function $g(x \mid x_0)$ of Eq. (4.4.14) in a Fourier series. But g satisfies the equation of motion (4.4.9),

$$\frac{d^2g}{dx^2} + \left(\frac{\omega}{c}\right)^2 g = -\delta(x - x_0)$$

The boundary conditions on g (and on Y) are that they go to zero at $x = 0$ and $x = l$. According to the arguments leading up to Eq. (4.3.13), it must be expressible in terms of a Fourier series,

$$g(x \mid x_0) = \sum_{n=0}^{\infty} B_n \sin \frac{\pi n x}{l}$$

Inserting this in the equation for g, we have

$$\sum_{m=0}^{\infty} B_m \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi m}{l}\right)^2\right] \sin \frac{\pi m x}{l} = -\delta(x - x_0)$$

Multiplying both sides by $\sin(\pi n x/l)$ and integrating, utilizing Eqs. (4.3.11) and (1.3.24), we find that

$$\frac{l}{2}\left[\left(\frac{\pi n}{l}\right)^2 - \left(\frac{\omega}{c}\right)^2\right] B_n = \sin \frac{\pi n x_0}{l}$$

so that the Green's function of Eq. (4.4.14) equals the Fourier series.

$$G(x \mid x_0) = 2l \sum_{n=0}^{\infty} \frac{\sin(\pi n x_0/l) \sin(\pi n x/l)}{(\pi n)^2 - (\omega l/c)^2}$$

This form again demonstrates the fact that G is symmetric to interchange of x and x_0, thus satisfying the principle of reciprocity.

The series can be inserted in the integral equation (4.5.36), giving

$$Y(x) = 2i \frac{\omega l}{c} \sum_{n=0}^{\infty} \frac{\sin(\pi n x/l)}{(\pi n)^2 - (\omega l/c)^2} \int_0^l \zeta(x)\, Y(x) \sin \frac{\pi n x}{l}\, dx \qquad (4.5.39)$$

This equation can be modified to obtain the allowed frequencies of free vibration, and also to obtain the series expansion for the shapes of the corresponding standing waves of the loaded string.

For example, if $\zeta l \ll 1$, the load cannot distort the shape very much; so the allowed values of ω must be close to the values $\pi n c/l$ which hold for an unloaded string. Therefore, for the mth allowed frequency, $\omega l/c$ must be close to πm in value, and thus the mth term in the series for Y must be very much larger than any of the others. Consequently, $Y_m(x)$ must be approximately equal to $A \sin(\pi m x/l)$, and Eq. (4.5.39) is approximately equivalent to

$$A \sin \frac{\pi m x}{l} \simeq 2i \frac{\omega l}{c} \frac{\sin(\pi m x/l)}{(\pi m)^2 - (\omega l/c)^2} \int_0^l \zeta(x) A \sin^2 \frac{\pi m x}{l}\, dx$$

which can be rearranged into an equation for the value $\omega_m/2\pi$ of the mth

natural frequency, with

$$\omega_m{}^2 \simeq \left(\frac{\pi mc}{l}\right)^2 - 2i\frac{\omega_m c}{l} D_{mm}$$

or

$$\frac{\omega_m l}{c} \simeq \pi m - iD_{mm} - \frac{(D_{mm})^2}{2\pi m} - \cdots$$

where

$$D_{nm} = \int_0^l \sin\left(\frac{\pi nx}{l}\right)\zeta(x)\sin\frac{\pi mx}{l}\,dx \tag{4.5.40}$$

With these first approximations, we can go on to higher approximations. Inserting $Y \simeq A\sin(\pi mx/l)$ and (4.5.40) into the right-hand side of (4.5.39), we obtain

$$Y_m(x) \simeq A\left[\sin\frac{\pi mx}{l} + 2i\pi m\sum_{n\neq m}\frac{\sin(\pi nx/l)}{(\pi n)^2 - (\pi m)^2} D_{nm}\right]$$

Inserting this for $Y_m(x)$ on the left-hand side of (4.5.39) and $Y_m(x) = A\sin(\pi mx/l)$ in all but the $n = m$ term in the right-hand series, we obtain

$$A\sin\frac{\pi mx}{l} \simeq 2i\frac{\omega l}{c}\frac{\sin(\pi mx/l)}{(\pi m)^2 - (\omega l/c)^2}\int_0^l \sin\frac{\pi mx}{l}\,\zeta(x)\,Y_m(x)\,dx$$

which allows us to calculate the next approximation for ω_n.

$$\left(\frac{\omega_n l}{c}\right)^2 \simeq (\pi n)^2 - 2i\frac{\omega_n l}{c}\left[D_{nn} + 2i(\pi n)\sum_{m\neq n}\frac{D_{nm}D_{mn}}{(\pi m)^2 - (\pi n)^2}\right]$$

or

$$\frac{\omega_n l}{c} \simeq \pi n - iD_{nn} - \frac{1}{2\pi n}(D_{nn})^2 - 2\pi n\sum_{m\neq n}\frac{D_{mn}D_{nm}}{(\pi n)^2 - (\pi m)^2} \tag{4.5.41}$$

Returning again to the load $\zeta(x)$, which is equal to $Z/\epsilon c(b-a)$ between $x = a$ and $x = b$ and is zero elsewhere, the matrix components are

$$D_{nn} = \frac{Z}{2\epsilon c}\left[1 - \frac{l}{b-a}\frac{\sin(2n\beta) - \sin(2n\alpha)}{2\pi n}\right] \qquad \alpha = \frac{\pi a}{l};\quad \beta = \frac{\pi b}{l}$$

$$D_{mn} = \frac{Z}{\epsilon c}\frac{l}{b-a}\left[\frac{\sin\beta(m-n) - \sin\alpha(m-n)}{2\pi(m-n)} - \frac{\sin\beta(m+n) - \sin\alpha(m+n)}{2\pi(m+n)}\right]$$

so that, to the first approximation,

$$\frac{\omega_m l}{c} \simeq \pi m - \frac{iZ}{2\epsilon c}\left[1 - \frac{l}{b-a}\frac{\sin(2m\beta) - \sin(2m\alpha)}{2\pi m}\right]$$

$$\to \pi m - \frac{iZ}{\epsilon c}\sin^2\frac{\pi ma}{l} \qquad b - a \ll l$$

The limiting form for $b - a \ll l$ agrees with the first-order-approximation solution of Eq. (4.5.34). If the load is masslike ($Z = -i\omega m$), each natural frequency is reduced, and the higher allowed frequencies are no longer integral multiples of the fundamental. Introduction of a resistance brings an imaginary part to ω_n, which corresponds to an exponential damping of the vibrations.

Problems

1 A string clamped at one end is struck at a point a distance D from the clamp by a hammer of width $D/4$. The head of the hammer is shaped so that the initial velocity given the string is maximum at the center of the head and is zero at the edge, the initial "velocity shape" of this portion being like an inverted V. Plot the shapes of the string at the times $t = 0$, $D/2c$, D/c, $3D/2c$, $2D/c$. Draw a curve showing the vertical component of the force on the clamp as a function of time.

2 A harp string is plucked so that its initial velocity is zero and its initial shape is

$$y_0(x) = \begin{cases} \dfrac{20h}{9l}\,x & 0 < x < \dfrac{9l}{20} \\ \dfrac{20h}{l}\left(\dfrac{l}{2} - x\right) & \dfrac{9l}{20} < x < \dfrac{11l}{20} \\ \dfrac{20h}{9l}\,(x - l) & \dfrac{11l}{20} < x < l \end{cases}$$

Plot the successive shapes of the string during one cycle of the motion. Draw a curve showing the vertical component of the force on one of the supports as a function of time.

3 What are the total energies of the two strings shown in Fig. 4.11?

4 An infinite string with tension T and wave velocity c is subjected to a transient transverse force of the form

$$f(x,t) = \begin{cases} A \sin\dfrac{\pi t}{\tau} \cos\dfrac{\pi x}{L} & 0 < t < \tau;\ -\tfrac{1}{2}L < x < \tfrac{1}{2}L \\ 0 & \text{otherwise} \end{cases}$$

Show that the resulting motion of the string is given by

$$y(x,t) = A\,\frac{c\tau}{\pi T}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} \cos\frac{\pi x_0}{L}\; Y\left(t - \frac{x - x_0}{c}\right) dx_0$$

where

$$Y(z) = \begin{cases} 0 & z < 0 \\ \sin^2\dfrac{\pi z}{2T} & 0 < z < \tau \\ 1 & z > \tau \end{cases}$$

5 A uniform string with no friction is stretched between rigid supports a distance l apart. It is driven by a force $F_0 e^{-i\omega t}$ concentrated at its midpoint. Show that the amplitude of motion of the midpoint is $(F_0/2\epsilon c\omega)\tan(\omega l/2c)$. What is the amplitude of motion of the point $x = l/4$?

6 A uniform string of small electric resistance is stretched between rigid supports a distance l apart, in a uniform magnetic field of B webers per sq m perpendicular to the string. A current $I_0e^{-i\omega t}$ amp is sent through the string; what is the force on the string per unit length? Show that the velocity shape of the string (assume zero friction) is

$$v = \frac{BI_0e^{-i\omega t}}{i\omega\epsilon}\left\{\frac{\cos\left[(\omega/c)(x - \frac{1}{2}l)\right]}{\cos(\omega l/2c)} - 1\right\} \text{ m per sec}$$

Use the formula $E = \int Bv\,dx$ volts to compute the motional emf induced in the string by the motion.

7 A condenser is discharged through the string of Prob. 6, producing a current of $\delta(t)$ amp. Compute the subsequent shape of the string.

8 The string of Prob. 5 is acted on by a force $F(\xi)u(t)$ concentrated at the point $x = \xi$ [see Eq. (1.2.16) for a definition of u, and Eq. (2.3.10) for its use]. Show that the shape of the string after $t = 0$ is

$$y_u(\xi,x) = \frac{lF(\xi)}{\pi^2\epsilon c^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{\pi n\xi}{l}\sin\left(\frac{\pi nx}{l}\right)\left(1 - \cos\frac{\pi nct}{l}\right)$$

From this formula compute the shape of the string when it has been subjected to the constant (independent of time) force $F(\xi)$ up to $t = 0$ and then released to vibrate freely.

9 A length D of an infinite string is in a magnetic field B, perpendicular to the string; the rest of the string is not in a field. A current $Ie^{-i\omega t}$ is sent through the string, which is conductive. What is the displacement of the string? Suppose the string is finite in length and is stretched between rigid supports a distance L apart, with the magnetic field centered along its length ($D < L$). What, then, is the displacement of the string if a current $Ie^{-i\omega t}$ is passed through it?

.0 An electrically conducting string of length l and total electric resistance R is clamped at both ends. The ends are connected electrically by a wire of negligible resistance. The string is set into oscillation transverse to a magnetic field $B(x)$, thus generating a current in the string, which in turn gives rise to a force on the string. Show that the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = -\frac{B(x)}{\epsilon R}\int_0^l \frac{\partial y}{\partial t}B(x_0)\,dx_0 \qquad c^2 = \frac{T}{\epsilon}$$

When both y and B are expanded in Fourier series,

$$y = \sum_m y_m(t)\sin\frac{\pi mx}{l} \qquad B(x) = \sum_n B_n\sin\frac{\pi nx}{l}$$

show that the equations for the time-dependent amplitudes are given by the set of coupled equations

(*a*) $$\frac{d^2y_m}{dt^2} = -\left(\frac{\pi mc}{l}\right)^2 y_m - \sum_n a_{mn}\frac{dy_n}{dt}$$

where the mode-coupling coefficients are given by

$$a_{mn} = \frac{l}{2\epsilon R}B_mB_n$$

For the case where $y = Y(x)e^{-i\omega t}$, show that a variational expression

(*b*)
$$V(Y) = \frac{\int_0^l (dY/dx)^2\,dx - (i\omega/TR)\left(\int_0^l BY\,dx\right)^2}{\int_0^l Y^2\,dx}$$

has a stationary value equal to $(\omega/c)^2$, the shape of Y for which V is stationary being the solution of the equation of motion. For $B = B_1 \sin(\pi x/l)$, compute the frequency and damping factor for the lowest resonance, first by using equation (*a*), then by using equation (*b*) and setting $Y = \sin(\pi x/l)$; compare the results. Carry out the same calculations and comparisons, to second order in B_n, for the case when $B(x)$ is constant from $x = 0$ to $x = l$.

11 A string of mass ϵ per unit length and length L is whirled about one end, with angular velocity ω_0, so that the motion is in a plane (neglect gravity). Show that the tension in the string is $\frac{1}{2}\epsilon\omega_0^2(L^2 - x^2)$, where x is the distance from the stationary end. Show that the equation of motion of a string with variable tension $T(x)$ is

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = \epsilon\frac{\partial^2 y}{\partial t^2}$$

and thus that the equation for simple-harmonic motion of the string perpendicular to the plane of rotation is

$$\frac{d}{dz}\left[(1 - z^2)\frac{\partial\Psi}{\partial z}\right] + 2\left(\frac{\omega}{\omega_0}\right)^2\Psi = 0 \qquad \begin{matrix} x = lz \\ y = \Psi(z)e^{-i\omega t}\end{matrix}$$

Show that when $\omega = \omega_0\sqrt{n(2n-1)}$ $(n = 1, 2, 3, \ldots)$ the displacement is proportional to the Legendre function $P_{2n-1}(z)$ [see Eq. (7.2.4)] and that no other frequencies are possible. Is the general motion periodic?

12 A string of length l and mass ϵ g per cm is hung from one end, so that gravitational forces are the only ones acting. Show that if the free end of the string is taken as origin, the normal modes of vibration have the form

$$y = J_0\left(\pi\beta_n\sqrt{\frac{x}{l}}\right)e^{-2\pi i\nu_n t}$$

where $\nu_n = (\beta_n/4)\sqrt{g/l}$; the function $J_0(z)$ is given in Eq. (1.2.5) and further in Eqs. (5.2.21); and the constants β_n are the solutions of the equation $J_0(\pi\beta_n) = 0$ [Eq. (5.2.23)]. Is the motion periodic in general? What are the ratios of the lowest three allowed frequencies to the frequency of oscillation that the string would have if all its mass were concentrated at its lower end?

13 Utilizing the general formulas of Eqs. (4.3.13) and (5.2.26), obtain formulas for the shape of the string of Prob. 12 when driven by a periodic force and when struck by an impulsive force.

14 Choosing suitable values for the properties of the string of Prob. 12, plot the shape of the string when it is vibrating at its lowest three allowed frequencies. Plot its shape when driven at its free end at a frequency 1.5. times its fundamental.

15 The tip of the string of Prob. 11 is struck an impulsive blow $\delta(t)$ in a direction perpendicular to the equilibrium plane of motion. Calculate the series representing the subsequent displacement. What is the amplitude of motion of the lowest mode?

16 A string carrying an electric current J is stretched along the lines of a homogeneous magnetic field B (in the x direction). As a result, the z and y motions of the string will be coupled. (*a*) Obtain the equations of motion describing this coupled motion, for small amplitudes. (*b*) Show that the solution predicts two characteristic modes of motion, representing a right- and left-hand "circularly polarized" motion of the string, with displacements y and z related either as $y = iz$ or $y = -iz$. Calculate the phase velocity for these two modes and show that for small B they are related as $\sqrt{T/\epsilon} - \delta$ and $\sqrt{T/\epsilon} + \delta$. Determine δ.

17 A uniform string of length l and tension T is fastened at $x = l$ to a support having transverse mechanical resistance $R = \epsilon c \tanh(\pi\alpha - \frac{1}{2}i\pi)$ and zero reactance. The string is originally at equilibrium, and the end at $x = 0$ is suddenly acted on by a transverse impulsive force $\delta(t)$. Utilize the Laplace transform to obtain the formula for the shape of the string.

$$y(x,t) = \frac{1}{\pi\epsilon c}\sum_{n=0}^{\infty}\frac{e^{-(\pi\alpha/l)(x+ct)}}{(n+\frac{1}{2})^2+\alpha^2}\left\{(n+\tfrac{1}{2})\sin\left[\frac{\pi}{l}(n+\tfrac{1}{2})(x+ct)\right]\right.$$
$$\left. + \alpha\cos\left[\frac{\pi}{l}(n+\tfrac{1}{2})(x+ct)\right]\right\}$$
$$-\frac{1}{\pi\epsilon c}\sum_{n=0}^{\infty}\frac{e^{-(\pi\alpha/l)(x-ct)}}{(n+\frac{1}{2})^2+\alpha^2}\left\{(n+\tfrac{1}{2})\sin\left[\frac{\pi}{l}(n+\tfrac{1}{2})(x-ct)\right]\right.$$
$$\left. + \alpha\cos\left[\frac{\pi}{l}(n+\tfrac{1}{2})(x-ct)\right]\right\} + \frac{1}{\epsilon c}\tanh(\pi\alpha)$$

18 A string of density 0.1 g per cm is stretched with a tension of 10^5 dynes from a support at one end to a device for producing transverse periodic oscillations at the other end. When the driving frequency has a given value, it is noted that the points of minimum amplitude are 10 cm apart, that the amplitude of motion of the minimum is 0.557 times the amplitude at the maximum, and that the nearest maximum is 6 cm from the support. What is the driving frequency and what is the value of the transverse impedance of the support?

19 A string is stretched between two supports having transverse mechanical resistance R large compared with ϵc, and zero reactance. The string is driven by a periodic force concentrated at $x = \xi$. What are the amplitudes of motion of the end supports, and by how much do they lag behind the driving force?

20 The string of Prob. 19 is struck an impulsive blow at the point $x = \xi$. Compute the subsequent motion of the two end supports.

21 An infinite string has a distribution of additional mass corresponding to the impedance ratio

$$\xi(x) = \begin{cases} \dfrac{Z(x)}{\epsilon c} = i\mu\dfrac{\omega}{c}\cos\dfrac{\pi x}{a} & -\tfrac{1}{2}a < x < \tfrac{1}{2}a \\ 0 & \text{otherwise} \end{cases}$$

By use of Eq. (4.5.17), obtain the Born approximation for the ratio between the amplitude of the wave reflected from the load to the amplitude of the incident wave. By use of Eqs. (4.5.25) and (4.5.26), obtain a variational value of this ratio, using the trial functions $\chi = e^{i\omega x/c}$, $\chi^* = e^{-i\omega x/c}$. Compare the results.

22 Show that an alternative variational expression to Eq. (4.5.37), equivalent to Eq. (4.5.11) plus the boundary conditions $Y = 0$ at $x = 0$ and $x = l$, is

$$V(Y) = \frac{\int_0^l [(dY/dx)^2 - i(\omega/c)\zeta Y^2]\, dx}{\int_0^l Y^2\, dx}$$

where the shape of $Y(x)$ is varied, and the stationary values of V are the allowed values of $(\omega/c)^2$.

23 A string of length l is loaded at its center by a point mass equal to $\frac{1}{2}\epsilon l$, half the total mass of the string, so that $Z = -\frac{1}{2}i\omega\epsilon l$ and $\zeta = -i(\omega l/2c)\,\delta(x - \frac{1}{2}l)$. Use Eq. (4.5.34) to show that the lowest frequency of the system is $\omega_1/2\pi$, where $\omega_1 l/c = 2.1418\ldots$. Plot the shape of the string. Show that the variational formula of Eq. (4.5.37), using $Y = \sin(\pi x/l)$, yields the same value of $\omega_1 l/c$. Show that the variational formula of Prob. 22, again using $Y = \sin(\pi x/l)$ and the same ζ, gives $\omega_1 l/c \simeq 2.22$. Show that the first-approximation formula (4.5.40) also yields $\omega_1 l/c \simeq 2.22$.

24 Integral equation (4.5.39) is homogeneous, so that the amplitude of the displacement $Y(x)$ is undetermined. Suppose we "normalize" this amplitude so that

$$\frac{2i(\omega l/c)}{\pi^2 - (\omega l/c)^2} \int_0^l \zeta Y(x) \sin\frac{\pi x}{l}\, dx = 1$$

in which case there results an inhomogeneous integral equation

$$Y(x) = \sin\left(\frac{\pi x}{l}\right) + 2\left(\frac{\omega l}{c}\right) \sum_{n=2}^{\infty} \frac{U_n(Y)}{(\pi n)^2 - (\omega l/c)^2} \sin\frac{\pi n x}{l}$$

where

$$U_n(Y) = i\int_0^l \zeta Y(x) \sin\frac{\pi n x}{l}\, dx$$

which is particularly useful in computing the standing wave of lowest frequency. (Can a similar equation be obtained for the nth wave?) Show that this equation can be manipulated to yield still another variational form,

$$V(Y) = 2U_1(Y) - i\int_0^l \zeta Y^2(x)\, dx + 2\frac{\omega l}{c} \sum_{n=1}^{\infty} \frac{U_n^2(Y)}{n^2\pi^2 - (\omega l/c)^2}$$

Show that, when Y is varied to make V stationary, the resulting value of V is the correct value of U_1. This can then be inserted in the normalizing equation, which becomes

$$\left(\frac{\omega l}{c}\right)^2 + 2\frac{\omega l}{c} U_1 = \pi^2$$

and which thus will result in the exact value of $\omega l/c$ for the lowest frequency of the system. Show that if a trial function $Y \simeq u(x)$, with one or more parameters, is inserted in the variational expression, setting the derivatives of V with respect to these parameters will produce a "best" form for u and a "best" value for U_1 from which to compute a value of $\omega l/c$ which will be good to second order in the difference between u and Y.

25 For the point load of Prob. 23 ($Z = -\frac{1}{2}i\omega\epsilon l$), use the variational technique of Prob. 24 to determine the lowest eigenvalue $\omega_1 l/c$. Use $u = A \sin(\pi x/l)$ as trial function, and use the relation

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2 - 1} = \tfrac{1}{4}$$

to show that the "best" value of A is $[1 - (\omega l/2\pi c)^2]^{-1}$. Thence, by use of the normalizing equation, show that the "best" value of $\omega_1 l/c$ is 2.145. Compare this with the other solutions discussed in Prob. 23.

26 A string of length l, stretched between rigid supports with tension T, has mass density ϵ between $x = 0$ and $x = \alpha l$, mass density $\beta^2\epsilon$ between $x = \alpha l$ and $x = l$. By joining appropriate sine functions in value and slope at $x = \alpha l$, obtain an implicit equation for the natural frequencies of oscillation. For $\alpha = \frac{1}{2}$ and $\beta = 2$, compute the lowest frequency. Use Eqs. (4.5.37) and (4.5.41) to obtain approximate expressions for this lowest frequency. How close do they come to the exact value?

27 A coil spring has a relaxed length l_0; when it is stretched to length l, the tension in the spring is $K(l - l_0)$. Accounting for the fact that the mass per unit length $\epsilon = m/l$ depends on the stretched length of the spring, show that the ratio between the speed of transverse and longitudinal waves on the spring is

$$\frac{c_t}{c_l} = \sqrt{1 - \frac{l_0}{l}}$$

and hence show that the transverse wave velocity is always less than the longitudinal wave velocity.

28 A string of mass density ϵ, under tension T, extends at equilibrium along the x axis from $x = 0$ to $x = \infty$. The $x = 0$ end is attached to a mass M, which slides on a frictionless guide bar in the y direction, the mass also being attached to a spring with stiffness constant $K = M\omega_0^2$, tending to keep it at $y = 0$.

(*a*) With the system originally in equilibrium, show that, if the mass is given an initial velocity v_0 at $t = 0$, the transverse displacement of the string thereafter is

$$y(x,t) = \frac{v_0}{k_f} \exp\left[-\frac{\epsilon}{2M}(ct - x)\right] \sin[k_f(ct - x)]$$

where

$$k_f^2 = \left(\frac{\omega_0}{c}\right)^2 \left[1 - \left(\frac{\epsilon c}{2M\omega_0}\right)^2\right]$$

(*b*) If an incident step wave $Au(ct + x)$ comes from $+\infty$ along the string, impinging on the mass originally at rest, show that the string displacement is

$$y(x,t) = Au(ct + x) - Au(ct - x) + A\frac{2\epsilon}{Mk_t} \exp\left[-\frac{\epsilon}{2M}(ct - x)\right] \sin[k_f(ct - x)]$$

(*c*) In case *a*, what is the energy in the string after the mass has come to rest?

