s \chi & Y^- &\simeq \lambda_s \chi^* \\ \Phi(-\omega) &\simeq \frac{1}{2A^2} V_s = \frac{C^2}{2(D+G)} \\ \Phi(\omega) &= \frac{1}{2A} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{i\omega x/c} \zeta Y dx \simeq \frac{CE}{2(D+G)} & E &= \int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta \chi e^{i\omega x/c} dx\end{aligned}\quad (4.5.25)$$

where we have inserted the “best” form for Y in Eq. (4.5.14) to obtain $\Phi(\omega)$. Since $\lambda_s \chi$ differs by first order in ϵ from Y (see page 157), this expression for $\Phi(\omega)$ is not as accurate as the one for $\Phi(-\omega)$, which is the stationary value for V , and thus is good to second order in ϵ . A value of $\Phi(\omega)$ of the same accuracy as that for $\Phi(-\omega)$ can be obtained by multiplying Eq. (4.5.13) by $(1/2A^2)\zeta Y$ and integrating.

$$\Phi(\omega) = \frac{1}{2A^2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} Y^2 \zeta dx + \frac{1}{4A^2} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} Y \zeta dx \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{i\omega |x-x_0|/c} \zeta Y dx_0$$

Inserting $\lambda_s \chi$ for Y in this, we obtain

$$\Phi(\omega) \simeq \frac{C^2}{2(D+G)^2} \left(\int \chi^2 \zeta dx + \frac{1}{2} \int \zeta \chi dx \int e^{i\omega |x-x_0|/c} \chi \zeta dx_0 \right) \quad (4.5.26)$$

which is of the same order of approximation as the formula for V .

More complex trial functions, with more than one parameter, can be used, of course. One could try $u = \lambda \chi(x) + \mu \psi(x)$, or have another parameter implicit in χ . The “best” values are then determined by requiring that $\partial V / \partial \lambda = 0$ and $\partial V / \partial \mu = 0$ and solving the resulting simultaneous equations.

An example

To indicate the range of validity of the various approximate formulas we have presented here, we should consider a case which can be solved exactly, so as to compare exact with various approximate results. Suppose the string, which extends from $-\infty$ to $+\infty$, is covered with a uniform load having mechanical impedance $z(x) = \epsilon c \zeta$ per unit length in the range $-\frac{1}{2}a < x < +\frac{1}{2}a$, and zero elsewhere. The equation of motion (4.5.11) of the string is thus

$$\frac{d^2 Y}{dx^2} + \left(\frac{\omega}{c}\right)^2 Y = \begin{cases} 0 & x < -\frac{1}{2}a \\ -i \frac{\omega}{c} \zeta Y & -\frac{1}{2}a < x < +\frac{1}{2}a \\ 0 & x > +\frac{1}{2}a \end{cases}$$

which can be solved exactly by joining solutions, valid for each region, at the boundaries $x = -\frac{1}{2}a$ and $x = +\frac{1}{2}a$. For a solution representing an incident

wave coming from $-\infty$, being reflected in part and in part being transmitted through the load, on to $+\infty$, the three solutions must have the following form:

$$y(x) = \begin{cases} Ae^{i\omega x/c} + Re^{-i\omega x/c} & x < -\frac{1}{2}a \\ Be^{i\beta\omega x/c} + Ce^{-i\beta\omega x/c} & -\frac{1}{2}a < x < +\frac{1}{2}a \\ Pe^{i\omega x/c} & x > +\frac{1}{2}a \end{cases}$$

where $\beta^2 = 1 + i(c\zeta/\omega)$. Since there are no concentrated loads anywhere, both the string displacement and its slope must be continuous everywhere. Thus the three solutions must match values and slopes at their joining points, which suffices to determine R , B , C , and P in terms of A , the incident amplitude.

$$\begin{aligned} R &= -2\zeta a \frac{A \sin(\beta\omega a/c)}{Q \omega a/c} e^{-i\omega x a/c} = -A\Phi(\omega) \\ P &= 4\beta \frac{A}{Q} e^{-i\omega a/c} = A[1 - \Phi(-\omega)] \quad \beta^2 = 1 + i \frac{c\zeta}{\omega} \quad (4.5.27) \\ B &= 2(\beta + 1) \frac{A}{Q} \quad C = 2(\beta - 1) \frac{A}{Q} \\ Q &= (\beta + 1)^2 e^{-i\beta\omega a/c} - (\beta - 1)^2 e^{i\beta\omega a/c} \end{aligned}$$