CHAPTER

5

BARS, MEMBRANES, AND PLATES

5.1 THE FLEXIBLE BAR

Although the primary subject of study of this book is acoustic wave motion in fluids, the first five chapters are concerned with the vibration of solid objects. The reasons for the roundabout approach are twofold. Vibrating systems produce sound waves in the medium surrounding them, or they are affected by sound waves striking them, and any analysis of the waves must take this into account. But also, many computational techniques are more easily explained in connection with vibrating systems than with acoustic waves. Although our discussion of vibration has been lengthy, it has been limited to examples which conform to one or both of these reasons.

This restraint in choice of material will be particularly noticeable in this chapter. A detailed discussion of wave motion in elastic solids would be a lengthier task than the one set for this book. All we can do here is to touch on those aspects of elasticity which will be needed later in discussing sound waves in fluids.

It must have been rather obvious in the preceding chapter that we were analyzing the motions of a somewhat idealized string. In the first place, we assumed that the string was perfectly flexible; that the only restoring force was due to the tension. Second, we made no mention of the possibility of longitudinal motion of alternate compression and tension, which can be set up in any actual string, as well as in any other piece of solid material. This longitudinal wave motion will be disregarded for a while longer; we shall spend the rest of this book discussing it. However, we can no longer put off studying the effect of stiffness on the string's motion. And we shall begin the study by discussing the transverse vibrations of bars.

There is no sharp distinction between what we mean by a bar and what we mean by a string. In general, tension is more important as a restoring force than stiffness for a string, and stiffness is more important for a bar; but there is a complete sequence of intermediate cases from stiff strings to bars under tension. The perfectly flexible string is one limiting case, where

the restoring force due to stiffness is negligible compared with that due to the tension. The rod or bar under no tension is the other limiting case, the restoring force being entirely due to stiffness. The first limiting case was studied in the preceding chapter. The second case, the bar under no tension, will be studied in the first part of this chapter, and the intermediate cases will be dealt with in a later part.

Stresses in a bar

To start with, we shall study the bending of a straight bar, with uniform cross section, symmetrical about a central plane. The motion of the bar is supposed to be perpendicular to this plane, and we shall call the displacement from equilibrium of the plane y. (See Fig. 5.1.)

When the bar is bent, its lower half is compressed and its upper half stretched (or vice versa). This bending requires a moment M, whose relation to the amount of bending we must find. To compress a rod of cross-sectional area S and length l by an amount dl requires a force $QS(dl/l)$, where Q is a constant, called *Young's modulus*. Now, imagine the bar to be a bundle of fibers of cross-sectional area dS, all running parallel to the center plane of the bar. If the bar is bent by an angle ϕ in a length dx, then the fibers which are a distance z down from the center surface (it is no longer a plane now that the bar is bent) will be compressed by a length $z\phi$; the force required to compress each fiber will be $Q\,dS(z\phi/dx)$; and the moment of this force about the center line of the bar's cross section will be $(Q\phi/dx)z^2\,dS$. The total moment of these forces required to compress and to stretch all the fibers in the bar will be

$$M = \frac{Q\phi}{dx}\int z^2\,dS \tag{5.1.1}$$

where the integration is over the whole area of the cross section.

We define a constant κ, such that $\kappa^2 = (1/S)\int z^2\,dS$, where S is the area of the cross section. This constant is called the *radius of gyration* of the

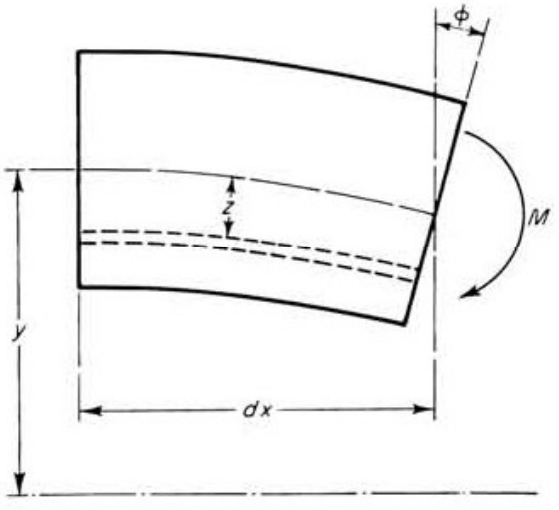


FIGURE 5.1
Moment acting on a bent element of a bar.

cross section, by analogy with the radius of gyration of solids. Its values for some of the simpler cross-sectional shapes are as follows.

Rectangle, length parallel to center line b, width perpendicular to center line a:

$$\kappa = \frac{a}{\sqrt{12}}$$

Circle, of radius a:

$$\kappa = \frac{a}{2}$$

Circular ring, outer radius a, inner radius b:

$$\kappa = \tfrac{1}{2}\sqrt{a^2 + b^2}$$

Bending moments and shearing forces

Equation (5.1.1), giving the moment required to bend a length dx of rod by an angle ϕ, is then

$$M = \frac{Q\phi S\kappa^2}{dx} \tag{5.1.2}$$

If the rod does not bend much, we can say that ϕ is practically equal to the difference between the slopes of the axial line of the rod at the two ends of the element dx:

$$\phi = -\left(\frac{\partial y}{\partial x}\right)_{x+dx} + \left(\frac{\partial y}{\partial x}\right)_x = -dx\left(\frac{\partial^2 y}{\partial x^2}\right)$$

Therefore the bending moment is

$$M = -QS\kappa^2 \frac{\partial^2 y}{\partial x^2} \tag{5.1.3}$$

This bending moment is not the same for every part of the rod; it is a function of x, the distance from one end of the rod. In order to keep the element of bar in equilibrium, we must have the difference in the moments acting on the two ends of the element balanced by a shearing force represented by F in Fig. 5.2 (moment and shear are, of course, two different aspects of the single stress which is acting on the bar). The moment of the shearing force is $F\,dx$, and this must equal dM for equilibrium, which means that

$$F = \frac{\partial M}{\partial x} = -QS\kappa^2 \frac{\partial^3 y}{\partial x^3} \tag{5.1.4}$$

This equation is not exactly true when the bar is vibrating (since a certain part of the moment must be used in getting the element of bar to turn as it bends), but it is very nearly correct when the amplitude of vibration is not large compared with the length of bar.

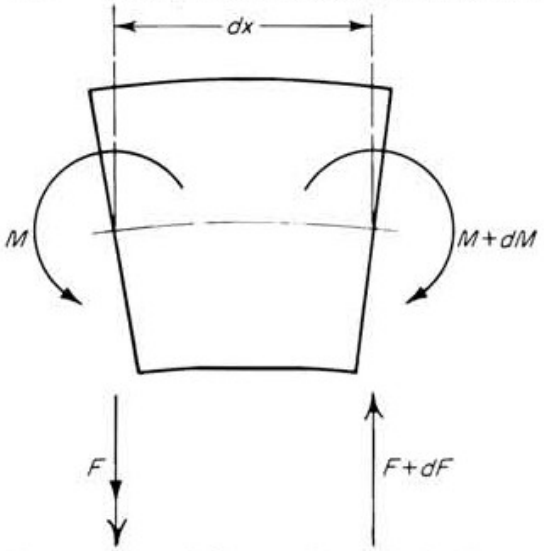


FIGURE 5.2
Bending moments and shearing forces to balance.

Properties of the motion of the bar

The shearing force F is also a function of x and may be different for different ends of the element of bar. This leaves a net force $dF = (\partial F/\partial x)\,dx$ acting on the element, perpendicular to the bar's axis; and this force must equal the element's acceleration times its mass $\rho S\,dx$, where ρ is the density of the material of the rod. Therefore the equation of motion of the bar is $dx\,(\partial F/\partial x) = \rho S\,dx\,(\partial^2 y/\partial t^2)$, or

$$\frac{\partial^4 y}{\partial x^4} = -\frac{\rho}{Q\kappa^2}\frac{\partial^2 y}{\partial t^2} \tag{5.1.5}$$

This equation differs from the wave equation in that it has a fourth derivative with respect to x instead of a second derivative. The general function $F(x + ct)$ is not a solution, so that a bar, satisfying Eq. (5.1.5), cannot have waves traveling along it with constant velocity and unchanged shape. In fact, the term "wave velocity" has no general meaning in this case, although it can be given certain special meanings. For instance, a simple-harmonic solution of Eq. (5.1.5) is

$$y = Ce^{2\pi i(\mu x - \nu t)} = A\cos\left[2\pi(\mu x - \nu t) - \Phi\right] \tag{5.1.6}$$

where
$$\mu = \left(\frac{\nu^2\rho}{4\pi^2 Q\kappa^2}\right)^{\frac{1}{4}} \qquad \nu = 2\pi\mu^2\kappa\sqrt{\frac{Q}{\rho}}$$

This represents a sinusoidal wave traveling in the positive direction, of just the sort one finds on strings. There is an important difference between the two waves, however: in the case of the bar, the velocity of the wave $u = \nu/\mu = (4\pi^2 Q\kappa^2/\rho)^{\frac{1}{4}}\sqrt{\nu}$ *depends on the frequency of oscillation of the wave,* whereas in the case of the string it does not. The velocity of propagation of a simple-harmonic wave is called the *phase velocity*, and is one of the special kinds of "velocity" that have meaning for a bar. In the case of the string, the phase velocity is independent of ν and is equal to the velocity of all waves c. For the bar the phase velocity depends on ν, and there is no

general velocity for all waves. This is analogous to the case of the transmission of light through glass, where light of different frequencies (i.e., of different colors) travels with different velocities and dispersion results. A bar is sometimes said to be a *dispersing* medium for waves of bending.

Wave motion of an infinite bar

Although the phase velocity is not a constant, nevertheless we should be able to build up any sort of wave out of a suitably chosen combination of sinusoidal waves of different frequencies, in a Fourier integral. At present we are considering the bar to be infinite in length, so that all frequencies are allowed, and the sum is an integral

$$y = \int_{-\infty}^{\infty} e^{2\pi i\mu x}[B(\mu)\cos(\gamma^2\mu^2 t) + C(\mu)\sin(\gamma^2\mu^2 t)]\,d\mu \tag{5.1.7}$$

where $\gamma^2 = 4\pi^2\kappa\sqrt{Q/\rho}$, and the functions $B(\mu)$ and $C(\mu)$, corresponding to the coefficients B_n and C_n of a Fourier series, are determined by the initial conditions. This integral is analogous to the Fourier integral of Eq. (1.3.16). If the initial shape and velocity shape of the bar are $y_0(x)$ and $v_0(x)$, Eq. (1.3.16) shows that

$$\begin{aligned} B(\mu) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} y_0(x)e^{-2\pi i\mu x}\,dx \\ C(\mu) &= \frac{1}{4\pi^2\gamma^2\mu^2}\int_{-\infty}^{\infty} v_0(x)e^{-2\pi i\mu x}\,dx \end{aligned} \tag{5.1.8}$$

This may seem a rather roundabout way of obtaining solutions of Eq. (5.1.5) which satisfy given initial conditions; but since the useful functions $F(x - ct)$ and $f(x + ct)$ are not solutions of (5.1.5), there is no other feasible method.

The case $y_0 = e^{-x^2/4a^2}$, $v_0 = 0$, is one that can be integrated, and can be used to show the change in shape of the wave as it traverses the bar. Solving Eqs. (5.1.8) and (5.1.7), we obtain

$$y = \left(1 + \frac{\gamma^4 t^2}{a^4}\right)^{-\frac{1}{4}} e^{-\frac{a^2x^2}{4(a^4+\gamma^4t^2)}} \cos\left[\frac{\gamma^2 t x^2}{4(a^4 + \gamma^4 t^2)} - \tfrac{1}{2}\tan^{-1}\frac{\gamma^2 t}{a^2}\right] \tag{5.1.9}$$

We can now see the utility of Eq. (5.1.7); we can easily see that (5.1.9) is a solution of the equation of motion (5.1.5) by direct substitution; we can also see that it would be exceedingly difficult to obtain (5.1.9) from (5.1.5) directly.

The shape of this function, at successive instants of time, is shown by the solid lines in Fig. 5.3. For comparison, the sequence of dotted lines shows the variation in shape of a flexible string with equal initial conditions. In the case of the string, the two partial waves move outward with unchanged shape. The shape of each "partial wave" for the bar, however, changes

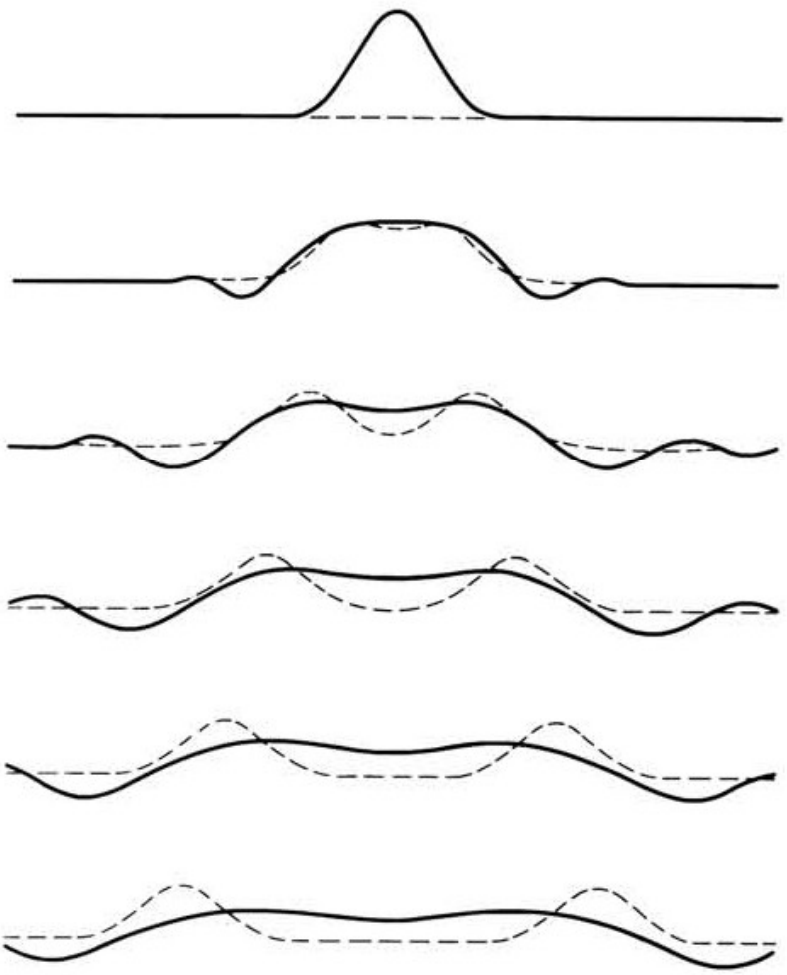


FIGURE 5.3
Comparison between the motion of waves on a bar (solid line) and on a string (dotted line), both of infinite length. Each sequence shows the shapes at successive instants, after starting from rest in the form given at the top of the sequence. The constant γ was chosen to make the average "velocity" of the waves on the bar approximately equal to c, the velocity of the waves on the string.

continuously as the wave travels outward. In particular, notice the formation of subsidiary "ripples" ahead of the principal "crest" of each wave. This is due to the fact, which we have already noted, that the high-frequency short-wavelength parts of the wave spread outward faster than the rest.

Simple-harmonic motion

We cannot pursue our study of the motions of the bar much further without examining its normal modes of vibration. As with the string, we must ask if there are any ways in which the bar can vibrate with simple-harmonic motion. We try setting $y = Y(x)e^{-2\pi ivt}$ and find that Y must satisfy the equation

$$\frac{d^4Y}{dx^4} = 16\pi^4\mu^4 Y \qquad \mu^4 = \frac{\rho v^2}{4\pi^2 Q\kappa^2} = 4\pi^2\frac{v^2}{\gamma^2} \tag{5.1.10}$$

The general solution of this is

$$y = C_1 e^{2\pi\mu x} + C_2 e^{-2\pi\mu x} + C_3 e^{2\pi i\mu x} + C_4 e^{-2\pi i\mu x}$$
$$= a\cosh(2\pi\mu x) + b\sinh(2\pi\mu x) + c\cos(2\pi\mu x) + d\sin(2\pi\mu x) \quad (5.1.11)$$

where $\cosh u = \cos(iu)$ and $\sinh u = -i\sin(iu)$. See Eq. (1.2.10) and Tables I and II.

This general solution satisfies Eq. (5.1.10) for any value of the frequency ν. It is, of course, the boundary conditions that pick out the set of allowed frequencies.

Bar clamped at one end

For example, if we have a bar of length l clamped at one end $x = 0$, the boundary conditions at this end are that *both* y and its slope $\partial y/\partial x$ must be zero at $x = 0$. The particular combination of the general solution (5.1.11) that satisfies these two conditions is the one with $c = -a$ and $d = -b$.

$$Y = a[\cosh(2\pi\mu x) - \cos(2\pi\mu x)] + b[\sinh(2\pi\mu x) - \sin(2\pi\mu x)] \quad (5.1.12)$$

If the other end is free, y and its slope will not be zero, but the bending moment $M = QS\kappa^2(d^2Y/dx^2)$ and the shearing force $F = -QS\kappa^2(d^3Y/dx^3)$ must both be zero, since there is no bar beyond $x = l$ to cause a moment or a shearing stress. We see that *two* conditions must be specified for each end instead of just one, as in the string. This is due to the fact that the equation for Y is a fourth-order differential equation, and its solution involves four arbitrary constants whose relations must be fixed, instead of two for the string. It corresponds to the physical fact that whereas the only internal stress in the string is tension, the bar has two, bending moment and shearing force, each depending in a different way on the deformation of the bar.

The two boundary conditions at $x = l$ can be rewritten as $\dfrac{1}{4\pi^2\mu^2}\dfrac{d^2Y}{dx^2} = 0$ and $\dfrac{1}{8\pi^3\mu^3}\dfrac{d^3Y}{dx^3} = 0$ at $x = l$. Substituting expression (5.1.12) in these, we obtain two equations that fix the relationship between a and b and between μ and l:

$$a[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] + b[\sinh(2\pi\mu l) + \sin(2\pi\mu l)] = 0$$

$$a[\sinh(2\pi\mu l) - \sin(2\pi\mu l)] + b[\cosh(2\pi\mu l) + \cos(2\pi\mu l)] = 0$$

or

$$b = a\frac{\sin(2\pi\mu l) - \sinh(2\pi\mu l)}{\cos(2\pi\mu l) + \cosh(2\pi\mu l)} = -a\frac{\cos(2\pi\mu l) + \cosh(2\pi\mu l)}{\sin(2\pi\mu l) + \sinh(2\pi\mu l)} \quad (5.1.13)$$

By dividing out a and multiplying across, we obtain an equation for μ:

$$[\cosh(2\pi\mu l) + \cos(2\pi\mu l)]^2 = \sinh^2(2\pi\mu l) - \sin^2(2\pi\mu l)$$

Utilizing some trigonometric relationships, this last equation can be reduced to two simpler forms:

$$\cosh(2\pi\mu l)\cos(2\pi\mu l) = -1 \qquad \text{or} \qquad \coth^2(\pi\mu l) = \tan^2(\pi\mu l) \tag{5.1.14}$$

where $\coth z = \cosh z/\sinh z$.

The allowed frequencies

We shall label the solutions of this equation in order of increasing value. They are $2\pi\mu_1 l = 1.8751$, $2\pi\mu_2 l = 4.6941$, $2\pi\mu_3 l = 7.8548$, etc. To simplify the notation, we let $1/\pi$ times the numbers given above have the labels β_n, so that

$$\mu_n = \frac{\beta_n}{2l} \tag{5.1.15}$$

where $\beta_1 = 0.597$, $\beta_2 = 1.494$, $\beta_3 = 2.500$, etc. It turns out that β_n is practically equal to $n - \frac{1}{2}$ when n is larger than 2.

By fixing μ, we fix the allowed values of the frequency. Using Eq. (5.1.10), we have

$$\nu_n = \frac{\gamma^2\mu_n^2}{2\pi} = \frac{\pi}{2l^2}\sqrt{\frac{Q\kappa^2}{\rho}}\,\beta_n^2 \tag{5.1.16}$$

or

$$\nu_1 = \frac{0.55966}{l^2}\sqrt{\frac{Q\kappa^2}{\rho}} \qquad \begin{aligned} \nu_2 &= 6.267\nu_1 \\ \nu_3 &= 17.548\nu_1 \\ \nu_4 &= 34.387\nu_1 \\ &\cdots\cdots\cdots \end{aligned}$$

Notice that the allowed frequencies depend on the inverse *square* of the length of the bar, whereas the allowed frequencies of the string depend on the inverse first power.

Equation (5.1.16) shows how far from harmonics are the overtones for a vibrating bar. The first overtone has a higher frequency than the sixth harmonic of a string of equal fundamental. If the bar were struck so that its motion contained a number of overtones with appreciable amplitude, it would give out a shrill and nonmusical sound. But since these high-frequency overtones are damped out rapidly, the harsh initial sound will quickly change to a pure tone, almost entirely due to the fundamental. A tuning fork can be considered to be two vibrating bars, both clamped at their lower ends. The fork exhibits the preceding behavior, the initial metallic "ping" rapidly dying out and leaving an almost pure tone.

The characteristic functions

The characteristic function corresponding to the allowed frequency ν_n is given by the equation

$$\psi_n = a_n\left(\cosh\frac{\pi\beta_n x}{l} - \cos\frac{\pi\beta_n x}{l}\right) + b_n\left(\sinh\frac{\pi\beta_n x}{l} - \sin\frac{\pi\beta_n x}{l}\right) \tag{5.1.17}$$

where

$$-b_n = a_n \frac{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}{\sinh(\pi\beta_n) + \sin(\pi\beta_n)} = a_n \frac{\sinh(\pi\beta_n) - \sin(\pi\beta_n)}{\cosh(\pi\beta_n) + \cos(\pi\beta_n)}$$

We shall choose the value of a_n so that $\int_0^l \psi_n^2\, dx = l/2$, by analogy with the sine functions for the string. The resulting values for a_n and b_n are $a_1 = 0.707$, $b_1 = -0.518$, $a_2 = 0.707$, $b_2 = -0.721$, $a_3 = 0.707$, $b_3 = -0.707$, etc. For n larger than 2, both a_n and b_n are practically equal to $1/\sqrt{2}$. Some of the properties of these functions that will be of use are

$$\int_0^l \psi_m(x)\psi_n(x)\, dx = \begin{cases} 0 & m \neq n \\ \frac{l}{2} & m = n \end{cases} \qquad \psi_n(l) = (-1)^{n-1}\sqrt{2}$$

$$\left(\frac{d\psi_1}{dx}\right)_{x=l} = 1.040\,\frac{\pi\beta_1}{l} \qquad \left(\frac{d\psi_2}{dx}\right)_{x=l} = -1.440\,\frac{\pi\beta_2}{l} \tag{5.1.18}$$

$$\left(\frac{d\psi_n}{dx}\right)_{x=l} \simeq (-1)^{n-1}\sqrt{2}\,\frac{\pi\beta_n}{l} \qquad \text{and} \qquad \beta_n \simeq n - \tfrac{1}{2} \qquad n > 2$$

$$\psi_n \simeq \frac{1}{\sqrt{2}}\left[e^{-\pi\beta_n x/l} + (-1)^{n-1}e^{\pi\beta_n(x-l)/l}\right] + \sin\left(\frac{\pi\beta_n x}{l} - \frac{\pi}{4}\right) \qquad n > 2$$

The shapes of the first five characteristic functions are shown in Fig. 5.4. Note that for the higher overtones most of the length of the bar has the sinusoidal shape of the corresponding normal mode of the string, with the nodes displaced toward the free end. In terms of the approximate form given above for ψ_n, the sine function is symmetrical about the center of the bar; the first exponential alters the sinusoidal shape near $x = 0$ enough to make ψ_n have zero value and slope at this point; and the second exponential adds enough near $x = l$ to make the second and third derivatives vanish. Note also that the number of nodal points in ψ_n is equal to $n - 1$, as it is for the string.

In accordance with the earlier discussion of series of characteristic functions, we can now show that a bar started with the initial conditions, at $t = 0$, of $y = y_0(x)$ and $\partial y/\partial t = v_0(x)$ will have a subsequent shape given by the series

$$y = \sum_{n=1}^{\infty} \psi_n(x)[B_n \cos(2\pi\nu_n t) + C_n \sin(2\pi\nu_n t)] \tag{5.1.19}$$

where

$$B_n = \frac{2}{l}\int_0^l y_0(x)\psi_n(x)\, dx$$

$$C_n = \frac{1}{\pi\nu_n l}\int_0^l v_0(x)\psi_n(x)\, dx$$

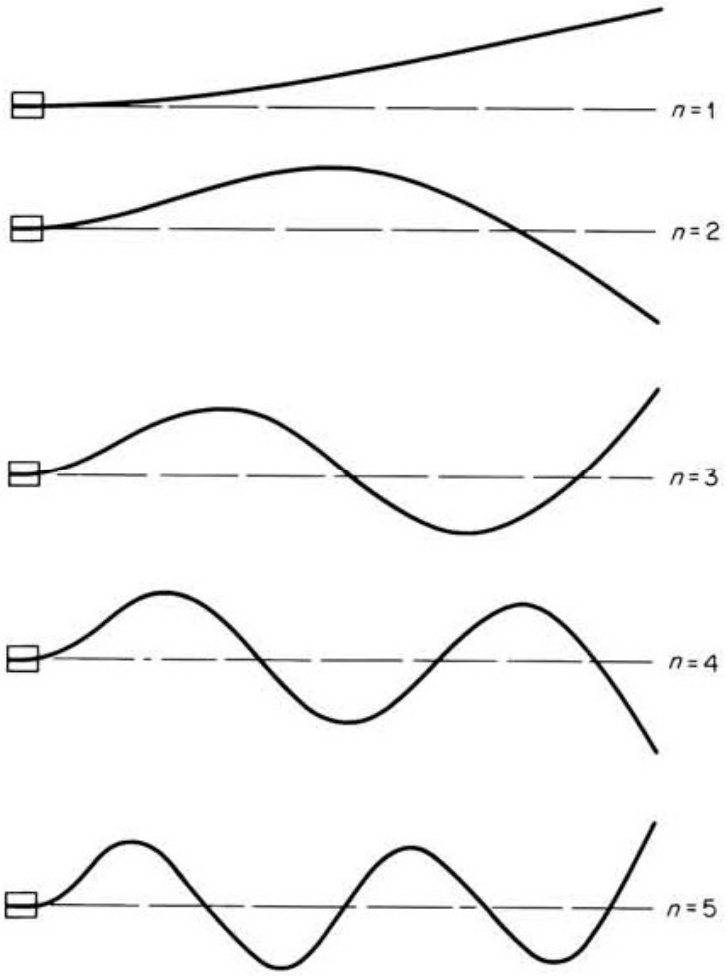


FIGURE 5.4
Shapes of the first five characteristic functions for a vibrating bar clamped at one end and free at the other.

Plucked and struck bar

Two examples of such calculations will be given. The first example is that of a rod suddenly released from an initial shape $y_0 = hx/l$, an undesirable case in actual practice (for it bends the bar quite severely at $x = 0$) but one that can be easily solved. The values of the coefficients B_n turn out to be

$$\begin{aligned} B_n &= \frac{2h}{l^2}\int_0^l x\psi_n(x)\,dx \\ &= \frac{2h}{\pi^2\beta_n{}^2}\{a_n[2 + \pi\beta_n(\sinh \pi\beta_n - \sin \pi\beta_n) - \cosh \pi\beta_n - \cos \pi\beta_n] \\ &\qquad + b_n[\pi\beta_n(\cosh \pi\beta_n + \cos \pi\beta_n) - \sinh \pi\beta_n - \sin \pi\beta_n]\} \\ &= \frac{4h}{\pi^2\beta_n{}^2}\,a_n \end{aligned}$$

when the ratios between a_n and b_n, given in Eq. (5.1.17), are used to simplify the expression. Since the overtones are not harmonics, the motion is not periodic.

The other case to be considered is that of a rod struck at its free end in such a manner that its initial velocity is zero everywhere except at $x = l$,

where it is very large, large enough so that $\int v_0\,dx = U$. This case corresponds to that of a tuning fork struck at one end. If the total impulse given to the end of the fork is P, then $U = P/\rho S$. The coefficients B_n are zero and

$$C_n = \frac{1}{\pi \nu_n l}\int_0^l v_0\psi_n\,dx = \frac{U}{\pi\nu_n l}\psi_n(l) = (-1)^{n-1}\frac{lU}{\pi^2\beta_n{}^2}\sqrt{\frac{8\rho}{Q\kappa^2}}$$

when the value given for $\psi_n(l)$ in Eqs. (5.1.18) is used. This motion is also nonperiodic, as are all motions of the bar that correspond to more than one normal mode of vibration.

Clamped-clamped and free-free bars

Other boundary conditions will give rise to other characteristic functions and frequencies. The bar may be clamped at both ends, in which case expression (5.1.12) must again be used for Y, but instead of Eq. (5.1.13) we must have

$$b = a\frac{\sin(2\pi\mu l) + \sinh(2\pi\mu l)}{\cos(2\pi\mu l) - \cosh(2\pi\mu l)} = -a\frac{\cos(2\pi\mu l) - \cosh(2\pi\mu l)}{\sin(2\pi\mu l) - \sinh(2\pi\mu l)}$$

and instead of Eq. (5.1.14) we have

$$\cosh(2\pi\mu l)\cos(2\pi\mu l) = 1 \qquad \text{or} \qquad \tanh^2(\pi\mu l) = \tan^2(\pi\mu l)$$

The allowed frequencies can be obtained from the formula

$$\nu_n = \frac{\gamma^2\mu_n{}^2}{2\pi} = \frac{\pi}{2l^2}\sqrt{\frac{Q\kappa^2}{\rho}}\,\beta_n{}^2 \qquad \begin{aligned} \beta_1 &= 1.5056 \\ \beta_2 &= 2.4997 \\ \beta_n &\simeq n + \tfrac{1}{2} \quad (n > 2) \end{aligned} \qquad (5.1.20)$$

Still another set of boundary conditions is the one for a completely free bar, where the second and third derivatives must be zero at both ends. The characteristic functions (which we can call ψ_n'') for this case can be obtained from the ones for the bar clamped at both ends (which we can call ψ_n, to distinguish them from ψ_n'') by simply differentiating twice with respect to x, $\psi_n'' = \partial^2\psi_n/\partial x^2$. By using Eq. (5.1.10) we can show that $\partial^2\psi_n''/\partial x^2 = (4\pi^2\nu_n{}^2\rho/Q\kappa^2)\psi_n$, so that if ψ_n and $\partial\psi_n/\partial x$ are both zero at $x = 0$ and $x = l$, then $\partial^2\psi_n''/\partial x^2$ and $\partial^3\psi_n''/\partial x^3$ are both zero at the end points. The functions ψ_n'' therefore satisfy the boundary conditions for a completely free bar. It is not difficult to see that ψ_n'' is a solution of the equation of motion, corresponding to the same allowed frequency as does the function ψ_n. The allowed values are given in Eq. (5.1.20).

We thus obtain the rather surprising result that the allowed frequencies for a free bar are the same as those for a similar bar clamped at both ends and that the corresponding characteristic functions are related by simple differentiation, although their shapes are quite different.

Energy of vibration

It is sometimes useful to know the energy of vibration of a bar. This energy can be computed by using the expressions obtained in the previous section. We saw there [Eqs. (5.1.2) and (5.1.3)] that the moment required to bend an element dx of the bar by an angle ϕ is $M = QS\kappa^2(\phi/dx)$. The amount of work required to bend it from equilibrium to this angle is

$$\int_0^\phi M\, d\phi = \frac{\frac{1}{2}QS\kappa^2\phi^2}{dx} = \tfrac{1}{2}QS\kappa^2\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$$

Therefore the potential energy of the bar, the work required to bend the bar into its final instantaneous shape $y(x,t)$, is

$$V = \tfrac{1}{2}QS\kappa^2\int_0^l\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$$

The kinetic energy is, of course,

$$U = \tfrac{1}{2}\rho S\int_0^l\left(\frac{\partial y}{\partial t}\right)^2 dx$$

so that the total energy turns out to be

$$E = \tfrac{1}{2}\rho S\int_0^l\left[\left(\frac{\partial y}{\partial t}\right)^2 + \frac{Q\kappa^2}{\rho}\left(\frac{\partial^2 y}{\partial x^2}\right)^2\right] dx \tag{5.1.21}$$

The energy of a bar, subject to one or another of the various boundary conditions discussed above, is obtained by substituting $y = \Sigma A_n\psi_n \cos(2\pi\nu_n t - \Omega_n)$ in Eq. (5.1.21). This results in a superfluity of terms, each containing a product of two trigonometric functions of time and an integral of a product of two characteristic functions, or two second derivatives of these functions. The kinetic-energy terms, containing integrals of the products of two functions, can be integrated by means of the first of Eqs. (5.1.18) and reduced to the single sum

$$\Sigma\pi^2\nu_n^2\rho lSA_n^2\sin^2(2\pi\nu_n t - \Omega_n)$$

The integrals in the potential-energy terms can be integrated by parts twice:

$$\int_0^l \frac{\partial^2\psi_n}{\partial x^2}\frac{\partial^2\psi_m}{\partial x^2}\,dx = \left[\frac{\partial\psi_n}{\partial x}\frac{\partial^2\psi_m}{\partial x^2}\right]_0^l - \left[\psi_n\frac{\partial^3\psi_m}{\partial x^3}\right]_0^l + \int_0^l \psi_n\frac{\partial^4\psi_m}{\partial x^4}\,dx$$

$$= \frac{4\pi^2\nu_m^2\rho}{Q\kappa^2}\int_0^l \psi_n\psi_m\,dx$$

by using the equation of motion. The terms in the square brackets are zero no matter which of the boundary conditions discussed above are used. The resulting potential-energy series is therefore $\Sigma\pi^2\nu_n^2\rho lSA_n^2\cos^2(2\pi\nu_n t - \Omega_n)$.

The $\sin^2$ of the kinetic-energy terms combines with the $\cos^2$ in the potential-energy terms and gives for the total energy the simple series

$$E = \sum_{n=1}^{\infty} 2\pi^2 \frac{\rho S l}{2} \nu_n^2 A_n^2 \tag{5.1.22}$$

Note the similarity with the corresponding expression for the energy of the string, given in Eq. (4.3.12).

Forced motion

We can discuss the forced motion of a uniform bar, now that we know the characteristic functions. If the bar is driven by a force $F(x)e^{-2\pi i\nu t}$ dynes per cm length, the equation for shape of the bar during the steady state is

$$\frac{Q\kappa^2}{\rho}\frac{d^4Y}{dx^4} - 4\pi^2\nu^2 Y = f(x) \qquad f = \frac{F(x)}{\rho S} \qquad y = Y(x)e^{-2\pi i\nu t} \tag{5.1.23}$$

If we expand Y and f in series of characteristic functions,

$$Y = \sum h_n \psi_n(x) \qquad f = \sum g_n \psi_n(x) \qquad g_n = \frac{2}{\rho S l}\int_0^l F(x)\psi_n(x)\,dx$$

we obtain for the steady-state motion

$$y = \frac{1}{4\pi^2}\left[\sum_{n=1}^{\infty} \frac{g_n}{\nu_n^2 - \nu^2}\psi_n(x)\right] e^{-2\pi i\nu t} \tag{5.1.24}$$

Resonance occurs whenever the driving frequency ν equals one of the natural frequencies ν_n, unless the corresponding g_n is zero.

These results are analogous to those discussed at the beginning of Sec. 4.4. The effects of friction can be handled as they were handled for the string on page 142 and in Sec. 4.3. The results are so similar that we do not need to go into detail about them here. Some cases will be taken up in the problems.

The stiff string

When a string is under a tension of T dynes, and also has stiffness, its equation of motion is

$$T\frac{\partial^2 y}{\partial x^2} - QS\kappa^2\frac{\partial^4 y}{\partial x^4} = \rho S\frac{\partial^2 y}{\partial t^2} \tag{5.1.25}$$

This equation can be obtained by combining the derivations in Secs. 4.1 and 5.1. The constant S is the area of cross section, κ its radius of gyration, and ρ and Q are the density and modulus of elasticity of the material.

Sinusoidal waves can travel along such a wire, for if we set $y = Ce^{2\pi i(\mu x - \nu t)}$, we obtain an equation relating μ and ν:

$$T\mu^2 + 4\pi^2 QS\kappa^2\mu^4 = \rho S\nu^2$$

or

$$\mu^2 = -\frac{T}{8\pi^2 QS\kappa^2} + \sqrt{\left(\frac{T}{8\pi^2 QS\kappa^2}\right)^2 + \frac{\nu^2\rho}{4\pi^2 Q\kappa^2}}$$

If the frequency is very small (i.e., if ν^2 is very much smaller than $T^2/16\pi^2\rho QS^2\kappa^2$), we expand the radical by the binomial theorem and keep only the first two terms, and μ^2 turns out to be approximately equal to $\rho S\nu^2/T$. The phase velocity ν/μ is practically equal to the constant value $\sqrt{T/\rho S}$, which would be the velocity of every wave if the wire had no stiffness (if $QS\kappa^2$ were zero). The phase velocity for the wire is not constant, however. It increases with increasing ν until for very high frequencies it is $\sqrt{2\pi\nu}\,(Q\kappa^2/\rho)^{\frac{1}{4}}$, the phase velocity given in Eq. (5.1.6) for a stiff bar without tension. In other words, the wire acts like a flexible string for long wavelengths and like a stiff bar for short ones. This is not surprising.

The boundary conditions

The usual boundary conditions correspond to clamping the wire at both ends, making $y = 0$ and also $\partial y/\partial x = 0$ at $x = 0$ and $x = l$. Setting $y = Y(x)e^{-2\pi i\nu t}$, we have for Y

$$\frac{d^4Y}{dx^4} - 8\pi^2\beta^2\frac{d^2Y}{dx^2} - 16\pi^4\gamma^4 Y = 0 \qquad \beta^2 = \frac{T}{8\pi^2 QS\kappa^2}$$
$$\gamma^2 = \frac{\nu}{2\pi}\sqrt{\frac{\rho}{Q\kappa^2}} \tag{5.1.26}$$

Setting $Y = Ae^{2\pi\mu x}$, we obtain an equation for the allowed values of μ:

$$\mu^4 - 2\beta^2\mu^2 - \gamma^4 = 0$$

This equation has two roots for μ^2 and therefore four roots for μ:

$$\mu = \pm\mu_1 \qquad \mu_1^2 = \sqrt{\beta^4 + \gamma^4} + \beta^2$$
or $$\mu = \pm i\mu_2 \qquad \mu_2^2 = \sqrt{\beta^4 + \gamma^4} - \beta^2 \tag{5.1.27}$$
$$\mu_1^2 = 2\beta^2 + \mu_2^2 \qquad \mu_1\mu_2 = \gamma^2$$

The general solution of Eq. (5.1.26) can then be written

$$Y = \begin{cases} ae^{2\pi\mu_1 x} + be^{-2\pi\mu_1 x} + ce^{2\pi i\mu_2 x} + de^{-2\pi i\mu_2 x} \\ A\cosh(2\pi\mu_1 x) + B\sinh(2\pi\mu_1 x) \\ \qquad + C\cos(2\pi\mu_2 x) + D\sin(2\pi\mu_2 x) \end{cases} \tag{5.1.28}$$

If the boundary conditions are symmetrical, it will be useful to place the point $x = 0$ midway between the supports. The normal functions will then be even functions, $\psi(-x) = \psi(x)$; or they will be odd ones, $\psi(-x) = -\psi(x)$. In either case, if we fit the boundary conditions at one end, $x = l/2$, they will also fit at the other end, $x = -(l/2)$. The even functions are built up out of

the combination

$$Y = A \cosh (2\pi\mu_1 x) + C \cos (2\pi\mu_2 x)$$

and the odd functions from the combination

$$Y = B \sinh (2\pi\mu_1 x) + D \sin (2\pi\mu_2 x)$$

The boundary conditions $Y = 0$ and $dY/dx = 0$ at $x = l/2$ correspond to the following equations for the even functions:

$$A \cosh (\pi\mu_1 l) = -C \cos (\pi\mu_2 l) \qquad \mu_1 A \sinh (\pi\mu_1 l) = \mu_2 C \sin (\pi\mu_2 l)$$

The allowed frequencies

By the use of Eq. (5.1.27), these reduce to

$$\tan (\pi l\mu_2) = -\sqrt{1 + \frac{2\beta^2}{\mu_2^2}} \tanh (\pi l\sqrt{\mu_2^2 + 2\beta^2}) \tag{5.1.29}$$

which can be solved for the allowed values of μ_2. The allowed values of the frequency are obtained from the equation

$$\nu = 2\pi\gamma^2 \sqrt{\frac{Q\kappa^2}{\rho}} = 2\pi\mu_2 \sqrt{(\mu_2^2 + 2\beta^2)\frac{Q\kappa^2}{\rho}}$$

These allowed values of ν can be labeled $\nu_1, \nu_3, \nu_5, \ldots$, in order of increasing size.

The corresponding equation for μ_2 for the odd functions is

$$\sqrt{1 + \frac{2\beta^2}{\mu_2^2}} \tan (\pi l\mu_2) = \tanh (\pi l\sqrt{\mu_2^2 + 2\beta^2}) \tag{5.1.30}$$

The allowed values of ν obtained from this equation can be labeled $\nu_2, \nu_4, \nu_6, \ldots$; and then the whole sequence of allowed values will be in order of increasing size, ν_1 being the smallest (the fundamental), ν_2 the next (the first overtone), and so on.

When $\beta\ (= \sqrt{T/8\pi^2 QS\kappa^2})$ goes to zero (i.e., when the tension is zero), the equations reduce to those of a bar clamped at both ends, with the following allowed values of ν,

$$\nu_1 = \frac{3.5608}{l^2}\sqrt{\frac{Q\kappa^2}{\rho}} \qquad \begin{aligned} \nu_2 &= 2.7565\nu_1 \\ \nu_3 &= 5.4039\nu_1 \\ \nu_4 &= 8.9330\nu_1 \\ &\cdots\cdots\cdots \end{aligned} \tag{5.1.31}$$

obtained from Eq. (5.1.20). When β is infinity (i.e., when the stiffness is zero), the equations reduce to those of a flexible string, and the allowed values of ν are

$$\nu_1 = \frac{0.5000}{l}\sqrt{\frac{T}{\rho S}} \qquad \begin{aligned} \nu_2 &= 2\nu_1 \\ \nu_3 &= 3\nu_1 \\ \nu_4 &= 4\nu_1 \\ &\cdots\cdots\cdots \end{aligned} \tag{5.1.32}$$

If β is large but not infinite (i.e., if the wire is stiff but the tension is the more important restoring force), it is possible to obtain an approximate expression for the allowed frequencies. When β is large, $\tanh(\pi l\sqrt{2\beta^2 + \mu_2^2})$ is very nearly unity, and $\sqrt{1 + (2\beta^2/\mu_2^2)}$ is a large quantity for the lower overtones (i.e., as long as μ_2^2 is small compared with β). Equation (5.1.29) for the even functions then states that $\tan(\pi l\mu_2)$ is a very large negative number, which means that $\pi l\mu_2$ is a small amount larger than $\pi/2$ or a small amount larger than $3\pi/2$. In general, $\pi l\mu_2 = (m + \frac{1}{2})\pi + \delta$, where δ is a small quantity and m is some integer. Expanding both sides of Eq. (5.1.29) and retaining only the first terms, we obtain

$$\frac{1}{\delta} = \sqrt{2\beta^2}\,\frac{l}{(m + \frac{1}{2})}$$

or

$$\delta = \pi(2m + 1)\sqrt{\frac{QS\kappa^2}{Tl^2}} \quad .$$

where $2m + 1$ is any odd integer.

Equation (5.1.30) for the odd functions states that $\tan(\pi\mu_2 l)$ must be very small, which means that for these cases $\pi l\mu_2$ is a small amount larger than π or a small amount larger than 2π, or in general, $\pi l\mu_2 = k\pi + \delta$, where δ is small and k is some integer. The corresponding equation for δ is $\delta = \pi(2k)\sqrt{QS\kappa^2/Tl^2}$, where $2k$ is any even integer.

We have shown that for either even or odd functions we must have $\pi l\mu_2 = (n\pi/2) + \delta$, where n is any integer and where $\delta = \pi n\sqrt{QS\kappa^2/Tl^2}$. The even solutions correspond to the odd values of n ($n = 2m + 1$), and the odd functions correspond to the even values of n ($n = 2k$). The allowed values of μ_2 are given approximately by

$$\mu_2 \simeq \frac{n}{2l}\left(1 + \frac{2}{l}\sqrt{\frac{QS\kappa^2}{T}}\right)$$

and the allowed values of the frequency are

$$\nu_n \simeq \frac{n}{2l}\sqrt{\frac{T}{\rho S}}\left(1 + \frac{2}{l}\sqrt{\frac{QS\kappa^2}{T}}\right)$$

A more accurate formula can be obtained by retaining the next terms in the series expansions that we have made:

$$\nu_n \simeq \frac{n}{2l}\sqrt{\frac{T}{\rho S}}\left[1 + \frac{2}{l}\sqrt{\frac{QS\kappa^2}{T}} + \left(4 + \frac{n^2\pi^2}{2}\right)\frac{QS\kappa^2}{Tl^2}\right] \qquad n = 1, 2, 3, \ldots \quad (5.1.33)$$

This formula is valid as long as n^2 is smaller than $l^2T/\pi^2QS\kappa^2$.

When the "stiffness constant" $QS\kappa^2$ becomes negligible compared with l^2T, this formula reduces to (5.1.32), the equation for the allowed frequencies of a flexible string. As the stiffness is increased, the allowed frequencies increase, the frequencies of the overtones increasing relatively somewhat more rapidly than the fundamental, so that they are no longer harmonics.

To obtain values for the higher overtones or for the cases where $QS\kappa^2$ is the same size as, or larger than, l^2T, we must solve Eqs. (5.1.29) and (5.1.30) numerically.

5.2 WAVE MOTION IN MEMBRANES

Before going on to study wave motion in air, it also is useful to consider some examples of two-dimensional wave motion, as illustrated by the vibrations of a flexible membrane under tension or of an elastic plate. We shall find that the addition of another dimension introduces a number of additional phenomena, which are most easily exhibited in the case of the membrane.

For example, the tension in the membrane, which tends to return the membrane to its equilibrium configuration, cannot be expressed in terms of a single magnitude, as was the case with the string. Here the tension at a point must be given in terms of the pull across an elementary length of a line drawn through the point, equaling the force tending to split the membrane along this line. The pull $T\,ds$ will be proportional to the length ds of the line element; the proportionality factor T, the force per unit length of line, is called a *stress*. It is a vector, which may or may not be perpendicular to the line element. Thus the stress in the membrane, in general, is a function of the position of the line element on the membrane, but also of its orientation.

We shall not enter into any discussion of the general case, but confine our attention to membranes in which the tensile stress is the same at every point on the membrane and for every orientation of the line element, being always perpendicular to this line element. In this special case the stress can be specified by a single quantity T, which we can call the tension. For this to be the case, the membrane must be elastic and deformable, like a sheet of rubber rather than a sheet of paper. Also, the outward pull across each unit length of the membrane's perimeter must be T.

It is possible for a membrane of this sort to behave like an assemblage of parallel strings, having waves whose crests are in parallel lines, perpendicular to their direction of propagation (like waves on the ocean). The behavior of such waves is exactly like the behavior of waves on a flexible string: the waves travel with unchanged shape (when friction is neglected), and every such wave travels with equal speed. But this is only the simplest form of wave motion that the membrane can have. It can have circular waves, radiating out from a point or in toward it; it can have elliptical ones, going out from or in to a line segment; and so on. This more complicated sort of

wave motion does not conserve its form as it travels, nor does it have a speed independent of its form. Much of it is so complicated in behavior that we are unable to analyze it at all. In this book we shall content ourselves with a treatment of parallel and circular waves.

If the sheet of material is not perfectly flexible (i.e., is a plate), the motion is still more complicated than that for a membrane. We shall deal only with the simplest possible case of such motion later in this chapter.

Forces on a membrane

Our first task is to set up the equation of motion for the membrane. The procedure is similar to that for the string. We shall find that any part of the membrane having a bulge facing away from the equilibrium plane will be accelerated toward the plane, and vice versa. In general, the acceleration of any portion is proportional to its "bulginess" and opposite in direction. We must find a quantitative measure of this bulginess.

Suppose that the membrane has a density of σ g per sq cm and that it is pulled evenly around its edge with a tension T dynes per cm length of edge. If it is perfectly flexible, this tension will be distributed evenly throughout its area, the material on opposite sides of any line segment of length dx being pulled apart with a force of $T\,dx$ dynes. The displacement of the membrane from its equilibrium position will be called η. It is a function of time and of the position on the membrane of the point in question. If we use rectangular coordinates to locate the point, η will be a function of x, y, and t.