For purposes of comparison with various approximate solutions, we write the expansion, in powers of ζ , to second order, of the expressions for $\Phi(\omega)$ and $\Phi(-\omega)$ and their limiting values, as $\omega a/c$ goes to zero.

$$\Phi(\omega) = -\frac{R}{A} \simeq \frac{c}{2\omega} \frac{\sin(\omega a/c) + \frac{1}{2}i\zeta a \cos(\omega a/c)}{1 + \frac{1}{2}\zeta a + \frac{1}{2}i(\zeta c/\omega)} \rightarrow \frac{\zeta a}{2 + \zeta a} \quad \frac{\omega a}{c} \rightarrow 0 \quad (4.5.28)$$

$$\begin{aligned} \Phi(-\omega) &= 1 - \frac{P}{A} \simeq \frac{1}{2}\zeta a \frac{1 + \frac{1}{4}\zeta a + \frac{1}{4}i(\zeta c/\omega) + \frac{1}{8}(\zeta c^2/a\omega^2)(e^{2i\omega a/c} - 1)}{1 + \frac{1}{2}\zeta a + \frac{1}{2}i(\zeta c/\omega)} \\ &\rightarrow \frac{\zeta a}{2 + \zeta a} \quad \frac{\omega a}{c} \rightarrow 0 \end{aligned}$$

The ratios $|R/A|^2 = |\Phi(\omega)|^2$ and $|P/A|^2 = |1 - \Phi(-\omega)|^2$, the ratios between the power reflected and transmitted by the load to the incident power, may be called the reflection and transmission ratios, respectively. If $z = \epsilon c\zeta$ is purely reactive, the sum of the ratios will equal unity; no energy is lost to the load in that case.

Both ratios oscillate as $\beta\omega a/c$ increases, because of interference between reflections at the discontinuities at $-\frac{1}{2}a$ and $+\frac{1}{2}a$. If z has a resistive part, then β is complex and the sum of the two ratios will be less than unity. The solid lines of Fig. 4.20 are plots of the reflection ratio for the simple case of pure mass loading, $z = -i\omega\mu\epsilon$, $\zeta = -i\mu(\omega/c)$, the mass density of the string being $(1 + \mu)\epsilon$ in the range $-\frac{1}{2}a < x < +\frac{1}{2}a$ and ϵ elsewhere. One curve

shows the dependence of the reflection ratio on μ , the mass parameter; the other curve displays its dependence on the frequency parameter $\eta = \omega a/c = 2\pi a/\lambda$, where λ is the wavelength of the waves in the unloaded part of the string. This curve shows the effect of interference by dropping to zero at $\beta\omega a/c = \pi$, or $\eta\sqrt{1+\mu} = \pi$.

This exact solution is now to be compared with the results of various approximate solutions. The Born approximation, for example, consists in inserting the incident wave $Ae^{i\omega x/c}$ into the integral of Eq. (4.5.13) and thus