Referring to the analogous argument on page 98 for the string and to the first drawing of Fig. 5.5, we see that the net force on an element $dx\,dy$ of the

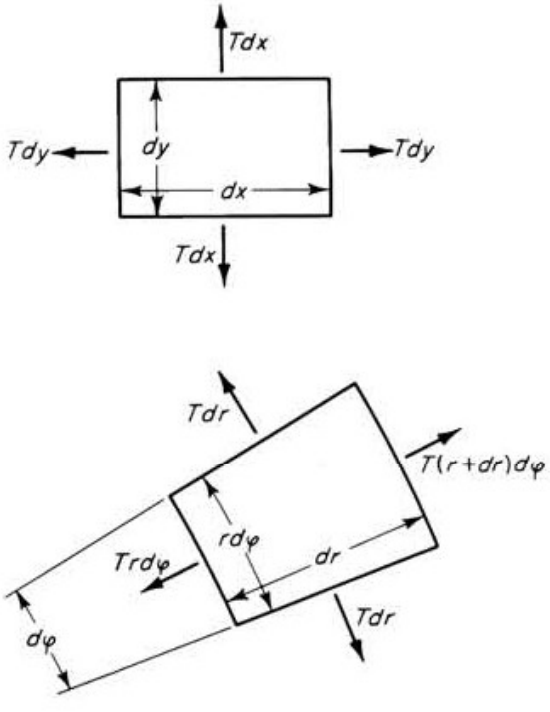


FIGURE 5.5
Forces on an element of a membrane in rectangular and polar coordinates.

membrane due to the pair of tensions $T\,dy$ will be

$$T\,dy\left[\left(\frac{\partial\eta}{\partial x}\right)_{x+dx}-\left(\frac{\partial\eta}{\partial x}\right)_{x}\right]=T\frac{\partial^2\eta}{\partial x^2}\,dx\,dy$$

and that due to the pair $T\,dx$ will be $T(\partial^2\eta/\partial y^2)\,dx\,dy$. The sum of these is the net force on the element and must equal the element's mass $\sigma\,dx\,dy$ times its acceleration. The wave equation for the membrane is therefore

$$\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2}\qquad c=\sqrt{\frac{T}{\sigma}} \tag{5.2.1}$$

The Laplacian operator

The left-hand side of this equation is the expression giving the measure of the bulginess (or rather, the negative bulginess) of the portion of the membrane under consideration.

We now encounter one of the extra difficulties of the two-dimensional analysis; for we find that if we had picked polar instead of rectangular coordinates, so that η was a function of r and ϕ, then the resulting wave equation would have a different aspect. The net force due to the tensions perpendicular to the radius (shown in the second drawing of Fig. 5.5) is

$$T\,dr\left[\left(\frac{\partial\eta}{r\,\partial\phi}\right)_{\phi+d\phi}-\left(\frac{\partial\eta}{r\,\partial\phi}\right)_{\phi}\right]=\frac{T}{r^2}\frac{\partial^2\eta}{\partial\phi^2}\,r\,dr\,d\phi$$

and that due to the tensions parallel to the radius is

$$T\,d\phi\left[\left(r\frac{\partial\eta}{\partial r}\right)_{r+dr}-\left(r\frac{\partial\eta}{\partial r}\right)_{r}\right]=\frac{T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\eta}{\partial r}\right)r\,dr\,d\phi$$

The resulting equation of motion in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\eta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\eta}{\partial\phi^2}=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2} \tag{5.2.2}$$

The left-hand side of this equation has a different form from the left-hand side of (5.2.1). This does not mean that it represents a different property of the membrane; for the value of the left side of (5.2.2) for some point on a given membrane will have the same value as the left side of (5.2.1) for the same point on the same membrane. It simply means that when we wish to measure the bulginess of a portion of a membrane by using polar coordinates, we must go at it differently from the way we should have gone at it if we had used rectangular coordinates.

We often emphasize the fact that the left side of both equations represents the same property of the surface η, by writing the wave equation as

$$\nabla^2\eta=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2} \tag{5.2.3}$$

where the symbol ∇^2 is called the *Laplacian operator*, or simply the *Laplacian*. It stands for the operation of finding the bulginess of the surface at some point. In different coordinates the operator takes on different forms:

$$\nabla^2 = \begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} & \text{(rectangular coordinates)} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \phi^2} & \text{(polar coordinates)} \end{cases}$$

The fact that we have different forms for the Laplacian operator corresponds to the fact that the membrane can have different sorts of waves. The form for rectangular coordinates is the natural one for parallel waves; that for polar coordinates is best for circular waves; and so on. Although the Laplacian is a measure of the same property of the membrane, no matter what coordinate system we use, nevertheless there is a great variation in the facility with which we can solve the wave equation in different coordinate systems. In fact, the known methods for its solution are successful only in the case of a few of the simpler coordinate systems.

Tension and shear in the membrane

As with the string, the equation of motion (5.2.3) is valid only when the displacement is small enough so that $|\partial\eta/\partial x|$ and $|\partial\eta/\partial y|$ are everywhere small compared with unity, and even then only when the membrane is flexible enough so that resistance to bending and shear is negligible compared with the transverse force produced by the tension acting on a bulge in the membrane. The membrane must be flexible in bending (as when a sheet of paper is rolled up); it also must not resist the sort of shear distortion produced by a bulge (such as that which causes a sheet of paper to wrinkle or tear when it is forced to conform to a sphere). A small circle drawn on the membrane at equilibrium is no longer a circle when the membrane is displaced if the curvatures $\partial^2\eta/\partial x^2$ and $\partial^2\eta/\partial y^2$ differ in value. In addition, if a circle is drawn on the displaced surface $\eta(x,y)$, the ratio between its length of perimeter l and its radius s, measured along the curved membrane surface, is

$$\frac{l}{s} = 2\pi\left[1 - \frac{s^2}{6}\frac{\partial^2\eta}{\partial x^2}\frac{\partial^2\eta}{\partial y^2} - \frac{s^2}{6}\left(\frac{\partial^2\eta}{\partial x\,\partial y}\right)^2 + \text{higher-order terms}\right]$$

This usually differs from the ratio 2π for a plane surface, implying a distortion of the surface, which material like paper, for example, would resist with internal forces, independent of any tension. For a membrane to move in accordance with Eq. (5.2.3), these forces must be negligible compared with the force of tension.

This limitation is not trivial. Although the shape distortion is second order in the derivatives of η, compared with the first-order terms produced

by the tension, the deflection of the membrane produced by a transverse force can be quite large if the force is applied to a small area of the membrane. To show this we work out the static deflection of a circular membrane, subject to a transverse force F, distributed evenly over a circular area of radius s at the center of the membrane, as shown in Fig. 5.6. The equation for static deflection, under the influence of a transverse applied force of density $f(x,y)$ per unit area, is, according to Eq. (5.2.3),

$$T\nabla^2\eta = -f(x,y)$$

For the membrane of Fig. 5.6, the force density is $F/\pi s^2$ for $0 < r < s$ and zero for $s < r < R$. Therefore the equation for η is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\eta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\eta}{\partial\phi^2} = \begin{cases} -\dfrac{F}{\pi s^2 T} & 0 \leqslant r < s \\ 0 & s < r \leqslant R \end{cases}$$

with the boundary condition that $\eta = 0$ at $r = R$.

Symmetry requires that η be independent of ϕ. The solution in the region of application of the force is $\eta(r) = \Delta + (F/4\pi s^2 T)(s^2 - r^2)$, and in the outer region it is $A \ln (R/r)$ (which goes to zero at $r = R$), where Δ and A must be adjusted so that η and $d\eta/dr$ are continuous at $r = s$. The result is

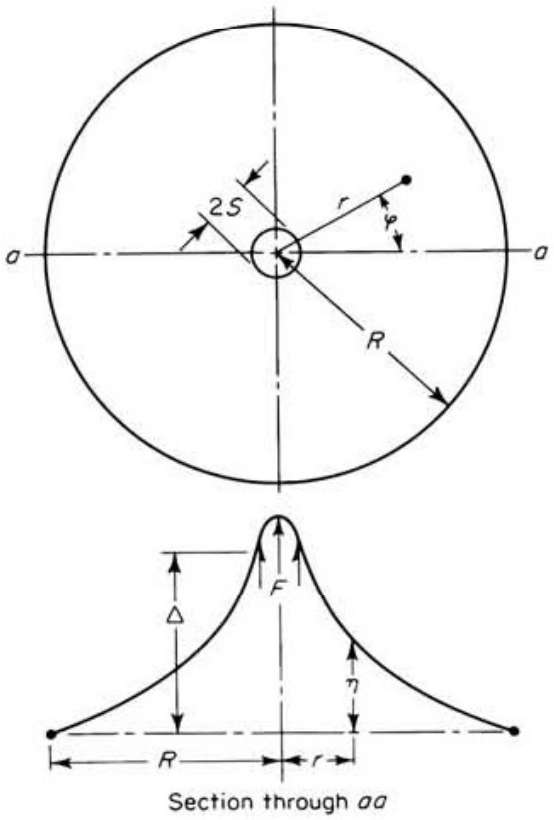


FIGURE 5.6
Top and sectional view of circular membrane displaced by force F concentrated on a small circular area at its center.

$A = F/2\pi T$ and $\Delta = (F/2\pi T)\ln(R/s)$, so that

$$\eta = \begin{cases} \dfrac{F}{2\pi T}\left[\ln\dfrac{R}{s} + \dfrac{1}{2s^2}(s^2 - r^2)\right] & 0 \leqslant r < s \\ \dfrac{F}{2\pi T}\ln\dfrac{R}{r} & s < r \leqslant R \end{cases} \tag{5.2.4}$$

The displacement Δ of the region of application of the force goes to infinity as $s \to 0$, as the area of application of the force is made small enough. No matter how small the applied force F is, compared with the tension T, the deflection caused by F can be made as large as we wish by concentrating its push into a small enough area of the membrane.

This is in contrast with the flexible string. A transverse force F concentrated at a point x of the string is balanced by the discontinuity in slope of the string.

$$F = T\lim_{\delta\to 0}\left[\left(\frac{dy}{dx}\right)_{x-\delta} - \left(\frac{dy}{dx}\right)_{x+\delta}\right]$$

If F is small compared with T, the discontinuity is small and the slopes themselves are small. But to balance a force F concentrated within a circle of radius s requires a downward slope at the edge of the circle of

$$F = -T\int_0^{2\pi}\left(\frac{\partial\eta}{\partial r}\right)_s s\,d\phi \qquad \text{or} \qquad \left\langle\frac{\partial\eta}{\partial r}\right\rangle_s \equiv \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial\eta}{\partial r}\right)_s d\phi = -\frac{F}{2\pi sT}$$

If radius s is made small enough, the mean downward slope $-\langle\partial\eta/\partial r\rangle_s$ at the edge of the circle can be large even if F is much smaller than T. The reason is that the tension is applied over a length of perimeter, and if this length $2\pi s$ is small, the mean slope must be large to balance the force F.

As a result, Eq. (5.2.3) is not adequate to describe the actual motion of a membrane if the forces involved are concentrated or if the membrane is not flexible to shear as well as to bending. The formulas which follow are all subject to this limitation.

Energy and momentum density

The energy and stress densities in the membrane are given by expressions similar to Eqs. (4.1.9) and (4.1.12) for the string. In continuation of this earlier discussion, and in preparation for the similar discussion for sound waves, we shall work out in some detail the formalism of the Lagrange equation for the membrane. If the mass of the membrane is σ per unit area, its kinetic energy of transverse motion of an element $dx\,dy$ of its area can be written $\frac{1}{2}\sigma\eta_t^2 dx\,dy$, where we have written η_t instead of $\partial\eta/\partial t$, to save space. To place a membrane which is elastic enough to satisfy the requirements mentioned in the previous pages under a tension T will stretch its dimensions

laterally by an amount proportional to T, even when the membrane is not displaced transversally (i.e., when $\eta = 0$). Transverse displacement from equilibrium ($\eta \neq 0$) will increase its area still further, and thus will require further work, equal to the increase in area times the tension per unit length. If the displacement is not large, the increase in area caused by the displacement η is the integral of

$$dx\,dy(\sqrt{1 + \text{grad}^2\,\eta} - 1) \simeq \tfrac{1}{2}(\eta_x{}^2 + \eta_y{}^2)\,dx\,dy$$

where $\eta_x = \partial\eta/\partial x$ and $\eta_y = \partial\eta/\partial y$. Because the additional increase in area is small (if η_x and η_y are small), the tension is not changed, to the first order, so that the work required to displace the membrane is the integral of $\frac{1}{2}T(\eta_x{}^2 + \eta_y{}^2)\,dx\,dy$, which is the potential energy of displacement.

The Lagrange density L is, as before, the difference between the kinetic and the potential energy densities,

$$L(\eta,\eta_t,\eta_x,\eta_y) = \tfrac{1}{2}\sigma\eta_t{}^2 - \tfrac{1}{2}T(\eta_x{}^2 + \eta_y{}^2) \tag{5.2.5}$$

and the total Lagrange function for the membrane is the integral of L over the whole area of the membrane. Hamilton's principle of dynamics states that the displacement of the system (in this case η) adjusts itself in shape and velocity so that the integral of this Lagrange function over time, from some initial time t_0 to a specified final time t_1, is minimal.

For example, if $\eta_c(x,y,t)$ is the correct solution, any slight deviation away from this form, such as that given by $\eta = \eta_c + \delta\zeta(x,y,t)$, which does not change the membrane's boundary conditions or its shape at t_0 or t_1 (i.e., where ζ is zero at t_0 and t_1 and along the membrane's boundary) will produce *no first-order change* in the integral of L over t and over area. Any deviation in the value of η from its correct value η_c will increase the kinetic energy over certain parts of the region and increase the potential energy over some other parts, and if the departure from correct shape is not large, the two kinds of changes will cancel to the first order. Thus Hamilton's principle provides us with a mathematical test, which we can apply to some assumed η, to tell whether the assumed η does satisfy the equations of motion.

In fact, Hamilton's principle is equivalent to the equation of motion of the membrane, plus the boundary and initial conditions, as the following argument will show. We substitute $\eta = \eta_c + \delta\zeta$ in the form for $L(\eta,\eta_t,\eta_x,\eta_y)$ and expand L in a power series in δ; the change in L caused by the change in η is then

$$\begin{aligned}\delta L &\equiv L(\eta_c + \delta\zeta, \eta_{ct} + \delta\zeta_t, \eta_{cx} + \delta\zeta_x, \eta_{cy} + \delta\zeta_y) - L(\eta_c,\eta_{ct},\eta_{cx},\eta_{cy}) \\ &= \delta\Bigg[\zeta\left(\frac{\partial L}{\partial\eta}\right)_{\eta=\eta_c} + \frac{\partial\zeta}{\partial t}\left(\frac{\partial L}{\partial\eta_t}\right)_{\eta=\eta_c} + \frac{\partial\zeta}{\partial x}\left(\frac{\partial L}{\partial\eta}\right)_{\eta=\eta_c} \\ &\qquad + \frac{\partial\zeta}{\partial y}\left(\frac{\partial L}{\partial\eta_y}\right)_{\eta=\eta_c} + \text{higher-order terms}\Bigg]\end{aligned}$$

where $\eta_t = \partial\eta/\partial t$, etc. The integral of this difference can be simplified by integration by parts. For example,

$$\int_{t_0}^{t_1} \frac{\partial\zeta}{\partial t}\left(\frac{\partial L}{\partial \eta_t}\right)dt = \left[\zeta\left(\frac{\partial L}{\partial \eta_t}\right)\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \zeta\,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \eta_t}\right)dt$$

and the first term on the right-hand side is zero because ζ goes to zero at t_0 and t_1. Therefore

$$\iiint \delta L\,dx\,dy\,dt = \delta\iiint \zeta\left[\frac{\partial L}{\partial \eta} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \eta_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \eta_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \eta_y}\right)\right]dx\,dy\,dt$$

plus terms in higher powers of δ. For this to be zero to the first order in δ, η_c must satisfy the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \eta_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \eta_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \eta_y}\right) - \frac{\partial L}{\partial \eta} = 0 \qquad (5.2.6)$$

which is the *Lagrange-Euler equation* for the membrane.

If, in addition, a transverse driving force $f(x,y,t)\,dx\,dy$ is applied to the element of membrane at x, y, the Lagrange density has a term $f\eta$ added to it and the Lagrange-Euler equation is still Eq. (5.2.6). For a Lagrange density of the form of Eq. (5.2.5), plus $f\eta$, with the first part quadratic in the partials of η and only the $f\eta$ term dependent on η itself, this becomes

$$\frac{\partial}{\partial t}(\sigma\eta_t) - \frac{\partial}{\partial x}(T\eta_x) - \frac{\partial}{\partial y}(T\eta_y) \equiv \sigma\frac{\partial^2\eta}{\partial t^2} - T\nabla^2\eta = f \qquad (5.2.7)$$

which is, of course, the wave equation.

We note that if we had used polar coordinates r, φ instead of x, y, the integral would have been $\iiint \delta L r\,dr\,d\varphi\,dt$, and the integration by parts over the space derivative part of δL is

$$\delta\iint\left[\zeta_r r\left(\frac{\partial L}{\partial \eta_r}\right) + \zeta_\varphi r\left(\frac{\partial L}{\partial \eta_\varphi}\right)\right]dr\,d\varphi = -\delta\iint\zeta\left[\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\partial L}{\partial \eta_r}\right) + \frac{\partial}{\partial\varphi}\left(\frac{\partial L}{\partial \eta_\varphi}\right)\right]r\,dr\,d\varphi$$

which leads, when $L = \frac{1}{2}\sigma\eta_t^2 - \frac{1}{2}T\eta_r^2 - \frac{1}{2}(T/r^2)\eta_\varphi^2$ is used, to

$$\frac{\partial}{\partial t}(\sigma\eta_t) - \frac{1}{r}\frac{\partial}{\partial r}(Tr\eta_r) - \frac{\partial}{\partial\varphi}\left(\frac{T\eta_\varphi}{r^2}\right) \equiv \sigma\frac{\partial^2\eta}{\partial t^2} - T\nabla^2\eta = f(r,\varphi,t)$$

precisely the modified form for the differential operator ∇^2 which was given following Eq. (5.2.3).

The stress-energy tensor

We now can extend the symbolism of Eqs. (4.1.8) to (4.1.12) to obtain the energy density and energy flux for wave motion. We define

$$W_{tt} = \eta_t\left(\frac{\partial L}{\partial \eta_t}\right) - L = \tfrac{1}{2}\sigma\left(\frac{\partial \eta}{\partial t}\right)^2 + \tfrac{1}{2}T\left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right] = H \quad (5.2.8)$$

which is called the *Hamiltonian density*. It is the sum of the kinetic and potential energy densities of transverse motion of the element $dx\,dy$ of membrane at x, y. Related to this is the two-dimensional vector $\mathbf{W}_t$, with components

$$W_{tx} = \eta_t\left(\frac{\partial L}{\partial \eta_x}\right) = -T\left(\frac{\partial \eta}{\partial t}\right)\left(\frac{\partial \eta}{\partial x}\right) \qquad W_{ty} = \eta_t\left(\frac{\partial L}{\partial \eta_y}\right) = -T\left(\frac{\partial \eta}{\partial t}\right)\left(\frac{\partial \eta}{\partial y}\right) \quad (5.2.9)$$

which is the generalization of the energy flux W_{tx} of Eq. (4.1.9) for the string. The direction of $\mathbf{W}_t$ is the direction of the energy flow; the magnitude is the rate of energy flow, along the membrane, per unit length perpendicular to the flow.

That $\mathbf{W}_t$ is actual energy flux is shown in several ways. In the first place, the dimensions of $\mathbf{W}_t$ are force per unit length times velocity, in other words, power per unit length. More importantly, however, is the relationship between $\mathbf{W}_t$ and the energy density W_{tt}. If $\mathbf{W}_t$ is really the energy flux vector, the divergence of $\mathbf{W}_t$,

$$\begin{aligned} \operatorname{div}(\mathbf{W}_t) &= \frac{\partial W_{tx}}{\partial x} + \frac{\partial W_{ty}}{\partial y} && \text{(for rectangular coordinates)} \\ &= \frac{1}{r}\frac{\partial r W_{tr}}{\partial r} + \frac{1}{r}\frac{\partial W_{t\varphi}}{\partial \varphi} && \text{(for polar coordinates)} \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

represents the net outflow of energy from a unit area of membrane, and thus must equal the rate of loss $-(\partial W_{tt}/\partial t)$ of energy from this unit area. The resulting equation is $(\partial W_{tt}/\partial t) + \operatorname{div}(\mathbf{W}_t) = Q$, where Q, the rate of introduction of energy into the area, is called the equation of continuity. It has been used before [Eq. (4.1.11)] and will be used many times again [see, for example, Eq. (6.1.12)]. That W_{tt} and $\mathbf{W}_t$ do satisfy the equation of continuity, and thus, that if W_{tt} is energy density, $\mathbf{W}_t$ must be energy flux, is demonstrated by inserting Eqs. (5.2.8) and (5.2.9) into

$$\begin{aligned} \frac{\partial W_{tt}}{\partial t} + \operatorname{div}\mathbf{W}_t &= \partial\eta_t\eta_{tt} + T(\eta_x\eta_{tx} + \eta_y\eta_{ty}) \\ &\quad - T(\eta_t\eta_{xx} + \eta_x\eta_{tx} + \eta_t\eta_{yy} + \eta_y\eta_{ty}) \\ &= \eta_t[\partial\eta_{tt} - T(\eta_{xx} + \eta_{yy})] = \eta_t f(x,y,t) \end{aligned}$$

where we have used the wave equation (5.2.7) to obtain the last expression. This is, of course, the rate of introduction of energy into the membrane by the applied force density f. If $f = 0$, then div $\mathbf{W}_t$ is just equal to $-(\partial W_{tt}/\partial t)$, which again demonstrates that $\mathbf{W}_t$ is energy flux.

The quantities

$$W_{xx} = \eta_x\left(\frac{\partial L}{\partial \eta_x}\right) - L = \tfrac{1}{2}\sigma\eta_t{}^2 - \tfrac{1}{2}T(\eta_x{}^2 - \eta_y{}^2)$$

$$W_{yy} = \eta_y\left(\frac{\partial L}{\partial \eta_y}\right) - L = \tfrac{1}{2}\sigma\eta_t{}^2 + \tfrac{1}{2}T(\eta_x{}^2 - \eta_y{}^2)$$

$$W_{xy} = \eta_x\left(\frac{\partial L}{\partial \eta_y}\right) = -T\eta_x\eta_y = W_{yx} = \eta_y\left(\frac{\partial L}{\partial \eta_x}\right)$$

are components of the tensor which represents the stress in the membrane surface produced by the deflection. For example, the force $\mathbf{W}_x\,dy = (W_{xx}\,dy, W_{xy}\,dy)$ is the stress across a line of length dy, perpendicular to the x axis, at point x, y. Its x component, $W_{xx}\,dy$, is the increment of tension in the x direction produced by the displacement; its y component, $W_{xy}\,dy$, is the component of shear, discussed on page 194. The force $\mathbf{W}_y\,dx = (W_{yx}\,dx, W_{yy}\,dx)$ is the related force across a line element perpendicular to y.

Corresponding to these stress components there is an along-the-membrane momentum vector, with components

$$W_{xt} = \eta_x\left(\frac{\partial L}{\partial \eta_t}\right) = \partial\eta_x\eta_t \qquad W_{yt} = \eta_y\left(\frac{\partial L}{\partial \eta_t}\right) = \sigma\eta_y\eta_t$$

analogous to the longitudinal momentum of the string, discussed on page 103. These quantities also satisfy equations of continuity

$$\left(\frac{\partial W_{xt}}{\partial t}\right) + \text{div } \mathbf{W}_x = \eta_x[\sigma\eta_{tt} - T(\eta_{xx} + \eta_{yy})] = \eta_x f$$

where $\eta_x f = (\partial\eta/\partial x)f(x,y,t)$ is the effective applied force density in the x direction at point x, y which produces momentum flow in the x direction (Fig. 4.6).

Beyond setting up the foregoing equations, to be compared with those for sound, it is not advantageous to investigate the motion of the membrane in much detail here. Motion of an actual membrane is strongly affected by its reaction with the air, and calculations of such coupled motion must be discussed later. Here we treat a few simple motions of membranes *in vacuo*.

Wave motion on an unlimited membrane

First let us neglect even the effects of the membrane boundaries, which support its perimeter and maintain its tension, and discuss traveling waves, as we did at the beginning of the discussion of the string.

As was mentioned earlier, a membrane can act like an assemblage of strings, each transmitting an identical wave, in parallel. For example, the function

$$\eta(x,y,t) = F_\theta(x \cos\theta + y \sin\theta - ct)$$

is a solution of the wave equation (5.2.1). This represents a wave traveling in a direction at angle θ with respect to the x axis, with velocity $c = \sqrt{T/\sigma}$. The wavefronts [for example, the line $x \cos\theta + y \sin\theta - ct = 0$ corresponding to $F(0)$] are all lines perpendicular to this direction; so it can be called a *linear wave*. The membrane bends like a sheet of paper, and there is no surface shear; the wave continues, without change of form until it strikes a boundary, just as does a wave on a flexible string.

If this were the only possible kind of wave, there would be little to say beyond what has been said for the string. But since the wave equation is linear in η, a superposition of waves, going in different directions, is also a solution. This superposition is also possible in the case of the string, of course, but in the case of the membrane, the waves may be in different directions, producing a motion which cannot be duplicated by waves in parallel strings. A general sort of wave can be built up by superposing a sequence of linear waves, each going in a different direction θ, and each, if we wish, having a different form, the combination represented by the integral

$$\eta(x,y,t) = \int_0^{2\pi} F_\theta(x \cos\theta + y \sin\theta - ct)\, d\theta \qquad (5.2.10)$$

Since each component of the integral is a solution of the wave equation, the integral is a solution. However, this solution is not a linear wave; it will exhibit properties differing from waves on a string.

We can see the difference more clearly if we express the waveform in terms of its initial conditions, as we did in Eq. (4.2.1), by specifically putting in the formula and limits for S and by integrating and differentiating y_0. The displacement of the string, in terms of its initial displacement $y_0(x)$ and its initial velocity $v_0(x)$, is

$$y(x,t) = \frac{1}{2c}\frac{\partial}{\partial t}\int_{x-ct}^{x+ct} y_0(X)\, dX + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(X)\, dX \qquad (5.2.11)$$

Point x is the *point of observation*, where the displacement of the string is observed at time t. Point X is an initial point used to specify the displacement and velocity of the string at $t = 0$. Equation (5.2.11) states that the displacement of the observation point at time t is a sum of the effects of all initial points within a distance ct of the observation point. When t is small, only the initial values for nearby parts of the string play a part; when t is large, the initial values of a long segment of the string contribute to the displacement at x.

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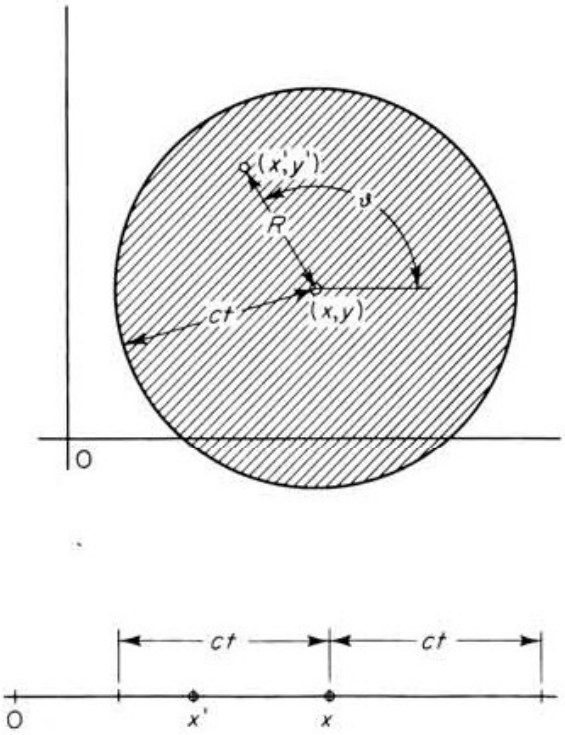


FIGURE 5.7
Dependence of the wave motion of a string and membrane on initial conditions. The displacement of the point (x,y) on the membrane at the time t depends on the initial conditions for that part of the membrane enclosed by a circle of radius ct with center (x,y). The displacement of point x on the string depends on the initial conditions for the portion within a distance ct on either side of x.

For the string, all initial values between $x - ct$ and $x + ct$ contribute with equal weight, which ensures that the waveform is unchanged as it progresses. For example, if y_0 and v_0 differ from zero only in the range $-\Delta < X < 0$, then at $x > 0$ the string will be at rest until time $t = x/c$, since neither of the integrals of Eq. (5.2.11) would include the active region of X before that time. Moreover, after $t = (1/c)(x + \Delta)$, the string at point x is again at rest. A simple example is shown in Fig. 5.8; we see that waves on the string propagate without change of form, leaving no "wake" behind.

The analogous expression for the displacement of a point x, y on a membrane at time t, in terms of the initial displacement η_0 and velocity v_0 of the point X, Y, is

$$\eta(x,y,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_0^{2\pi} d\theta \int_0^{ct} \frac{y_0(X,Y)}{\sqrt{c^2t^2 - R^2}}\, R\, dR + \frac{1}{2\pi c}\int_0^{2\pi} d\theta \int_0^{ct} \frac{v_0(X,Y)}{\sqrt{c^2t^2 - R^2}}\, R\, dR \quad (5.2.12)$$

where R is the magnitude of the vector joining the observation point x, y to the initial point X, Y, and θ is its direction, as shown in Fig. 5.7. Thus all initial points within a range ct of the observation point contribute to the displacement η at time t, but in contrast to the string, different points contribute different amounts, in accord with the factor $(c^2t^2 - R^2)^{-\frac{1}{2}}$. The initial points on the perimeter of the circle of radius ct, around the observation point, contribute the most; the points inside this circle contribute progressively

less, the closer they are to the observation point. If the nearest point on the membrane for which η_0 or v_0 is nonzero is a distance R_0 from the observation point, the observation point will be first displaced at time (R_0/c). Thus the initial rise of the circular wave generated by a localized disturbance travels

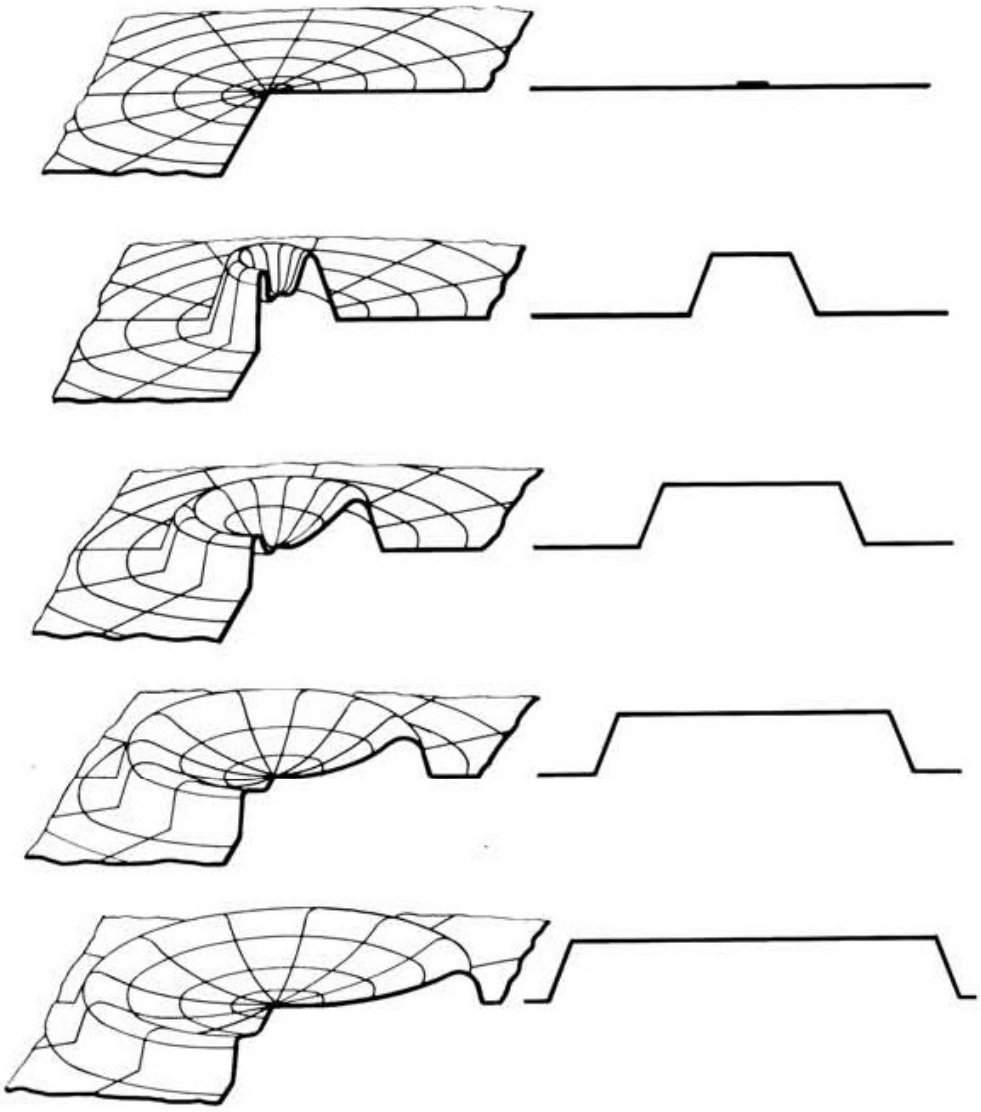


FIGURE 5.8
Comparison between the behavior of a struck string and a struck membrane. The initial displacements are zero, and the center portions are started upward; the lower sketches show the shapes at successive instants later. One-quarter of the membrane has been cut away to show the shape of the cross section.

with velocity c, as for the string. But for the membrane, the shape of the wave changes, its amplitude decreases as it spreads out, and the wave leaves a "wake" of residual motion behind it. Figure 5.8 illustrates this effect and compares it with the simpler behavior of waves on the string.

Rectangular membrane; standing waves

As with the string, when the motion has gone on long enough so that reflections from the membrane boundaries contribute to the motion, standing waves are set up, and the motion resembles that of a series of simple oscillators. For example, suppose the membrane is stretched between a rectangular frame of sides a and b, each unit length of boundary exerting a pull T on the membrane. The motion can most easily be expressed in terms of the rectangular coordinates, x being the axis parallel to the side of length a, and y being in the direction of the sides of length b, with the origin at one corner of the boundary.

The wave equation for the displacement $\eta(x,y,t)$ of the point x, y of the membrane, away from its equilibrium position, at time t is

$$\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2} \qquad c^2=\frac{T}{\sigma} \tag{5.2.13}$$

if we can assume that there is no force on the membrane other than that caused by the tension T, and if the magnitude of grad η is everywhere small compared with unity. To analyze the motion we can first ask whether the membrane can oscillate with simple-harmonic motion.

If we wish to fit rectangular boundaries, we must use a standing wave having a set of nodal lines (lines along which the displacement is zero) parallel to the x axis and another set parallel to the y axis. Now, the only way that we can have a nodal line parallel to the y axis (i.e., for η to be zero for a given value of x for all values of y) is to have a factor of η which is a function of x only and which goes to zero at the value of x corresponding to the nodal line. The nodal lines parallel to the x axis would require a factor depending on y only. Finally, if the motion is simple harmonic, the time dependence must come in the factor $e^{-2\pi i\nu t}$. Therefore an expression for the form of the membrane satisfying our requirements must be

$$\eta = X(x)\,Y(y)e^{-2\pi i\nu t}$$

where substitution in Eq. (5.2.13) shows that

$$Y\frac{\partial^2 X}{\partial x^2}=-\frac{4\pi^2\nu^2}{c^2}XY-X\frac{\partial^2 Y}{\partial y^2}$$

or

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\frac{1}{Y}\left(\frac{4\pi^2\nu^2}{c^2}Y+\frac{\partial^2 Y}{\partial y^2}\right)$$

The left-hand side of this equation is a function of x only, whereas the right-hand side is a function of y only. Now, a function of y cannot equal a function of x for all values of x and y if both functions really vary with x and y, respectively; so that the only possible way for the equation to be true is for both sides to be *independent of both* x and y, that is, to be a constant. Suppose that we call the constant $-(4\pi^2\zeta^2/c^2)$. Then the equation reduces to two

simpler equations:

$$\frac{\partial^2 X}{\partial x^2} = -\frac{4\pi^2\zeta^2}{c^2} X \qquad \frac{\partial^2 Y}{\partial y^2} = -\frac{4\pi^2}{c^2}(\nu^2 - \zeta^2) Y$$

The solution of this pair of equations is

$$\eta = A \cos\left(\frac{2\pi\zeta}{c} x - \Omega_x\right) \cos\left(\frac{2\pi\tau}{c} y - \Omega_y\right) \cos\left(2\pi t \sqrt{\zeta^2 + \tau^2} - \Phi\right) \tag{5.2.14}$$

where we have set $\tau = \sqrt{\nu^2 - \zeta^2}$.

The normal modes

If the boundary conditions are that η must be zero along the edges of a rectangle composed of the x and y axes, the line $x = a$ and the line $y = b$, it is not difficult to see that $\Omega_x = \Omega_y = \pi/2$, that $2\zeta a/c$ must equal an integer, and that $2\tau b/c$ must equal an integer (not necessarily the same integer). The characteristic functions, giving the possible shapes of the membrane as it vibrates with simple-harmonic motion, are

$$\eta = A\psi_{mn}(x,y) \cos\left(2\pi\nu_{mn} t - \Phi\right)$$

$$\psi_{mn}(x,y) = \sin\frac{\pi m x}{a} \sin\frac{\pi n y}{b} \tag{5.2.15}$$

where $$\nu_{mn} = \frac{1}{2}\sqrt{\frac{T}{\sigma}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \begin{matrix} m = 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \end{matrix}$$

The shapes of the first normal modes are shown in Fig. 5.9. We notice that the number of nodal lines parallel to the y axis is $m - 1$ and the number parallel to the x axis, $n - 1$.

The fundamental frequency is ν_{11}, depending on T and σ in a manner quite analogous to the case of the string. Among the allowed frequencies are all the harmonics of the fundamental, $\nu_{22} = 2\nu_{11}$, $\nu_{33} = 3\nu_{11}$, etc. But there are many more allowed frequencies which are not harmonics. When a is nearly equal to b, 2 extra overtones, ν_{12} and ν_{21}, come in between the first and second harmonic; 6 extra, ν_{13}, ν_{31}, ν_{23}, ν_{32}, ν_{14}, ν_{41}, all come between the second and third; 10 extra come between the third and fourth; and so on. No matter what the ratio between a and b is, it is possible to show that the average number of overtones between the nth and the $(n + 1)$st harmonics is $\frac{1}{2}\pi n(a^2 + b^2)/ab$, or that the average number of allowed frequencies in the frequency range between ν and $\nu + \Delta\nu$ is $(2\pi\nu ab/c^2)\Delta\nu$.

In the case of the string, the allowed frequencies are equally spaced along the frequency scale, but in the case of the membrane, the allowed frequencies get closer and closer together, the higher the pitch. The higher the pitch,

the more overtones there are in a range of frequency of a given size. This property is true of all membranes, no matter what shape their boundary has. It can be shown for any membrane that the average number of allowed frequencies between ν and $\nu + \Delta\nu$ is $(2\pi\nu/c^2)\Delta\nu$ times the area of the membrane. We shall show later how this can be proved.

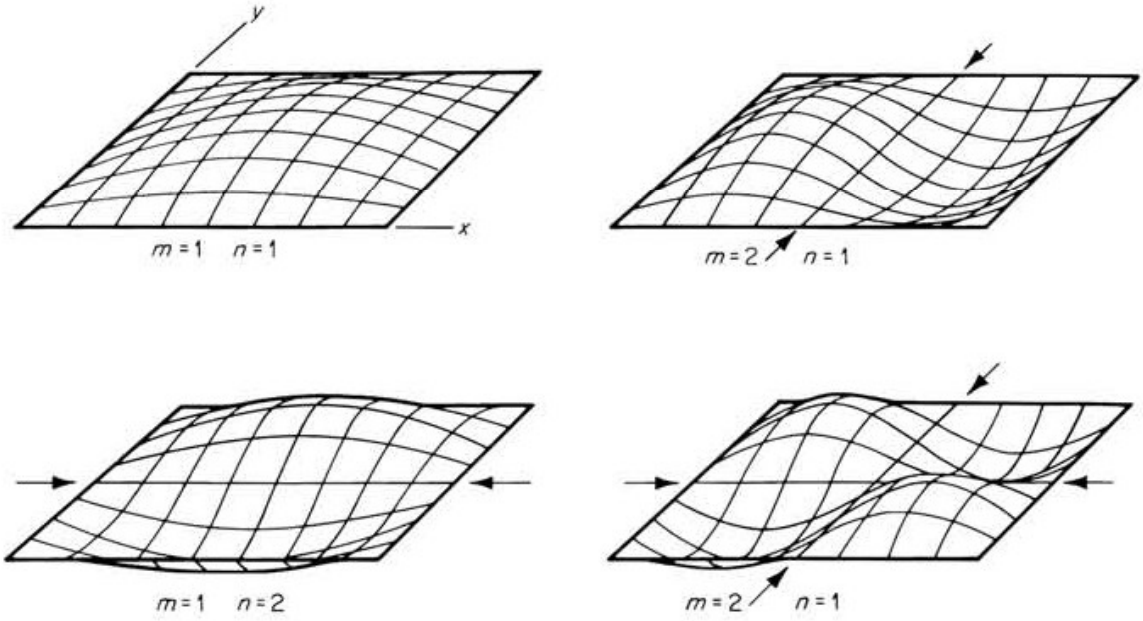


FIGURE 5.9
Shapes of the first four normal modes of a rectangular membrane. Arrows point to the nodal lines.

The degenerate case

An interesting phenomenon occurs when the rectangular membrane is a square one (that is, $a = b$), for then the allowed frequencies become equal in pairs, ν_{mn} being equal to ν_{nm}. There are fewer different allowed frequencies, but there are just as many characteristic functions as when a is not equal to b. This is called a condition of *degeneracy*. There are two different functions, ψ_{mn} and ψ_{nm}, each corresponding to the same frequency (except for the cases $n = m$, which are not degenerate). In such cases the membrane can vibrate with simple-harmonic motion of frequency ν_{mn} with any one of an infinite number of different shapes, corresponding to the different values of γ in the combination

$$\psi_{mn} \cos \gamma + \psi_{nm} \sin \gamma$$

Figure 5.10 shows the shapes of the normal modes of vibration of the square membrane corresponding to ν_{12} and ν_{13}, for different values of γ. The vibrations can be standing waves, corresponding to

$$\eta = (\psi_{mn} \cos \gamma + \psi_{nm} \sin \gamma) \cos (2\pi\nu_{mn}t)$$

The nodal lines have a different shape for each different value of γ. It is also possible to have traveling waves, corresponding to the expression

$$\eta = \psi_{mn} \cos(2\pi\nu_{mn}t) + \psi_{nm} \sin(2\pi\nu_{mn}t)$$

In this case the nodal lines go through the whole range of possible shapes during each cycle. It is only in degenerate cases that it is possible to have traveling waves of simple-harmonic motion in a membrane of finite size.

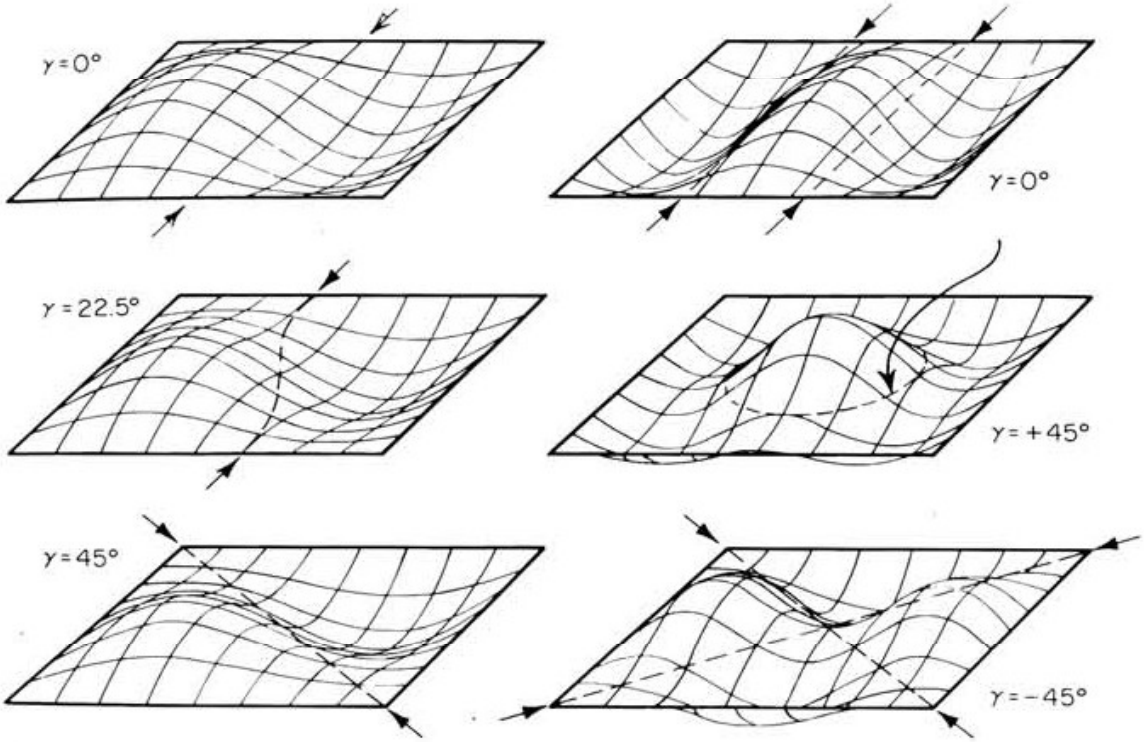


FIGURE 5.10
Various modes of simple-harmonic vibration of a square membrane, for the degenerate cases $\nu_{12} = \nu_{21}$ and $\nu_{13} = \nu_{31}$. Arrows point to the nodal lines.

The characteristic functions

The characteristic functions for the rectangular membrane have the following integral properties:

$$\int_0^a \int_0^b \psi_{mn}\psi_{m'n'}\,dx\,dy = \begin{cases} 0 & m' \neq m \quad \text{or} \quad n' \neq n \\ \dfrac{ab}{4} & m' = m \quad \text{and} \quad n' = n \end{cases} \tag{5.2.16}$$

The behavior of a membrane having an initial shape $\eta_0(x,y)$ and an initial velocity $v_0(x,y)$ is therefore

$$\eta = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}(x,y)[B_{mn} \cos(2\pi\nu_{mn}t) + C_{mn} \sin(2\pi\nu_{mn}t)] \tag{5.2.17}$$

where

$$B_{mn} = \frac{4}{ab} \iint \eta_0 \psi_{mn} \, dx \, dy$$

$$C_{mn} = \frac{2}{\pi \nu_{mn} ab} \iint v_0 \psi_{mn} \, dx \, dy$$

This expression is the correct one for both the degenerate and the non-degenerate cases.