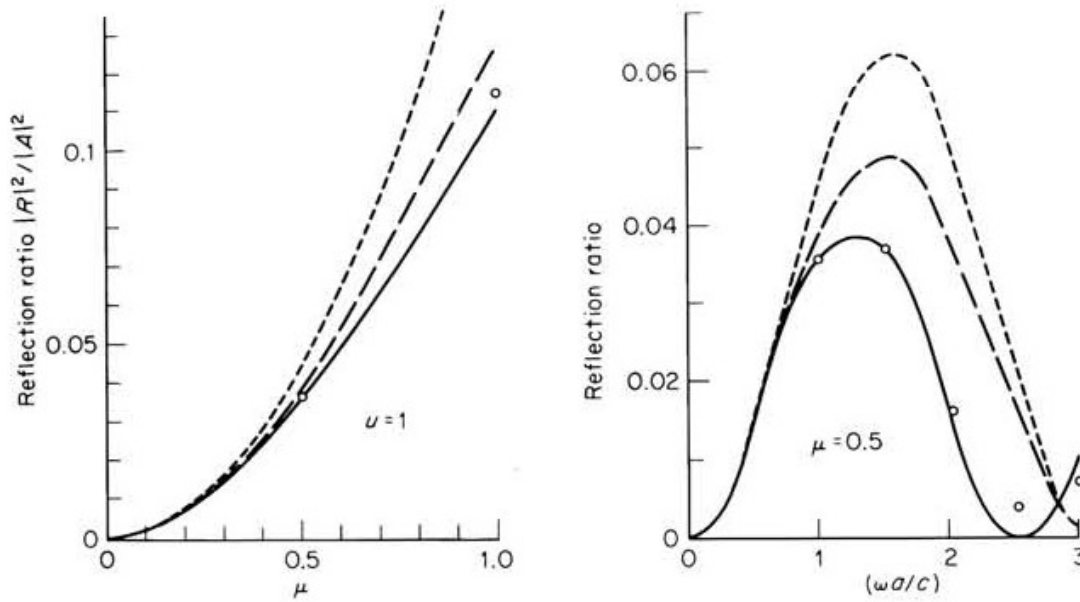


FIGURE 4.20

Ratio of power reflected to incident power for wave incident on an added mass load of density $\mu\epsilon$ over length a of string, as function of μ for $\eta = 1$ and of $\eta = 2\pi a/\lambda$ for $\mu = 0.5$. Solid lines are exact solution; dashed lines are first-order variational solution of Eq. (4.5.31); dotted lines are Born approximation; and circles are values from Eq. (4.5.32).

into the integral of Eq. (4.5.14) for the reflection factor Φ . For the problem under discussion, the approximations for these factors are

$$\begin{aligned}\Phi(\omega) &\simeq \frac{1}{2}\zeta \int_{-\frac{1}{2}a}^{\frac{1}{2}a} e^{2i\omega x/c} dx = \frac{1}{2}\zeta \frac{c}{\omega} \sin \frac{\omega a}{c} \\ \Phi(-\omega) &\simeq \frac{1}{2}\zeta \int_{-\frac{1}{2}a}^{\frac{1}{2}a} dx = \frac{1}{2}\zeta a \quad (\text{Born approximation})\end{aligned}\tag{4.5.29}$$

Comparison with Eqs. (4.5.27) indicates that the Born approximation is a first-order approximation in the quantities ζa and $\zeta c/\omega$. The dotted lines in Fig. 4.20 plot this approximation to $|\Phi(\omega)|^2$ for $\zeta = -i\mu(\omega/c)$, as function of μ and of $\eta = \omega a/c$. It turns out to be a fairly good approximation as long as the mass ratio μ is less than about 0.4 and the frequency parameter η is less

than about 0.6. The interference minimum comes at $\eta = \pi$ instead of $\beta\eta = \pi$ because the approximation neglects the change of wavelength in the loaded portion of the string.

A better approximation can be obtained by using the variational technique just outlined. If we assume a trial function $\chi = e^{i\omega x/c}$, which is again the unperturbed incident wave, but with an amplitude λ adjusted for stationarity of V , we obtain, from Eqs. (4.5.25),

$$\begin{aligned}\chi &= e^{i\omega x/c} & \chi^* &= e^{-i\omega x/c} & C &= D = \zeta a \\ G &= \frac{1}{2}\zeta^2 a^2 \left[1 + i\left(\frac{c}{\omega a}\right) - \frac{1}{2}\left(\frac{c}{\omega a}\right)^2 (e^{2i\omega x/c} - 1) \right] \\ E &= \frac{\zeta c}{\omega} \sin \frac{\omega a}{c}\end{aligned}$$

The "best" expression for the transmission factor, for this choice of trial function, is then

$$\Phi(-\omega) \simeq \frac{\frac{1}{2}\zeta a}{1 + \frac{1}{4}\zeta a + \frac{1}{4}i(\zeta c/\omega) - \frac{1}{8}(\zeta^2 c^2/a\omega^2)(e^{2i\omega a/c} - 1)} \quad (4.5.30)$$

A rearrangement of the second of Eqs. (4.5.28) so that the power series is all in the denominator shows that the variational expression for $\Phi(-\omega)$ agrees with the exact solution through the second-order terms in ζa and $\zeta c/\omega$. Furthermore, it stays finite as $\omega a/c$ goes to zero, approaching $\frac{1}{2}\zeta a/(1 + \frac{1}{2}\zeta a)$ in this limit, as does the exact solution. Thus this approximation is better than the Born approximation.