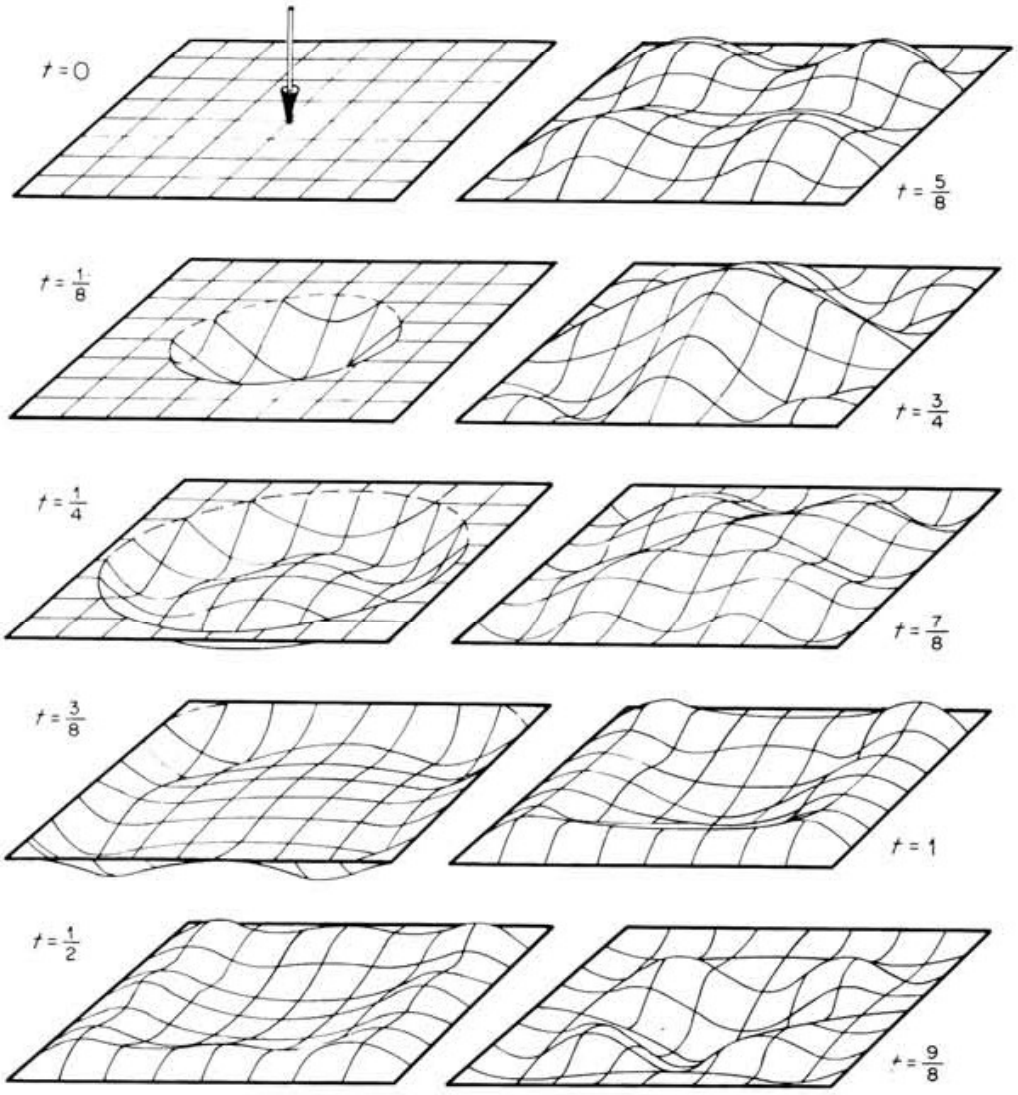


FIGURE 5.11
Successive shapes of a rectangular membrane struck at its center. Times are given in terms of fractions of the fundamental period of vibration.

Figure 5.11 shows the motion of a membrane that has been struck so that a small area around its center is started downward at $t = 0$. In this case $\eta_0 = 0$, and v_0 is zero except near the point $x = a/2$, $y = b/2$, where it has a large enough value so that $\iint v_0 \, dx \, dy = U$. The integrations for the

coefficients C_{mn} become simple, and the resulting expression for η,

$$\eta = \frac{2U}{\pi ab} \sum_{m,n} \frac{1}{\nu_{mn}} \psi_{mn}\left(\frac{a}{2}, \frac{b}{2}\right) \psi_{mn}(x,y) \sin (2\pi \nu_{mn} t)$$

The series does not converge if m and n both run to infinity, a corollary of the fact that a concentrated force produces an infinite displacement of the membrane. However, if the series does not continue for m larger than some number M or for n larger than N, there will be only a finite number of terms to be added together, and the result will never be infinite. Such a finite series corresponds approximately to a starting area of dimensions a/M by b/N.

The case shown in Fig. 5.11 is for $M = N = 10$. The initial shape and the shapes of successive eighths of the fundamental cycle are shown. We see the initial pulse spread out and then reflect back to pile up in the center in a "splash." Since not all the overtones are harmonics, the motion is not periodic. We notice that the shape of the pulse changes as it spreads out to the edge, as has been mentioned in connection with Fig. 5.8.

The circular membrane

Rectangular coordinates are useful to describe parallel waves and normal modes for a rectangular boundary, but for the discussion of circular waves and for the study of the normal modes for a circular boundary, we shall find it easiest to use polar coordinates. In this study we shall encounter more forcibly than before the essential differences between wave motion on a string and on a membrane. We have already seen, in Fig. 5.8, that circular waves, spreading out from a concentrated source, do not maintain shape or amplitude as they progress. We also have noted that the Laplace operator ∇^2, the analogue of the curvature operator for the string, has a form, for polar coordinates, quite different from that of a sum of second derivatives. For this reason even the standing waves in a circular membrane differ markedly from stringlike behavior.

Simple-harmonic waves

To show this we turn to the circular waves that vibrate with simple-harmonic motion. As in the rectangular case, we separate the wave equation. In polar coordinates the equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \eta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 \eta}{\partial t^2}$$

and by setting $\eta = R(r)\Phi(\phi)e^{-2\pi i \nu t}$, we obtain the equations for R and Φ:

$$\frac{d^2\Phi}{d\phi^2} = -\mu^2\Phi \qquad \Phi = \cos(\mu\phi) \quad \text{or} \quad \sin(\mu\phi) \tag{5.2.18}$$

$$\frac{d^2R}{dr_2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{4\pi^2\nu^2}{c^2} - \frac{\mu^2}{r^2}\right)R = 0 \tag{5.2.19}$$

For these circular waves one "boundary condition" is required even before the shape of the boundary line is decided upon. The requirement is

simply that the displacement η be a single-valued function of position; for the coordinate ϕ is a periodic one, repeating itself after an angle 2π, and we must have $\eta(r,\varphi)$ equal to $\eta(r, \varphi + 2\pi)$. This restricts the allowed values of μ to integers:

$$\Phi_{em} = \cos(m\phi) \qquad \Phi_{om} = \sin(m\phi) \qquad m = 0, 1, 2, 3, \ldots$$

This is not true for a membrane whose boundary is shaped like a sector of a circle, where ϕ cannot go from zero clear around to 2π, but such cases are of no great practical importance.

The foregoing requirement is the third different type of boundary condition that we have encountered, the first type being the fixing of the displacement or its slope (or both) at some point or along some line (as with the string, the bar, and the rectangular membrane), the second being simply that the displacement have no infinite values in the range of interest. The third, the condition of periodicity, will be used whenever any of the coordinates are angles that repeat themselves.

Bessel functions

Equation (5.2.19) for the radial factor is Bessel's equation (1.2.4). The general solution $J_m(z)$, where $z = 2\pi\nu r/c$ and m is an integer, has the following properties:

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{d}{dz}J_m\right) + \left(1 - \frac{m^2}{z^2}\right)J_m = 0$$

$$J_m(z) = \frac{1}{m!}\left(\frac{z}{2}\right)^m - \frac{1}{(m+1)!}\left(\frac{z}{2}\right)^{m+2} + \frac{1}{2!(m+2)!}\left(\frac{z}{2}\right)^{m+4} \cdots$$

$$J_m(z) \xrightarrow[z\to\infty]{} \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{2m+1}{4}\pi\right)$$

$$J_m(z) = \frac{1}{2\pi i^m}\int_0^{2\pi} e^{iz\cos w}\cos(mw)\,dw \tag{5.2.20}$$

$$J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z}J_m(z)$$

$$\frac{d}{dz}J_m(z) = \tfrac{1}{2}[J_{m-1}(z) - J_{m+1}(z)]$$

$$\frac{d}{dz}[z^m J_m(z)] = z^m J_{m-1}(z) \qquad \frac{d}{dz}z^{-m}J_m(z) = -z^{-m}J_{m+1}(z)$$

$$\int J_1(z)\,dz = -J_0(z) \qquad \int zJ_0(z)\,dz = zJ_1(z)$$

$$\int J_0^2(z)z\,dz = \frac{z^2}{2}[J_0^2(z) + J_1^2(z)] \tag{5.2.21}$$

$$\int J_m^2(z)z\,dz = \frac{z^2}{2}[J_m^2(z) - J_{m-1}(z)J_{m+1}(z)]$$

$$\int J_m(\alpha z)J_m(\beta z)z\,dz = \frac{z}{\alpha^2 - \beta^2}[\beta J_m(\alpha z)J_{m-1}(\beta z) - \alpha J_m(\beta z)J_{m-1}(\alpha z)]$$

All these properties are proved in books on Bessel functions. Values of J_0, J_1, and J_2 are given in Table V.

The function $J_m(2\pi\nu r/c)$ is not the only solution of Eq. (5.2.19), for it is a second-order equation, and there must be a second solution. This other solution, however, becomes infinite at $r = 0$, and so is of no interest to us at present. It will be discussed in Sec. 7.3. We should note, however, that the use of J_m is in accord with the boundary condition of the second kind, mentioned above.

The allowed frequencies

Coming back to our problem, we can now say that a simple-harmonic solution of the wave equation which is finite over the range from $r = 0$ to $r = \infty$, which is single-valued over the range from $\phi = 0$ to $\phi = 2\pi$, is

$$\eta = \cos m(\phi - \alpha)\, J_m\left(\frac{2\pi\nu r}{c}\right) \cos(2\pi\nu t - \Omega) \tag{5.2.22}$$

If the membrane is fastened along a boundary circle of radius a, the allowed frequencies must be those that make $J_m(2\pi\nu a/c) = 0$. For each value of m there will be a whole sequence of solutions. We shall label the allowed values of the frequency ν_{mn}, so that $\nu_{01}, \nu_{02}, \nu_{03}, \ldots$ are the solutions of $J_0(2\pi\nu a/c) = 0$; $\nu_{11}, \nu_{12}, \nu_{13}, \ldots$ are the solutions of $J_1(2\pi\nu a/c) = 0$; and so on. The values of ν_{mn} are given by the equations

$$\nu_{mn} = \frac{c}{2a}\beta_{mn} \qquad \begin{aligned} &\beta_{01} = 0.7655, \quad \beta_{02} = 1.7571, \quad \beta_{03} = 2.7546, \ldots \\ &\beta_{11} = 1.2197, \quad \beta_{12} = 2.2330, \quad \beta_{13} = 3.2383, \ldots \\ &\beta_{21} = 1.6347, \quad \beta_{22} = 2.6793, \quad \beta_{23} = 3.6987, \ldots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{5.2.23}$$

$$\beta_{mn} \simeq n + \frac{m}{2} - \frac{1}{4} \qquad \text{if } n \text{ is large}$$

The frequency ν_{01} is the fundamental. Another way of writing these results is

$$\nu_{01} = 0.38274\,\frac{1}{a}\sqrt{\frac{T}{\sigma}}, \quad \nu_{11} = 1.5933\nu_{01}, \quad \nu_{21} = 2.1355\nu_{01} \tag{5.2.24}$$

$$\nu_{02} = 2.2954\nu_{01}, \qquad \nu_{31} = 2.6531\nu_{01}, \quad \nu_{12} = 2.9173\nu_{01}, \ldots$$

It is to be noticed that none of these overtones is harmonic.

The characteristic functions

Corresponding to the frequency ν_{0n} is the characteristic function

$$\psi_{e0n}(r,\phi) = J_0\left(\frac{\pi\beta_{0n}r}{a}\right)$$

and corresponding to the frequency $\nu_{mn}(m > 0)$ are the two characteristic functions

$$\psi_{emn} = \cos(m\phi)\, J_m\left(\frac{\pi\beta_{mn}r}{a}\right) \qquad \psi_{0mn} = \sin(m\phi)\, J_m\left(\frac{\pi\beta_{mn}r}{a}\right) \tag{5.2.25}$$

Except for $m = 0$, the normal modes are degenerate, there being two characteristic functions for each frequency. The shapes of a few of the normal modes are shown in Fig. 5.12. We notice that the (m,n)th characteristic function has m diametrical nodal lines and $n - 1$ circular nodes.

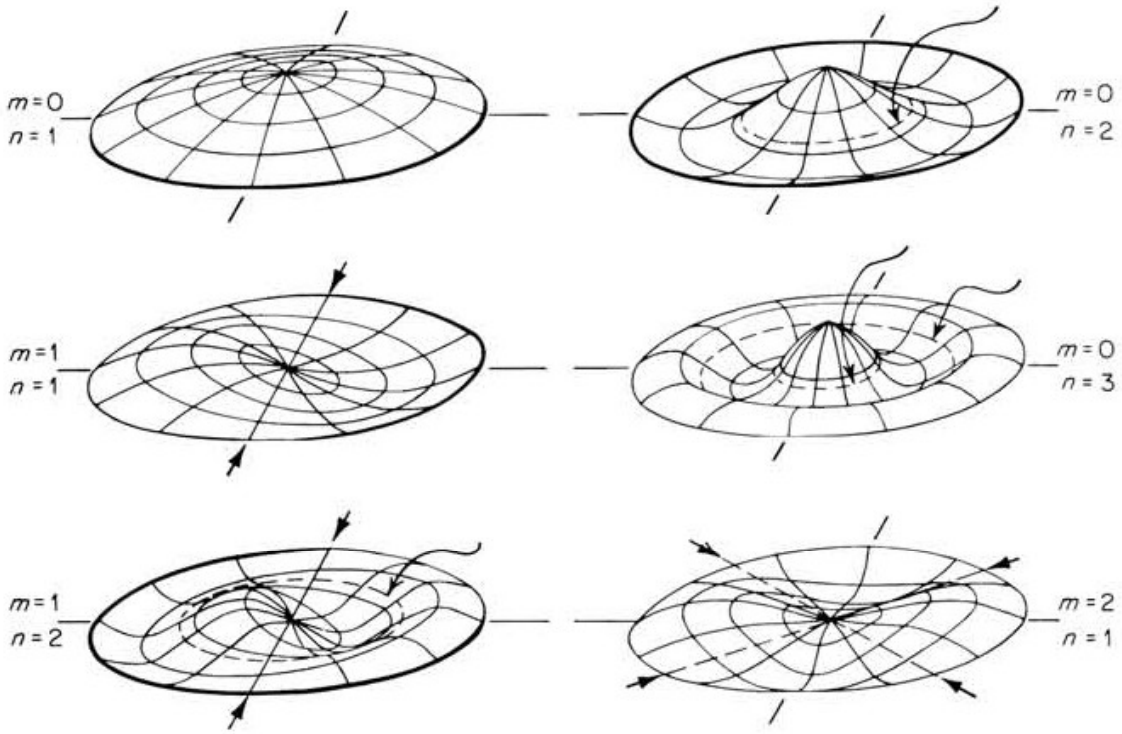


FIGURE 5.12
Shapes of some of the normal modes of vibration of the circular membrane. Arrows point to the nodal lines.

The integral properties of these functions are obtained from Eqs. (5.2.21):

$$\int_0^a\int_0^{2\pi} \psi_{emn}\psi_{em'n'} r\, dr\, d\phi = \int_0^a\int_0^{2\pi} \psi_{0mn}\psi_{0m'n'} r\, dr\, d\phi$$
$$= \begin{cases} 0 & m \neq m' \quad \text{or} \quad n \neq n' \\ \pi a^2 \Lambda_{mn} & m = m' \quad \text{and} \quad n = n' \end{cases} \tag{5.2.26}$$

where $\Lambda_{0n} = [J_1(\pi\beta_{0n})]^2 \qquad \Lambda_{mn} = \frac{1}{2}[J_{m-1}(\pi\beta_{mn})]^2 \qquad m > 0$

The values of the constants Λ_{mn} can be computed from the following values

of J_m:

$$J_1(\pi\beta_{01}) = +0.5191 \qquad J_1(\pi\beta_{02}) = -0.3403$$
$$J_1(\pi\beta_{03}) = +0.2715 \qquad J_1(\pi\beta_{04}) = -0.2325$$
$$J_0(\pi\beta_{11}) = -0.4028 \qquad J_0(\pi\beta_{12}) = +0.3001$$
$$J_0(\pi\beta_{13}) = -0.2497 \qquad J_0(\pi\beta_{14}) = +0.2184$$
$$J_1(\pi\beta_{21}) = -0.3397 \qquad J_1(\pi\beta_{22}) = +0.2714$$
$$J_1(\pi\beta_{23}) = -0.2324 \qquad J_1(\pi\beta_{24}) = +0.2066$$

With the values of the constants Λ known, it is possible to compute the behavior of a circular membrane when started with the initial shape $\eta_0(r,\varphi)$ and the initial velocity $v_0(r,\varphi)$. By methods that we have used many times before, we can show that this behavior is governed by the series

$$\eta = \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{\infty} \psi_{emn}[B_{emn} \cos(2\pi\nu_{mn}t) + C_{emn} \sin(2\pi\nu_{mn}t)] + \sum_{m=1}^{\infty} \psi_{0mn}[B_{0mn} \cos(2\pi\nu_{mn}t) + C_{0mn} \sin(2\pi\nu_{mn}t)] \right\} \quad (5.2.27)$$

where

$$B_{emn} = \frac{1}{\pi a^2 \Lambda_{mn}} \int_0^a \int_0^{2\pi} \eta_0 \psi_{emn} r \, dr \, d\phi$$

$$C_{emn} = \frac{1}{2\pi^2 \nu_{mn} a^2 \Lambda_{mn}} \int_0^a \int_0^{2\pi} v_0 \psi_{emn} r \, dr \, d\phi$$

A similar set of equations holds for B_{0mn} and C_{0mn}.

This completes our preliminary discussion of wave motion on a membrane. We shall return to the subject in Chaps. 9 and 10, when we have learned something of wave motion in air, and can thus take into account the interactions between membrane and surrounding air, which are so important in actual membrane behavior.

5.3 THE VIBRATION OF PLATES

The study of the vibrations of plates bears the same relation to the study of the membrane as the study of the vibrations of bars does to the study of the flexible string. The effect of stiffness in both cases increases the frequencies of the higher overtones more than it does those of the lower overtones, and so makes the fundamental frequency very much lower than all the overtones. However, the motions of a plate are very much more complicated than those of a bar; so much more complicated that we shall have to be satisfied with the study of one example, that of the circular plate, clamped at its edge and under no tension. The diaphragm of an ordinary telephone receiver is a plate of this type; so the study will have some practical applications.

The equation of motion

The increased complications encountered in the study of plates come partly from the increased complexity of wave motions in two dimensions over those of one, but also come about because of the complex sorts of stresses that are set up when a plate is bent. The bending of a plate compresses the material on the inside of the bend and stretches it on the outside. But when a material is compressed it tries to spread out in a direction perpendicular to the compressional force, so that when a plate is bent downward in one direction, there will be a tendency for it to curl up in a direction at right angles to the bend. The ratio of the sidewise spreading to the compression is called *Poisson's ratio*, and will be labeled by the letter s. It has a value about equal to 0.3 for most materials. This complication was not considered when we studied the vibration of bars, for we tacitly assumed that the bar was thin enough compared with its length so that the effects of a sidewise curl would be negligible.

The derivation of the wave equation for the plate involves more discussion than is worthwhile here (it is given in books on theory of elasticity). The equation is

$$\nabla^4\eta + \frac{3\rho(1-s^2)}{Qh^2}\frac{\partial^2\eta}{\partial t^2} = 0 \qquad (5.3.1)$$

where ρ is the density of the material, s its Poisson's ratio, Q its modulus of elasticity, and h the half-thickness of the plate.

We shall not spend any time discussing the general behavior of waves on a plate of infinite extent, but shall simply remark that, like the bar, the plate is a dispersive medium; waves of different wavelength travel with different velocities.

Simple-harmonic vibrations

To study the simple-harmonic motion of a plate, we insert the exponential dependence on time and separate the factors depending on the individual coordinates. The differential operator ∇^4 is difficult to separate in most coordinate systems, but for polar coordinates the results turn out to be sufficiently simple to justify our analyzing them in detail. Here, if we set $\eta = Y(r,\phi)e^{-2\pi ivt}$, where Y's dependence on ϕ is by the factor cos or sin $(m\phi)$, then the differential equation for Y can be written

$$(\nabla^2 - \gamma^2)(\nabla^2 + \gamma^2)Y = 0 \qquad \gamma^4 = \frac{12\pi^2\nu^2\rho(1-s^2)}{Qh^2} \qquad (5.3.2)$$

Therefore Y can be a solution either of $\nabla^2 Y + \gamma^2 Y = 0$ or of $\nabla^2 Y - \gamma^2 Y = 0$.

Since ∇^2 and Y are to be expressed in polar coordinates, the solution of the first equation, which is finite at $r = 0$, is

$$Y = \begin{matrix}\cos\\ \sin\end{matrix}(m\phi)J_m(\gamma r) \qquad \text{where } m \text{ is an integer}$$

This is the usual solution for the membrane, with γ instead of $2\pi\nu/c$. The solution of the second equation is obtained from the first by changing γ into $i\gamma$, and necessitates a little discussion of Bessel functions of imaginary values of the independent variable. Let us call these *hyperbolic Bessel functions* and define them by the equation $I_m(z) = i^{-m}J_m(iz)$. The properties of the function $I_m(z)$ can be obtained from Eqs. (5.2.20) and (5.2.21) for $J_m(z)$. The more useful formulas are

$$I_{m-1}(z) - I_{m+1}(z) = \frac{2m}{z} I_m(z) \qquad \frac{d}{dz} I_m(z) = \tfrac{1}{2}[I_{m-1}(z) + I_{m+1}(z)]$$

$$\int I_0(z) z\, dz = zI_1(z) \qquad \int I_1(z)\, dz = I_0(z) \tag{5.3.3}$$

Values of the functions I_s, I_1, and I_2 are given in Table IV.

The normal modes

Possible solutions for the simple-harmonic oscillations of a plate are therefore given by the expressions

$$Y(r,\phi) = \begin{matrix}\cos\\ \sin\end{matrix} (m\phi)[AJ_m(\gamma r) + BI_m(\gamma r)]$$

The boundary conditions corresponding to a circular plate of radius a, clamped at its edges, are that $Y(a,\phi) = 0$ and $(\partial Y/\partial r)_{r=a} = 0$. The first condition is satisfied by making

$$B = -A \frac{J_m(\gamma a)}{I_m(\gamma a)}$$

and the second condition is satisfied by requiring that γ have those values that make

$$I_m(\gamma a) \frac{d}{dr} J_m(\gamma r) - J_m(\gamma a) \frac{d}{dr} I_m(\gamma r) = 0 \qquad \text{at } r = a \tag{5.3.4}$$

This equation fixes the allowed values of the frequency, for γ depends on ν. We shall label the solutions of Eq. (5.3.4) by the symbols γ_{mn}, where $\gamma_{mn} = (\pi/a)\beta_{mn}$, and where

$$\begin{aligned} \beta_{01} &= 1.015, & \beta_{02} &= 2.007, & \beta_{03} &= 3.000 \\ \beta_{11} &= 1.468, & \beta_{12} &= 2.483, & \beta_{13} &= 3.490 \\ \beta_{21} &= 1.879, & \beta_{22} &= 2.992, & \beta_{23} &= 4.000 \end{aligned} \tag{5.3.5}$$

$$\beta_{mn} \xrightarrow[n\to\infty]{} n + \frac{m}{2}$$

The allowed values of the frequency are therefore

$$\nu_{mn} = \frac{\pi h}{2a^2}\sqrt{\frac{Q}{3\rho(1-s^2)}}\,(\beta_{mn})^2$$

$$\nu_{01} = 0.9342\,\frac{h}{a^2}\sqrt{\frac{Q}{\rho(1-s^2)}}$$

$$\nu_{11} = 2.091\nu_{01}, \qquad \nu_{21} = 3.426\nu_{01}, \qquad \nu_{02} = 3.909\nu_{01}$$

$$\nu_{12} = 5.983\nu_{01}, \ldots$$

The allowed frequencies are spread apart much farther than those for the membrane, given in Eq. (5.2.24). The overtones are not harmonic.

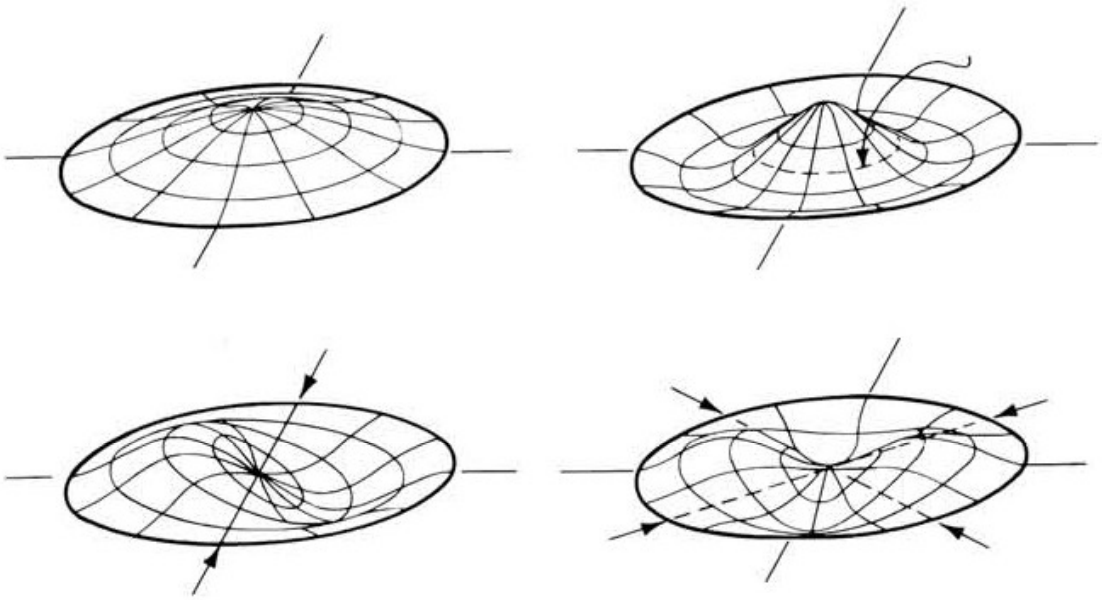


FIGURE 5.13
Shapes of a few of the normal modes of vibration of a circular plate clamped at its edge.

The characteristic functions are

$$\psi_{emn} = \cos(m\phi)\left[J_m\left(\frac{\pi\beta_{mn}r}{a}\right) - \frac{J_m(\pi\beta_{mn})}{I_m(\pi\beta_{mn})}\,I_m\left(\frac{\pi\beta_{mn}r}{a}\right)\right] \qquad (5.3.6)$$

and a similar expression for ψ_{omn} (for $m > 0$), where $\sin(m\phi)$ is used instead of $\cos(m\phi)$. Some of these functions are shown in Fig. 5.13.

The free vibrations of the plate corresponding to arbitrary initial conditions can be expressed in terms of a series of these characteristic functions.

To compute the coefficients of the series, we need the expressions for the integrals, over the area of the plate, of the squares of the characteristic

functions. These can be obtained by using Eqs. (5.2.21) and (5.3.4).

$$\int_0^{2\pi} d\phi \int_0^a [\psi_{\sigma mn}(\phi,r)]^2 r\, dr = \pi a^2 \Lambda_{mn} \qquad \sigma = e \text{ or } o;\ m, n \text{ integers}$$
$$\Lambda_{mn} = \frac{2}{\epsilon_m} \{[J_m(\pi\beta_{mn})]^2 + [J_m'(\pi\beta_{mn})]^2\} \tag{5.3.7}$$

where $J_m'(u) = (d/du)J_m(u)$.

Forced motion

As an example of the calculation of forced motion, let us consider the case of a diaphragm subject to a uniform simple-harmonic transverse driving force $F_\omega e^{-i\omega t}$ per unit area of its surface. The equation of motion is

$$2h\rho \frac{\partial^2 \eta}{\partial t^2} = -\frac{2Qh^3}{3(1-s^2)} \nabla^4 \eta + F_\omega e^{-i\omega t} \tag{5.3.8}$$

where $2h\rho$ is the mass of the plate per unit area, h being its half-thickness. Setting $\eta = Y_\omega(r,\phi)e^{-i\omega t}$ for the steady-state solution, we have

$$\nabla^4 Y_\omega - \gamma^4 Y_\omega = \frac{3(1-s^2)F_\omega}{2Qh^3} \qquad \gamma^4 = \frac{3\omega^2 \rho_p (1-s^2)}{Qh^2} \tag{5.3.9}$$

For a circular plate, the solutions of the homogeneous equation $\nabla^4 Y - \gamma^4 Y = 0$, which are finite at $r = 0$ and independent of ϕ, are the two Bessel functions $J_0(\gamma r)$ and $I_0(\gamma r)$. A combination of these would not satisfy the boundary conditions $Y = 0$ and $\partial Y/\partial r = 0$ at $r = a$, appropriate for a plate clamped at its edge, for an arbitrarily chosen value of ω. In fact, we have just shown that this is possible only for those frequencies given in Eq. (5.3.5). Only by adding a constant term can it satisfy these conditions for any frequency. Specifically, the combination

$$Y = A[I_1(\gamma a)J_0(\gamma r) + J_1(\gamma a)I_0(\gamma r) - I_1(\gamma a)J_0(\gamma a) - J_1(\gamma a)I_0(\gamma a)]$$

does satisfy the two boundary conditions at $r = a$, since

$$\gamma J_1(\gamma r) = -[dJ_0(\gamma r)/dr] \qquad \text{and} \qquad \gamma I_1(\gamma r) = dI_0(\gamma r)/dr$$

Of course, the combination does not satisfy the homogeneous equation; in fact, it satisfies the inhomogeneous equation

$$\nabla^4 Y - \gamma^4 Y = A\gamma^4[I_1(\gamma a)J_0(\gamma a) + J_1(\gamma a)I_0(\gamma a)]$$

But the equation we are to solve is not homogeneous; in fact, we need only adjust the value of A to satisfy Eq. (5.3.9). The final result is

$$Y_\omega = \frac{F_\omega}{2h\rho\omega^2} \frac{I_1(\gamma a)[J_0(\gamma r) - J_0(\gamma a)] + J_1(\gamma a)[I_0(\gamma r) - I_0(\gamma a)]}{I_1(\gamma a)J_0(\gamma a) + J_1(\gamma a)I_0(\gamma a)}$$
$$\to \frac{3F_\omega(1-s^2)}{32h^3Q}(a^2 - r^2)^2 \qquad \gamma a \to 0 \tag{5.3.10}$$

At certain frequencies, those for which the denominator $I_1(\gamma a)J_0(\gamma a) + J_1(\gamma a)I_0(\gamma a)$ is zero, the diaphragm resonates; the formula predicts an infinite amplitude because the frictional term has been omitted. These values of γ (and thus of frequency) for resonance are, of course, those specified by Eq. (5.3.4) for $m = 0$ for the standing waves of free vibration of the plate. We note that, as the driving frequency goes to zero, the static displacement of the diaphragm, given by the limiting form, is small.

The Green's function

If instead of being spread over the plate surface, the driving force is concentrated at the point r_0, ϕ_0, the steady-state motion is given in terms of the Green's function G, the solution of

$$\nabla^4 G - \gamma^4 G = \frac{1}{r}\,\delta(r - r_0)\,\delta(\phi - \phi_0) \qquad \gamma^4 = \omega^2 \frac{3\rho(1 - s^2)}{Qh^2} \tag{5.3.11}$$

which is the analogue of Eq. (4.4.9). This function can be expressed as a series of the characteristic functions $\psi_{\sigma mn}(\phi,r)$ of Eq. (5.3.6). Inserting $G = \sum A_{\sigma mn}\psi_{\sigma mn}$ in Eq. (5.3.11), we obtain

$$\sum_{skl} A_{skl}({\gamma_{kl}}^4 - \gamma^4)\psi_{skl}(\phi,r) = \frac{1}{r}\,\delta(r - r_0)\,\delta(\phi - \phi_0) \qquad \gamma_{mn} = \frac{\pi\beta_{mn}}{a}$$

Multiplying through by $\psi_{\sigma mn}(\phi,r)r\,d\phi\,dr$ and integrating over the diaphragm area, using Eq. (5.3.7) and the fact that the integral of $\psi_{\sigma mn}\psi_{skl}$ is zero unless $s = \sigma$, $k = m$, and $l = n$, and considering the property of the delta functions, we see that

$$G_\omega(\phi r \mid \phi_0 r_0) = \frac{1}{\pi a^2} \sum_{\sigma,m,n} \frac{\psi_{\sigma mn}(\phi,r)\psi_{\sigma mn}(\phi_0,r_0)}{\Lambda_{mn}({\gamma_{mn}}^4 - \gamma^4)} \tag{5.3.12}$$

This has properties similar to the Green's function of Eq. (4.4.14), for the string between rigid supports. For example, it obeys the principle of reciprocity since its value is unchanged when r, ϕ and r_0, ϕ_0 are interchanged. It also has resonance poles whenever $\omega = \pm 2\pi\nu_{mn}$, with ν_{mn} defined below Eq. (5.3.5). The series converges fairly rapidly, except at resonance, since ${\gamma_{mn}}^4 \simeq (\pi/a)^4(n + \frac{1}{2}m)^4$ for n large. From Eq. (5.3.9) the steady-state displacement of the plate, when driven by a force $F_\omega e^{-i\omega t}$ concentrated at the point ϕ_0, r_0, is $[3(1 - s^2)F_\omega/2Qh^3]G_\omega(\phi r \mid \phi_0 r_0)e^{-i\omega t}$, and the transverse velocity of the plate is $-i\omega$ times this. Therefore the mechanical impedance which the plate presents to a driving force of frequency $\omega/2\pi$, applied at r_0, ϕ_0, is

$$Z_m = \frac{F_\omega}{u} = \frac{i}{\omega}(2\pi a^2 h\rho)\left\{\sum_{\sigma mn} \frac{[\psi_{\sigma mn}(\phi_0,r_0)]^2}{\Lambda_{mn}({\omega_{mn}}^2 - \omega^2)}\right\}^{-1} \tag{5.3.13}$$

where ${\omega_{mn}}^2 = (2\pi\nu_{mn})^2 = [Qh^2/3\rho(1 - s^2)](\pi\beta_{mn}/a)^4$. At very low frequencies this behaves like a stiffness reactance, being inversely proportional to ω. At a resonance ($\omega = \omega_{mn}$), Z_m is zero; just above a resonance, Z_m is negative-imaginary, acting like a mass reactance. As long as we neglect

internal friction of the plate material and assume that the plate is moving in a vacuum, Z_m is purely reactive at all frequencies.

Green's function for an infinite plate

Later we shall be studying the reaction between a plate of very large extent and the acoustic waves in a medium in contact with the plate. At that time we shall need to know the expression for the deflection of a plate of infinite extent, driven by a simple-harmonic force concentrated at a single point. This is, of course, proportional to a Green's function, the solution of Eq. (5.3.11) for an infinite plate. Because there are no boundaries, we can place the origin anywhere; we might as well place it at the point of application of the force. Thus we rewrite Eq. (5.3.11) as

$$\nabla^4 g - \gamma^4 g = \delta(x)\,\delta(y)$$

the function on the right-hand side being zero everywhere but at the origin, where its integral is unity. (We habitually use the lowercase symbol g for the Green's function in an infinite medium.)

With the force symmetrically placed, the radiating waves must be symmetrical, so that g should be independent of ϕ. Thus, referring to the definition of ∇^2 in polar coordinates, given in Eq. (5.2.2), we have

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right)\right]\right\} - \gamma^4 g = \delta(x)\,\delta(y) \tag{5.3.14}$$

Except at the origin, function g must be a combination of Bessel functions of order zero, so arranged that the solution represents an outgoing wave, $\sqrt{1/r}\,e^{i\gamma r}$, when r is large. Part of the solution will be the two functions $J_0(\gamma r)$ and $I_0(\gamma r) = J_0(i\gamma r)$ we have already discussed. But these two functions by themselves cannot produce a traveling wave; to do this we must also use the second solution of the Bessel equation, mentioned on page 8. This solution, called the *Neumann function* of zero order, is defined by the following series and asymptotic formula:

$$\begin{aligned} N_0(z) &= \frac{2}{\pi}[\ln(z) - 0.11593\ldots]J_0(z) - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m(z/2)^{2m}}{(m!)^2} \\ &\qquad \times\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\right) \\ &\to \sqrt{\frac{2}{\pi z}}\sin(z - \tfrac{1}{4}\pi) \qquad z \to \infty \\ N_0(iz) &= iJ_0(iz) + \frac{2}{\pi}[\ln(z) - 0.11593\ldots]J_0(iz) \\ &\qquad - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(z/2)^{2m}}{(m!)^2}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\right) \\ &\to \sqrt{\frac{2}{\pi z}}\sinh(z + \tfrac{1}{4}i\pi) \qquad z \to \infty \end{aligned} \tag{5.3.15}$$

The function N_0 has a logarithmic infinity at $z = 0$; so we were not able to use it to describe the free vibrations of a circular plate or membrane. It must be used for the forced motion of a circular plate or membrane, however, in order to be able to express an outgoing wave. Referring to Eq. (5.2.20), we see that this combination must be

$$J_0(z) + iN_0(z) \to \sqrt{\frac{2}{\pi z}}\,[\cos(z - \tfrac{1}{4}\pi) + i\sin(z - \tfrac{1}{4}\pi)] = \sqrt{\frac{2i}{\pi z}}\,e^{iz}$$

and thus

$$J_0(iz) + iN_0(iz) \to \sqrt{\frac{2}{\pi z}}\,e^{-z} \qquad z \to \infty$$

Therefore the solution of Eq. (5.3.14) must be a combination of these two combinations. The first one must be used because it is the only combination of the Bessel functions which represents purely outgoing waves, with no term containing e^{-iz} ($z = \gamma r$). The second one must be used because it is the only combination of $J_0(i\gamma r)$ and $N_0(i\gamma r)$ which does not contain a term with $e^{+\gamma r}$ (which would go to infinity at $r \to \infty$). Both of these combinations have a logarithmic infinity at $r = 0$, but these can be canceled against each other. Therefore the only combination of the four which stays finite at the origin and at infinity and which represents a purely outgoing wave is

$$\begin{aligned} g &= A[J_0(\gamma r) + iN_0(\gamma r) - J_0(i\gamma r) - iN_0(i\gamma r)] \\ &\to A\sqrt{\frac{2i}{\pi\gamma r}}\,e^{i\gamma r} \qquad r \to \infty \\ &\to A\left\{1 - \tfrac{1}{4}(\gamma r)^2 + \frac{i}{\pi}(\gamma r)^2 + \frac{i}{\pi}(\gamma r)^2[\ln(\gamma r) - 0.1159]\right\} \qquad r \to 0 \end{aligned} \tag{5.3.16}$$

where we must determine the value of A to satisfy Eq. (5.3.14).

If we integrate both sides of Eq. (5.3.14) over the area of a circle of small radius α centered at the origin, the integral on the right will be 1, no matter how small α is. The integral of $\gamma^4 g$ will go to zero as α goes to zero, but the integral of the first term will be

$$\int_0^{2\pi} d\phi \int_0^{\alpha} dr\,\frac{d}{dr}\left\{r\,\frac{d}{dr}\left[\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{dg}{dr}\right)\right]\right\} = 2\pi\left\{r\,\frac{d}{dr}\left[\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{dg}{dr}\right)\right]\right\}_{r=\alpha}$$

From Eq. (5.2.20), however, if g has the form given in Eq. (5.3.16),

$$\begin{aligned} \frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{dg}{dr}\right) &= -A\gamma^2[J_0(\gamma r) + iN_0(\gamma r) + J_0(i\gamma r) + iN_0(i\gamma r)] \\ &\to -A\gamma^2\left\{1 - \tfrac{1}{4}(\gamma r)^2 + \frac{4i}{\pi}[\ln(\gamma r) - 0.1159]\right\} \qquad r \to 0 \end{aligned}$$

so that

$$2\pi r \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right)\right] \to -8i\gamma^2 A \qquad r = \alpha \to 0$$

Since this must be equal to the integral of the delta functions, which is 1, the value of A is determined, and the Green's function for a plate of infinite extent is

$$\begin{aligned} g_\omega(r \mid r_0) &= \frac{i}{8\gamma^2}\left[J_0(\gamma R) + iN_0(\gamma R) - J_0(i\gamma R) - iN_0(i\gamma R)\right] \\ &\to \frac{i}{8\gamma^2} \qquad R \to 0; \qquad \gamma^4 = \omega^2 \frac{3\rho(1-s^2)}{Qh^2} \\ &\to -\frac{e^{i\gamma R}}{\sqrt{32\pi i\gamma^5 R}} \qquad R \to \infty \end{aligned} \tag{5.3.17}$$

where R is the distance between the source point at r_0, ϕ_0 and the observation point at r, ϕ.

We first notice that the plate displacement is finite, even at the point of application of the force (the source point). The stiffness of the plate prevents the yielding to a point force which characterizes the elastic membrane, as illustrated in Fig. 5.6. Mathematically, the functions available to represent the membrane displacement are but two, $J_0(\omega r/c)$ and $N_0(\omega r/c)$, and these must be arranged, for an infinite membrane, in the combination $J_0 + iN_0$ in order to have outgoing waves. Thus the logarithmic infinity in the N_0 solution requires that any material which satisfies the wave equation in two dimensions exactly must exhibit a logarithmic singularity at the point of application of a point force. In contrast, the equation $\nabla^4 Y - \gamma^4 Y = 0$ allows both the combinations $J_0(\gamma r) + iN_0(\gamma r)$ and $J_0(i\gamma r) + iN_0(i\gamma r)$; between them it is possible to devise a solution which is finite at $r = 0$ and represents outgoing wave motion at $r \to \infty$.

If the driving force at the origin is $F_\omega e^{-i\omega t}$, the steady-state plate displacement is

$$\begin{aligned} \eta_\omega(r,t) = \frac{iF_\omega}{16\omega h^2\rho}\sqrt{\frac{3\rho(1-s^2)}{Q}}\,&[J_0(\gamma r) + iN_0(\gamma r) \\ &- J_0(i\gamma r) - iN_0(i\gamma r)]e^{-i\omega t} \end{aligned} \tag{5.3.18}$$

The transverse velocity of the plate is $-i\omega\eta$, and the mechanical impedance of the plate to a concentrated simple-harmonic force of frequency $\omega/2\pi$ is

$$Z_m = 16h^2\rho\sqrt{\frac{Q}{3\rho(1-s^2)}} \tag{5.3.19}$$

a pure resistance, in contrast to the impedance, given in Eq. (5.3.13) for the plate of finite size, clamped at the edges, which is purely reactive (if the plate is in a vacuum). The difference, of course, lies in the fact that energy can escape from the source if the plate is infinite in extent, but cannot escape

(after steady state has set in) from the plate of finite size in a vacuum. These formulas will be useful later, when we investigate the effect of the plate vibrations on the medium surrounding it, and the reaction, back on the plate, of the induced wave motion in the medium.

Problems

1 An annealed steel bar 20 cm long is clamped at one end. If its cross section is a square 1 cm on a side, what will be the lowest four frequencies of vibration? If the cross section is a circle 0.5 cm in radius, what will they be? If the cross section is a rectangle of sides a and $2a$, what must be the value of a to have the bar's lowest frequency be 250 cps? For steel, $\rho = 7.7$, $Q = 2 \times 10^{11}$ dynes per sq cm.

2 A bar of steel of length 10 cm, whose cross section is a rectangle of sides 1 cm and 0.5 cm, is clamped at one end. It is struck at the midpoint of one of its wider sides, so that $v_0(x) = 0$ except at $x = l/2$, and $\int v_0\, dx = 100$. What is the shape of the bar a time $T_1/4$ after the blow? (T_1 is the period of the bar's fundamental.) What is its shape at a time $T_1/2$? What is the motion of the end point of the bar? What is the amplitude of that part of the motion of the end point corresponding to the fundamental? To the first overtone? To the second overtone?

3 Plot the shapes of the first three normal modes of vibration of a bar clamped at both ends. Of a completely free bar.

4 What is the energy of vibration of the bar, struck at one end, which is discussed on page 185? What is the ratio between the energy of the fundamental and that of the first overtone? Of the second overtone?

5 A cylindrical bar of radius a, clamped at one end, is damped by the air by a force equal to $1.25 \times 10^{-9}\, a^4 \nu^3\, (\partial y/\partial t)$ dynes per cm length of bar, where ν is the frequency of vibration and $\partial y/\partial t$ the bar's velocity. Show that the modulus of decay of the nth allowed frequency is $16 \times 10^8\, (\pi\rho/a^2 \nu_n^3)$, where ν_n is given by Eq. (5.1.16).

6 The Green's function $g(x \mid x_0)$ for transverse oscillations of a bar satisfies the equation

$$\frac{d^4 g}{dx^4} - \gamma^4 g = \delta(x - x_0) \qquad \gamma^4 = (2\pi\mu)^4 = \frac{\rho\omega^2}{Q\kappa^2}$$

with requirements that g, dg/dx, and d^2g/dx^2 be continuous at $x = x_0$.

(*a*) Show that, for a bar of infinite extent,

$$g(x \mid x_0) = \frac{i}{4\gamma^3}\left(e^{i\gamma|x-x_0|} + ie^{-\gamma|x-x_0|}\right)$$

(*b*) When the bar is acted on by a transverse force $F(x)e^{-i\omega t}$ per unit length of bar, so that

$$\frac{d^4 Y}{dx^4} - \gamma^4 Y = \frac{F}{Q\kappa^2 S} \equiv f(x)$$

show that [Eq. (4.5.13)]

$$Y(x) = \int_a^b f(x_0) g(x \mid x_0)\, dx_0 + \left[Y(x_0)\frac{d^3 g}{dx_0^3} - \frac{dY}{dx_0}\frac{d^2 g}{dx_0^2} + \frac{dg}{dx_0}\frac{d^2 Y}{dx_0^2} - g(x \mid x_0)\frac{d^3 Y}{dx_0^3}\right]_{x_0=a}^{x_0=b}$$

where x is inside the region $a < x < b$, and a is considerably to the left of, and b considerably to the right of, the region where f differs from zero. Show that, if all the transverse motion of the bar is caused by force F, the quantity in brackets is zero at $x_0 = a$ and $x_0 = b$.

7 A distributed load of transverse mechanical impedance $z(x)$ $(a \ll x \ll b)$ per unit length is attached to the infinite bar. A transverse wave $A \exp(i\gamma x - i\omega t)$ is incident from the left. (*a*) Use the results of Prob. 6 to show that the displacement of the bar satisfies the integral equation

$$Y(x) = Ae^{i\gamma x} - \tfrac{1}{4}\int_a^b \zeta(x_0)\, Y(x_0)\, (e^{i\gamma|x-x_0|} + ie^{-\gamma|x-x_0|})\, dx_0$$

where $\zeta(x) = \gamma z(x)/\rho\omega S$ is the ratio between the impedance density $z(x)$ of the load and the characteristic impedance $\rho Su = \rho S\omega/\gamma$ of the bar for the frequency $\omega/2\pi$. (*b*) What integral equals the ratio $-\Phi(\omega)$ between the amplitude of the reflected wave and that of the incident wave? Compare with Eq. (5.3.15). (*c*) The impedance load is concentrated at the origin, so that $z(x) = [-i\omega M + R + i(K/\omega)]\delta(x)$. Show that the ratio between the wave power reflected from the point load to the incident power is

$$|\Phi|^2 = \frac{R^2 + [\omega M - (K/\omega)]^2}{[(4\omega\rho S/\gamma) + R + \omega M - (K/\omega)]^2 + [R - \omega M + (K/\omega)]^2}$$

Compare this with Eq. (4.5.8). (*d*) For $Q = \sqrt{MK}/R = 10$ and $\omega\rho S/\gamma = R\sqrt{\omega/\omega_0}$ and $\omega_0 = \sqrt{K/M}$, plot $|\Phi|^2$ as a function of ω/ω_0 from 0 to 2. Compare this with the dashed curves of Fig. 4.19. Why, as $\omega \to 0$, does the reflection ratio go to $\frac{1}{2}$, whereas for the string it goes to 1?

8 Show that the Green's function for the clamped-free bar is

$$G(x,x_0) = \frac{2Q\kappa^2}{l\rho}\sum_n \frac{\psi_n(x)\psi_n(x_0)}{\omega_n^2 - \omega^2}$$

where $\omega_n^2 = \pi^4 Q\kappa^2\beta_n^4/l^4\rho = (2\pi\nu_n)^2$, and the other quantities are defined in Eqs. (5.1.16) and (5.1.17). Show that when the bar is pushed sideward by a force $F(x)e^{-i\omega t}$, the steady-state displacement of the bar is $Y(x)e^{-i\omega t}$, where

$$Y(x) = \frac{1}{Q\kappa^2 S}\int_0^l G(x,x_0)F(x_0)\, dx_0$$

Compare this with Eq. (5.1.24). What is the corresponding formula for a clamped-clamped bar?