Formula (4.5.25) for the reflection factor $\Phi(\omega)$ gives

$$\Phi(\omega) \simeq \frac{(\zeta c/2\omega) \sin(\omega a/c)}{1 + \frac{1}{4}\zeta a + \frac{1}{4}i(\zeta c/\omega) - \frac{1}{8}(\zeta^2 c^2/a\omega^2)(e^{2i\omega a/c} - 1)} \quad (4.5.31)$$

The square of this is plotted as a dashed line in Fig. 4.20. As mentioned earlier, it does not check with the exact formula to second order; in particular, the interference minimum comes at $\eta = \pi$ instead of $\beta\eta = \pi$. Nevertheless, this formula fits the exact solution better than the Born approximation, and is satisfactory for η less than about 1 and μ less than about 0.8.

The more appropriate expression for the reflection factor is, of course, Eq. (4.5.26), which uses the integral equation to improve the shape of the strong displacement in the loaded region, before inserting it into the integral for $\Phi(\omega)$. In the present case we have

$$\Phi(\omega) \simeq \frac{\zeta c}{2\omega} \frac{\sin(\omega a/c) + \frac{1}{2}i\zeta a e^{-i\omega a/c}}{[1 + \frac{1}{4}\zeta a + \frac{1}{4}i(\zeta c/\omega) - \frac{1}{8}(\zeta^2 c^2/a\omega^2)(e^{2i\omega a/c} - 1)]^2} \quad (4.5.32)$$

which does check with Eq. (4.5.28) to second order. The circles in Fig. 4.20 correspond to this formula. The fit is excellent for μ and η less than 2. The formula greatly improves the interference effect; it has its minimum near $\beta\eta = \pi$, though the minimum is not zero.

Of course, if we had chosen better trial functions χ and χ^* to insert in Eqs. (4.5.25) and (4.5.26), for example, if we had let $\chi = e^{i\beta\omega x/c}$ and $\chi^* = e^{-i\beta\omega x/c}$, to allow for the difference in wavelength between the loaded and unloaded parts of the string, the resulting formulas would have produced curves which would be impossible to distinguish from the solid lines of Fig. 4.20, over the range there plotted. And of course, if we set $u = \lambda e^{i\beta\omega x/c} + \sigma e^{-i\beta\omega x/c}$ and $u^- = \lambda e^{-i\beta\omega x/c} + \sigma e^{i\beta\omega x/c}$ in Eq. (4.5.24) and had varied both λ and σ to find the stationary value of V , the resulting "best" form will be the exact solution.

Distortion of standing waves

When an extra load is attached to a string of finite length, stretched between rigid supports, the load changes the frequencies of free vibration of the string and also changes the shape of the standing waves of free vibration. If the string supports are at $x = 0$ and $x = l$ and the load is an impedance Z , attached at the point x_0 , then the string, when it is oscillating at frequency $\omega/2\pi$, experiences a reaction force

$$-Z(\partial y/\partial t)_{x=x_0} \delta(x - x_0) = i\omega Z Y(x) e^{-i\omega t} \delta(x - x_0)$$

where $y(x, t) = Y(x)e^{-i\omega t}$ for simple-harmonic motion. Reference to Eq. (5.1.14) indicates that the shape of the string will be

$$Y(x) = \frac{iZ}{\epsilon c} \frac{Y(x_0)}{\sin(\omega l/c)} G(x | x_0)$$

where

$$G(x | x_0) = \begin{cases} \sin \frac{\omega x}{c} \sin \left[\frac{\omega}{c} (l - x_0) \right] & 0 < x < x_0 < l \\ \sin \frac{\omega x_0}{c} \sin \left[\frac{\omega}{c} (l - x) \right] & 0 < x_0 < x < l \end{cases} \quad (4.5.33)$$

This is, of course, an equation from which both shape and allowed frequencies can be determined. Setting $x = x_0$ provides the equation for the resonance frequencies.

$$\sin \frac{\omega l}{c} = \frac{iZ}{\epsilon c} \sin \frac{\omega}{c} x_0 \sin \frac{\omega}{c} (l - x_0)$$

or

$$\tan \frac{\omega l}{c} = \frac{-(iZ/\epsilon c) \sin^2(\omega x_0/c)}{1 - (iZ/2\epsilon c) \sin(2\omega x_0/c)} \quad (4.5.34)$$

This can usually be solved by successive approximations, inserting a trial value of ω in the right-hand side and computing a better value by taking the inverse tangent of the result. If the load is a mass m , so that $Z = -i\omega m$, and if $m > \epsilon l$, then the lowest mode of oscillation has a natural frequency considerably lower than that of the fundamental, $\omega_1 = \pi c/l$, of the unloaded

string. Here we can assume that $\omega l/c \ll 1$, and the equation becomes, to the first approximation in the small quantity $\omega l/c$,

$$\frac{\omega l}{c} \simeq \frac{-(\omega m/\epsilon c)(\omega x_0/c)^2}{1 - (\omega m/2\epsilon c)(2\omega x_0/c)} \quad \text{or} \quad 1 \simeq \frac{m}{\epsilon} \left(\frac{\omega}{c}\right)^2 x_0 \left(1 - \frac{x_0}{l}\right)$$

or

$$\omega_0 \simeq \sqrt{\frac{T}{mx_0} \left(1 - \frac{x_0}{l}\right)^{-1}} \quad \text{where} \quad T = \epsilon c^2$$

This would be an exact expression for a mass load on a string of negligible mass, as a simple analysis will demonstrate. To this approximation the shape of the string is

$$Y(x) \simeq \begin{cases} A \frac{x}{x_0} & 0 < x_0 < x < l \\ A \frac{l-x}{l-x_0} & 0 < x < x_0 < l \end{cases}$$

A better approximation can be found by inserting the expression for ω_0 into the right-hand side of (4.5.34) and then finding the value of ω such that $\tan(\omega l/c)$ equals this right-hand side.

As a second example, we consider the case where the load is resistive, $Z = R = \epsilon c \rho$, where $\rho \ll 1$. Since the right-hand side of Eq. (4.5.34) is small in this case, the quantity $\omega l/c$ must be close to an integral multiple of π for $\tan(\omega l/c)$ to be small. We let $\omega l/c = \pi n + u_n$, where $u_n \ll 1$. Then, to the second order in the small quantities u_n and ρ , we have

$$u_n \simeq \frac{-i\rho[\sin^2(\pi n x_0/l) + 2(u_n x_0/l) \cos(\pi n x_0/l) \sin(\pi n x_0/l)]}{1 - i\rho \sin(\pi n x_0/l) \cos(\pi n x_0/l)}$$

Setting the first-order solution $u_n \simeq -i\rho \sin^2(\pi n x_0/l)$ into the second term in the numerator, we eventually obtain, to second order,

$$\omega_n \simeq \frac{\pi n c}{l} - \rho^2 \left(\frac{c}{l}\right) \left(2 \frac{x_0}{l} - 1\right) \sin^3 \frac{\pi n x_0}{l} \cos \left(\frac{\pi n x_0}{l}\right) - i\rho \left(\frac{c}{l}\right) \sin^2 \frac{\pi n x_0}{l} \quad (4.5.35)$$

for the allowed values of angular frequency ω_n . Once ω_n is determined, the corresponding shape of the string is obtained from Eq. (4.5.33), with the value of $Y(x_0)$ taking the part of an amplitude factor. We note that the real part of ω is affected by the load (though only to the second order in ρ) as well as the imaginary part, the damping constant of the free vibration. Whenever the load is attached at a node of the standing wave (i.e., whenever $\pi n x_0/l$ is an integer times π), the load has no effect on the motion.

Effect of a distributed load

If the inhomogeneity is distributed along a length of the finite string so that the equation of motion is given by (4.5.11), this equation can be changed into an integral equation, as was done for the infinite string. The integral equation for the motion of the finite string differs from Eq. (4.5.13) for the infinite string, however. Here there is no incident wave; so the integral equation is homogeneous; also, the Green's function (4.4.14) for a finite string between rigid supports differs from the Green's function for an infinite string.

We use the method outlined between Eqs. (4.5.11) and (4.5.13) to obtain the equation. The shape of the string, which is to be determined, is a solution of Eq. (4.5.11), going to zero at $x = 0$ and $x = l$. The solution of the auxiliary equation (4.4.9) which satisfies the same boundary conditions is the Green's function of Eq. (4.4.14). As before, we multiply Eq. (4.5.11) by g and subtract from it Eq. (4.4.9) multiplied by Y , interchange x and x_0 , and integrate over x_0 from 0 to l . In this case the terms $Y(dg/dx_0) - g(dY/dx_0)$ go to zero at 0 and l ; so the result is simply

$$\begin{aligned} Y(x) &= \frac{i\omega}{c} \int_0^l \zeta(x_0) g(x | x_0) Y(x_0) dx_0 \\ &= \frac{i}{\sin(\omega l/c)} \int_0^l \zeta(x_0) Y(x_0) G(x | x_0) dx_0 \end{aligned} \quad (4.5.36)$$

where

$$G(x | x_0) = \begin{cases} \sin \frac{\omega x}{c} \sin \left[\frac{\omega}{c} (l - x_0) \right] & 0 < x < x_0 < l \\ \sin \frac{\omega x_0}{c} \sin \left[\frac{\omega}{c} (l - x) \right] & 0 < x_0 < x < l \end{cases}$$

and where $\zeta(x)$ is the impedance density ratio $z(x)/\epsilon c$.

This equation is quite different in form from Eq. (4.5.13), yet both equations are for a solution of the same differential equation (4.5.11); the difference in the two lies in the boundary conditions, which are included in the integral equation. Solutions of (4.5.36) occur only for a discrete set of values of $\omega l/c$, which are $2\pi l/c$ times the allowed frequencies of oscillation of string-plus-load. Solutions may be obtained by successive approximations or by a variational procedure, as before.

For example, we can multiply the second form of Eq. (4.5.36) by $Y(x)\zeta(x)$ and integrate over x , obtaining a symmetric expression for the resonance factor $\sin(\omega l/c)$.

$$\sin \frac{\omega l}{c} = i \frac{\int_0^l Y(x) \zeta(x) dx \int_0^l G(x, x_0) \zeta(x_0) Y(x_0) dx_0}{\int_0^l Y(x) \zeta(x) Y(x) dx} \quad (4.5.37)$$

which is appropriate for a variational calculation of ω . Just as before, we can show that, if an approximate form is used in the integrals for $Y(x)$, the computed value of the fraction is a better approximation to the correct value of $\sin(\omega l/c)$ than is the approximation for Y . Also, if the form for Y has parameters $\alpha, \dots$ which change its form, then the form, for which the partials of the fraction with respect to $\alpha, \dots$ are all zero, comes closest to the correct form of Y , and the corresponding value of the fraction is closest to the correct value of $\sin(\omega l/c)$.

To show how the technique works, we take the case where

$$\zeta(x) = \begin{cases} 0 & 0 < x < a \\ \frac{Z}{\epsilon c(b-a)} & a < x < b \\ 0 & b < x < l \end{cases}$$

and use the crudest approximation possible for the shape of Y between a and b , namely, $Y = A$. The denominator of the fraction is then, simply, $A^2(Z/\epsilon c)$, and Eq. (4.5.37) becomes

$$\begin{aligned} \sin \frac{\omega l}{c} \simeq & -\frac{iZ}{\epsilon c} \frac{c}{\omega(b-a)} \left\{ \sin \left(\frac{\omega l}{c} \right) \left[1 - \frac{c}{\omega(b-a)} \left(2 \cos \frac{\omega a}{c} \sin \frac{\omega b}{c} \right. \right. \right. \\ & \left. \left. \left. - \cos \frac{\omega a}{c} \sin \frac{\omega a}{c} - \cos \frac{\omega b}{c} \sin \frac{\omega b}{c} \right) \right] \right. \\ & \left. \left. - \cos \left(\frac{\omega l}{c} \right) \frac{c}{\omega(b-a)} \left(\cos \frac{\omega a}{c} - \cos \frac{\omega b}{c} \right)^2 \right\} \right. \end{aligned}$$

or

$$\begin{aligned} \tan \frac{\omega l}{c} & \simeq \frac{-\left(\frac{iZ}{\epsilon c}\right) \left[\frac{c}{\omega(b-a)}\right]^2 \left[2 \sin \frac{\omega}{2c} (a+b) \sin \frac{\omega}{2c} (b-a)\right]^2}{1 - \left(\frac{iZ}{\epsilon c}\right) \frac{c}{\omega(b-a)} \left\{ \sin \frac{\omega}{c} (b-a) + \sin \frac{\omega}{c} (b+a) \left[1 - \cos \frac{\omega}{c} (b-a)\right] \right\}} \end{aligned} \quad (4.5.38)$$

When $b-a \rightarrow 0$ and $b+a \rightarrow 2x_0$, this formula reduces to formula (4.5.34), the exact expression for a concentrated load at x_0 . When $b-a$ is not small, the formula is only an approximate one. A better approximation for the n th natural frequency could be obtained by setting $Y = A \sin(\pi n x/l)$ in Eq. (4.5.37).

Fourier series expansions

An alternative way of calculating the effect of an impedance load $z(x) = \epsilon c \zeta(x)$ on the free vibrations of a finite string is to expand Eq. (4.5.36) in a Fourier series. To do this we must find an expansion of the Green's