9 A clamped-free bar is in a homogeneous magnetic field B perpendicular to the bar axis. A wire of negligible mass and stiffness is connected to the free end, so that a current $Ie^{-i\omega t}$ is passed along the bar. (*a*) Obtain the series expansion for the steady-state displacement. (*b*) Do the same for the clamped-clamped bar. Plot the shape of the bar when $\omega = \frac{1}{2}\omega_1$; when $\omega = 2\omega_1$ (to do this you will have to compute the integral of the square of ψ_n for the clamped-clamped bar for $n = 1, 2, 3$).

10 A point mass α times the total mass ρlS of the bar is fastened to the free end of a clamped-free bar. Use the result of Prob. 8 to show that the integral equation for the bar's displacement becomes

$$Y(x) = 2^{\frac{3}{2}}\omega^2\alpha\, Y(l)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\psi_n(x)}{\omega_n^2 - \omega^2}$$

remembering that $\psi_n(l) = (-1)^{n-1}\sqrt{2}$ for this bar. (*a*) Play the game of Chap. 4, Prob. 24, by "normalizing" Y so that $\omega^2[1 + 2\frac{3}{2}\alpha Y(l)] = \omega_1^2$, and thus obtain the implicit equation for $Y_1(l)$ (and thus for the lowest natural frequency of the loaded bar),

$$Y(l) = \sqrt{2}\left[1 - 4\alpha \sum_{n=2}^{\infty} \frac{(\omega_1/\omega_n)^2}{1 + 2\frac{3}{2}\alpha Y(l) - (\omega_1/\omega_n)^2}\right]^{-1}$$

where ω_n is given in Prob. 8 and Eq. (5.1.16). (*b*) For $\alpha = \frac{1}{2}$ and $\alpha = 1$ solve this for $Y(l)$, then compute the ratio (ω/ω_1) between the loaded and unloaded natural frequency for the lowest mode; finally, plot the corresponding $Y(x)$ for the two values of α. Are these exact solutions if the equation for $Y(l)$ is solved exactly?

11 An oscillating driving force of frequency ν is applied to the free end of a clamped-free bar. Show that the mechanical impedance of the bar is

$$Z = \tfrac{1}{4} i l \rho S \left(\sum_n \frac{\nu}{\nu_n^2 - \nu^2} \right)^{-1}$$

Plot $|4Z/\rho l S \nu_1|$ against ν/ν_1 from 0 to 4.

12 A steel wire ($\rho = 7.7$ g per cu cm, $Q = 2 \times 10^{11}$ dynes per sq cm) 2 mm in diameter is stretched between two rigid supports 100 cm apart, with a tension of 2×10^6 dynes. Calculate the ratios between the first three overtone frequencies and the fundamental.

13 A membrane is made of material of density 0.1 g per sq cm and is under a tension of 100,000 dynes per cm. It is wished to have the membrane respond best to sound of frequency 250 cps. If the membrane is square, what will be the length of one side? What will be the frequencies of the two lowest overtones?

14 A square membrane 20 cm on a side, of mass 1 g per sq cm, is under a tension of 10^8 dynes per cm. Its motion is opposed by a frictional force of $42(\partial\eta/\partial t)$ dynes per sq cm. Find the modulus of decay of the oscillations. What are the first four "frequencies" of the damped motion?

15 A square membrane b cm on a side, of density σ and under a tension T, is loaded at its center with a mass of M g. Show that the allowed frequencies are, approximately,

$$\nu = \left\{ \tfrac{1}{4} \frac{T}{\sigma} \left(\frac{m^2}{b^2} + \frac{n^2}{b^2} \right) \left[1 - \frac{4M}{\sigma b^2} \psi_{mn}^2 \left(\frac{b}{2}, \frac{b}{2} \right) \right] \right\}^{\frac{1}{2}}$$

as long as $4M/\sigma b^2$ is small. What will be the expression for the frequency if the membrane is rectangular but not square?

16 Show that the energy of vibration of a rectangular membrane of sides a and b is

$$W = \tfrac{1}{2}\sigma c^2 \int_0^a \int_0^b \left[\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 + \frac{1}{c^2} \left(\frac{\partial \eta}{\partial t} \right)^2 \right] dx\, dy$$

and that when the motion is given in terms of the series

$$\eta = \sum_{m,n} A_{mn} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b} \cos (2\pi \nu_{mn} t + \Phi_{mn})$$

the series for the energy is

$$W = \frac{\pi^2}{2} \sigma a b \sum_{m,n} \nu_{mn}^2 A_{mn}^2$$

17 A square membrane 20 cm on a side, with $\sigma = 1$ and $T = 10^6$, is started from rest at $t = 0$ with an initial shape $\eta = 10^{-5}x(20 - x)y(20 - y)$. What are the energies of vibration corresponding to the fundamental and the lowest three overtones, and what is the total energy of vibration of the membrane?

18 A rectangular membrane is pushed aside at the point (x_0,y_0) by a steady force F_0 and is then suddenly released at $t = 0$. What is the expression for the subsequent motion of the membrane? Neglect the reaction of the air.

19 The tensile strength of aluminum is 2.5×10^9 dynes per sq cm, and its density is 2.7 g per cc. What is the highest value of fundamental frequency that can be attained with an aluminum membrane stretched over a circular frame 3 cm in radius? If the aluminum is 0.005 cm thick, what will be the maximum tension attainable?

20 If the natural frequencies and characteristic functions for a membrane are $\omega_{mn}/2\pi = \nu_{mn}$ and $\psi_{mn}(\mathbf{r})$, where $\mathbf{r}$ stands for x, y or r, ϕ, etc., and where the mean square of ψ_{mn} is Λ_{mn}, show that the Green's function for the membrane is

$$G_\omega(\mathbf{r},\mathbf{r}_0) = \frac{c^2}{S}\sum_{m,n}\frac{\psi_{mn}(\mathbf{r})\psi_{mn}(\mathbf{r}_0)}{\Lambda_{mn}(\omega_{mn}{}^2 - \omega^2)}$$

If the membrane is acted on by a transverse force $F(\mathbf{r})e^{-i\omega t}$, show that the steady-state displacement of the membrane is

$$\eta = \frac{1}{\sigma c^2}\iint G_\omega(\mathbf{r},\mathbf{r}_0)\,F(\mathbf{r}_0)\,dS_0\,e^{-i\omega t}$$

where the integration is over the whole area S of the membrane.

21 A square membrane of side a is driven by a uniform force $Fe^{-i\omega t}$ per unit area. Show that the displacement of the center of the membrane is

$$\frac{16a^2F}{\pi^4\sigma c^2}\sum_{m,n}\left\{(2m+1)(2n+1)\left[(2m+1)^2 + (2n+1)^2 - 2\left(\frac{\omega}{\omega_{11}}\right)^2\right]\right\}^{-1}$$

times $e^{-i\omega t}$, where $\omega_{11}{}^2 = 2(\pi c/a)^2$ is the lowest natural frequency. Plot $\pi^4\sigma c_2/16a^2F$ times this as a function of ω/ω_{11} from 0 to 10.

22 A circular membrane is driven by a force per unit area equal to

$$F_0\cos(m\phi)J_m(qr/a)e^{-i\omega t}$$

Since $\nabla^2\cos(m\phi)J_m(qr/a) = -(q/a)^2\cos(m\phi)J_m(qr/a)$, show that the solution of the equation

$$\nabla^2\eta + \left(\frac{\omega}{c}\right)^2\eta = -\frac{F_0}{\sigma c^2}\cos(m\phi)J_m\left(\frac{qr}{a}\right)$$

for steady-state motion of the membrane is

$$\frac{A}{\sigma[(qc/a)^2 - \omega^2]}\left[J_m\left(\frac{qr}{a}\right) - \frac{J_m(q)}{J_m(\omega a/c)}J_m\left(\frac{\omega r}{c}\right)\right]\cos(m\phi)$$

which goes to zero at $r = a$ and goes to infinity at the resonant frequencies, when $J_m(\omega a/c) = 0$, but does *not* become infinite when $q = \omega a/c$. Thus, by comparison with the results of Prob. 20, show that

$$\int_0^{2\pi}d\phi\int_0^a\cos(m\phi_0)G_\omega(\mathbf{r},\mathbf{r}_0)J_m\left(\frac{qr}{a}\right)r\,dr$$
$$= \frac{a^2}{q^2 - (\omega a/c)^2}\left[J_m\left(q\,\frac{r}{a}\right) - \frac{J_m(q)}{J_m(\omega a/c)}J_m\left(\omega\,\frac{r}{c}\right)\right]\cos(m\phi)$$

Show that the Green's function for this equation is

$$G_\omega(\mathbf{r},\mathbf{r}_0) = \frac{1}{\pi}\sum_{m,n}\frac{\epsilon_m}{J_{m-1}^2(\pi\beta_{mn})}\cos m(\phi-\phi_0)\frac{J_m(\pi\beta_{mn}r/a)J_m(\pi\beta_{mn}r_0/a)}{(\pi\beta_{mn})^2-(\omega a/c)^2}$$

where $\epsilon_0 = 1$, $\epsilon_m = 2 \quad (m > 0)$.

23 By use of Eq. (1.2.9) show that a plane acoustic wave of velocity c_a, incident on the xy plane at an angle θ to the z axis, produces a pressure distribution on the xy plane of

$$P_0\exp\left[i\left(\frac{\omega}{c_a}\right)x\sin\theta - i\omega t\right] = P_0\sum_{m=0}^{\infty}\epsilon_m i^m\cos(m\phi)J_m\left(\frac{\omega r}{c_a}\sin\theta\right)e^{-i\omega t}$$

If this is the driving force on a circular membrane in the xy plane, show that its displacement is [see Eq. (6.3.7)]

$$\eta = \frac{P_0a^2/\sigma c^2}{(\omega a/c_a)^2[\sin^2\theta-(c_a/c)^2]}\sum_m\epsilon_m i^m\cos(m\phi)\left[J_m\left(\frac{\omega r}{c_a}\sin\theta\right) - \frac{J_m[(\omega a/c_a)\sin\theta]}{J_m(\omega a/c)}J_m\left(\frac{\omega r}{c}\right)\right]e^{-i\omega t}$$

where $c = \sqrt{T}/\sigma$ is the speed of transverse waves on the membrane.

24 A membrane is fastened over the top of a vessel of volume V_0 so that the integral of the membrane's displacement is equal to the net increase in volume δV occupied by the air in the vessel. If the oscillations are not too rapid, this will produce a change in pressure $-(\delta V/V_0\kappa_s)$, where $\kappa_s = 1/\rho_a c_a^2$ is the compressibility of the air in the vessel, and c_a the speed of sound. Show that the equation of motion for transverse motion of the membrane is

$$\frac{1}{c^2}\frac{\partial^2\eta}{\partial t^2} = \nabla^2\eta - \frac{\rho_a c_a^2}{V_0 T}\iint\eta\,dS \qquad T = \sigma c^2$$

where the integration is over the whole membrane area. If it is circular of radius a, show that the vibrations of the form $A\cos(m\phi)J_m(\pi\beta_{mn}r/a)$ $(m > 0)$ will have frequencies given by Eqs. (5.2.24). Show that, on the other hand, the radially symmetric vibrations $(m = 0)$ have different frequencies and that the form of the displacement is $\eta = A[J_0(\omega r/c) - J_0(\omega a/c)]e^{-i\omega t}$, the allowed frequencies being obtained by substituting this form in the equation of motion. Show that the resulting equation reduces to

$$(\omega a/c)^2J_0(\omega a/c) = -(\pi\rho_a c_a^2 a^4/V_0\sigma c^2)J_2(\omega a/c)$$

Solve this equation for the lowest allowed value of $\omega a/c$ when the fraction $\pi\rho_a c_a^2 a^4/V_0\sigma c^2 = 1$.

25 Carry through Prob. 22 when the circular diaphragm over the vessel is a plate, as given in Eq. (5.3.10) and preceding.

26 The circular plate of Eq. (5.3.6) is loaded with a point mass $M = 2\alpha\pi a^2 h\rho$ (α times the total mass of the plate) placed at its center. Use the methods of Prob. 10 [see also Eq. (5.3.12)] to compute the allowed frequencies and corresponding characteristic functions for the loaded plate.

CHAPTER

6

ACOUSTIC WAVE MOTION

6.1 THE DYNAMICS OF FLUID MOTION

We now come to the study of wave motion in air, the most important type of wave motion studied in the science of acoustics. Sound waves differ from the waves that we have discussed heretofore in several important respects. They are waves in three dimensions and as such can be more complicated in behavior than waves in two dimensions or in one. Sound waves also differ from waves on a string or on a membrane by being *longitudinal* waves. So far, we have been studying *transverse* waves, where the material transmitting the wave moves in a direction perpendicular to the direction of propagation of the wave. Each part of the string, for instance, moves in a direction at right angles to the equilibrium shape of the string, whereas the wave travels along the string. The molecules of air, however, move in the direction of propagation of the wave, so that there are no alternate crests and troughs, as with waves on the surface of water, but alternate compressions and rarefactions. The restoring force responsible for keeping the wave going is simply the opposition that the fluid exhibits against being compressed.

Since there are so many points of difference between the waves discussed earlier and the more complicated forms of sound waves, it is well not to introduce all the complications at once. Accordingly, we shall first study the motion of *plane waves* of sound, waves having the same direction of propagation everywhere in space, whose "crests" are in planes perpendicular to the direction of propagation. They correspond to the parallel waves on a membrane. Waves traveling along the inside of tubes of uniform cross section will usually be plane waves. Waves that have traveled unimpeded a long distance from their source will be, very nearly, plane waves.

The detailed properties of the acoustic wave motion in a fluid depend on the ratios between the amplitude and frequency of the acoustic motion and the molecular mean-free-path and the collision frequency, on whether the fluid is in thermodynamic equilibrium or not, and on the shape and

thermal properties of the boundaries enclosing the fluid. All these effects will be discussed in this or in later chapters. At present we shall make a start by looking at the simplest case, that of an idealized fluid, uniform and continuous in its properties, at rest in thermodynamic equilibrium, except for the motion caused by the sound waves themselves, and with this acoustic motion small enough in magnitude so that many nonlinear effects are negligible and the equation of motion takes on its simplest form. We shall find this simplified model of acoustic wave motion an adequate representation in the majority of the cases of practical interest.

General characteristics of sound

Fluids, such as air or water, have mass density and volume elasticity; thus they have many of the characteristics of the chain of masses and springs of Fig. 3.4. The elasticity causes the fluid to resist being compressed, tending to return to its original state, and the inertia of the mass density causes the motion to "overshoot," thus providing the two requisites of wave motion. At equilibrium the fluid has density ρ, in units of kilograms per cubic meter; is under a uniform pressure P in units of newtons per square meter (or in atmospheres, where 1 atm $= 10^5$ newtons per sq m); and is at a uniform temperature T°K. These three quantities are connected by an equation of state, which can be given either as an explicit equation, relating ρ, P, and T, or in terms of the partial derivatives, assuming density as a function of T and P.

The equation of state is usually expressed in terms of the volume V occupied by n moles of the fluid (a mole is M kg of a substance which has molecular weight M). If nM kg of the fluid occupies volume V, the density of the fluid is obviously $\rho = nM/V$, or $V = nM/\rho$. The two partial derivatives, which are usually employed to measure the equation of state of the fluid near the equilibrium state, are

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T \qquad \beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \tag{6.1.1}$$

Quantity κ_T, the fractional rate of change of volume (or density) with pressure at constant temperature, is called the *isothermal compressibility* of the fluid. As was indicated in Eq. (3.3.2), the compressibility of a spring is $\kappa = \frac{1}{l}\frac{\Delta l}{F} = \frac{1}{lK}$, where $F = K\,\Delta l$ is the force applied to shorten the spring by an amount Δl, and K is the stiffness constant. Quantity β, the fractional change in volume (or density) with temperature at constant pressure, is called the *coefficient of thermal expansion* of the fluid. Both κ_T and β are, of course, functions of P and T, though usually their rates of change are not very large. From κ_T and β we can compute the other partial derivatives of ρ, P, and T near the equilibrium state of the fluid; for example, if P, β, and κ_T are expressed as functions of ρ and T, then $(\partial P/\partial T)_\rho = \beta/\kappa_T$.

For a perfect gas the equation of state can be given as an explicit equation, $PV = nRT$, or $MP = RT\rho$, where $R = 8.3 \times 10^3$ joules per mole degree absolute. Therefore, for perfect gases, κ_T and β take on particularly simple values:

$$\kappa_T = \frac{1}{P} \qquad \text{and} \qquad \beta = \frac{1}{T}$$

For liquids, κ_T and β have much smaller values.

Kinetic theory indicates that the pressure of a perfect gas is caused by its molecular motion and that $P = \frac{1}{3}\rho\langle v^2\rangle$, where $\langle v^2\rangle$ is the mean-square molecular velocity and ρ is the gas density. The compressibility of a perfect gas is thus inversely proportional to the mean kinetic energy of the gas molecules. We noted in Eq. (3.3.2) that the velocity of propagation of waves along a spring-mass system is $\sqrt{1/\kappa\epsilon}$, where κ is the compressibility of the spring, and ϵ is the mass per unit length of the chain. We shall show that the velocity of propagation of sound waves in a fluid is $\sqrt{1/\kappa\rho}$, where κ is the compressibility and ρ is the density. For perfect gases, where $\kappa_T = 1/P = 3/\rho\langle v^2\rangle$, we see that the speed of sound

$$c = \sqrt{\frac{1}{\kappa\rho}} \simeq \sqrt{\frac{\langle v^2\rangle}{3}}$$

is approximately equal to the root-mean-square of the molecular speed. In liquids and solids, where intermolecular forces are large, the speed of sound is considerably larger than the root-mean-square speed of vibration of an atom about its equilibrium position. It can be shown, in fact, that c in solids is roughly equal to the mean speed a molecule would have if its amplitude of vibration were equal to the mean distance between molecules. The actual amplitudes are much smaller than this; otherwise the solid would melt or vaporize.

Thus the presence of a sound wave produces changes in density, pressure, and temperature in the fluid, each change being proportional to the amplitude of the wave. The pressure changes are usually the most easily measurable, though some sound detectors measure the motion of the fluid, caused by the sound. Direct measurement of temperature or density change is difficult, and the readings are not particularly accurate; so they are usually computed from the measured pressure change.

Pressure measurements are usually expressed in dynes per square centimeter ($= 10^{-1}$ newton per sq m $= 10^{-6}$ atm) above or below equilibrium pressure. Sometimes the pressure unit is 0.00022 dyne per sq cm, the amplitude of pressure change in a barely audible sound wave of 1,000 cps frequency [this is in terms of rms amplitudes; see Eq. (6.2.18)]. For simple-harmonic waves the pressure amplitude can be given in terms of the *pressure level* in *decibels*, 20 times the logarithm to the base 10 of the pressure amplitude in dynes per square centimeter, above or below the level of 1 dyne per

sq cm. Or it can be given as pressure level in decibels above or below 0.00022 dyne per sq cm; on this scale a pressure amplitude of 1 dyne per sq cm rms amplitude is 74 db.

In fluids with very large thermal conductivity the temperature of the fluid is practically unchanged by the passage of the sound wave. In this case the isothermal compressibility κ_T is directly applicable. If the pressure change caused by the sound is p (the *acoustic pressure*) and the equilibrium pressure is P, so that the total pressure in the presence of sound is $P + p$, then the density of the fluid is

$$\rho + \delta = \rho + \left(\frac{\partial \rho}{\partial P}\right)_T p = \rho + \rho \kappa_T p \qquad \text{or} \qquad \delta = (\kappa_T \rho) p \tag{6.1.2}$$

where ρ is the equilibrium density, and δ is the small increase in density produced by the sound.

For sound of frequencies below about 10^9 cps or so, in gases, however, it is a better approximation to assume that the compression is *adiabatic*, i.e., that the entropy content of the gas remains nearly constant during the compression [see the discussion following Eqs. (6.4.23)]. In this case the differential relations between density and pressure and between temperature and pressure are

$$\begin{aligned} \left(\frac{\partial \rho}{\partial P}\right)_s &= \rho \kappa_s = \frac{1}{\gamma}\left(\frac{\partial \rho}{\partial P}\right)_T \qquad \text{or} \qquad \kappa_s = \frac{\kappa_T}{\gamma} \\ \left(\frac{\partial T}{\partial P}\right)_s &= \frac{\gamma - 1}{\gamma}\left(\frac{\partial T}{\partial P}\right)_\rho = (\gamma - 1)\frac{\kappa_s}{\beta} \end{aligned} \tag{6.1.3}$$

so that $$\rho + \delta = \rho + \rho \kappa_s p \qquad T + \tau = T + (\gamma - 1)\frac{\kappa_s}{\beta} p$$

for adiabatic compression, where $\gamma = (C_p/C_v)$ is the ratio of the specific heat at constant pressure of the gas to the specific heat at constant volume, and τ is the small change in temperature caused by the sound wave. For a perfect monatomic gas (such as helium), $\gamma = \frac{5}{3}$; for a perfect diatomic gas (such as hydrogen or air), $\gamma = \frac{7}{5}$; for polyatomic gases, $\gamma = \frac{4}{3}$. Therefore, for perfect gases, where $MP = RT\rho$,

$$\begin{aligned} \kappa_s &= \frac{1}{\gamma P} \qquad \text{and} \qquad \left(\frac{\partial \rho}{\partial P}\right)_s = \frac{\rho}{\gamma P} = \frac{M}{\gamma RT} \\ \left(\frac{\partial T}{\partial P}\right)_s &= \frac{\gamma - 1}{\gamma}\frac{T}{P} = (\gamma - 1)\frac{M}{\gamma R \rho} \end{aligned} \tag{6.1.4}$$

and if the pressure in the presence of the sound is $P + p$, the density and

temperature are

$$\rho + \delta = \rho + \left(\frac{\rho}{\gamma P}\right)p \qquad \text{or} \qquad \delta = (\rho\kappa_s)p = \left(\frac{M}{\gamma RT}\right)p$$
$$T + \tau = T + \frac{(\gamma - 1)T}{\gamma P}\,p \qquad \text{or} \qquad \tau = \frac{(\gamma - 1)M}{\gamma R\rho}\,p \tag{6.1.5}$$

for adiabatic compression, for a perfect gas.

We see that, in this simple case, the ratio between the change in density δ and the acoustic pressure p is independent of the equilibrium pressure but inversely proportional to the equilibrium temperature of the gas. Also, the ratio between the change in temperature τ and the sound pressure p is independent of temperature but inversely proportional to the equilibrium density ρ. All these relationships are valid only to the first order in the small quantity p/P.

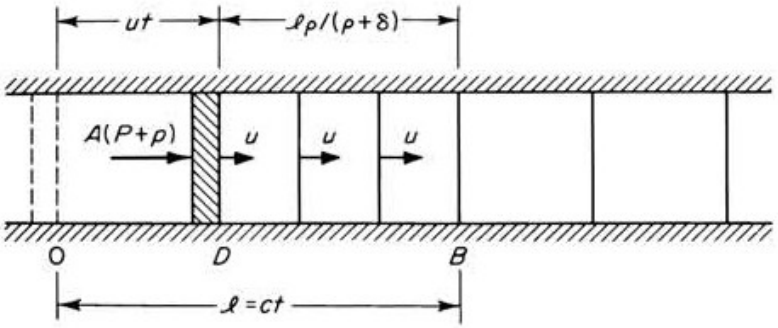


FIGURE 6.1
Pressure wave generated by motion of piston D in pipe of area A.

Acoustic energy and momentum

The discussion of the energy and momentum carried in a sound wave can be carried through in a manner quite parallel to that given on wave motion along a linear array of masses and springs, pictured in Fig. 3.5. Suppose the fluid is confined to the interior of a long pipe, of uniform cross section A, having rigid walls. At one end, as shown in Fig. 6.1, is a piston of negligible mass, which constrains the fluid to stay to its right. Originally, the piston was at rest at 0, held stationary by a force AP, counteracting the equilibrium pressure P of the fluid (this may, of course, be supplied by the presence of similar fluid to the left of the piston).

Now suppose, at time $t = 0$, the piston is started into motion to the right with constant velocity u. It will push the fluid ahead of it, compressing it somewhat and thus increasing its pressure to $P + p$. Thus, in order to get the piston into motion, the force on the piston must be increased from AP to $A(P + p)$, where p is related to the change in density δ of the fluid

just ahead of the piston by the equation $\delta = \kappa\rho p$, with κ either κ_T or κ_s, whichever is appropriate.

But δ is related to the change in volume of the compressed fluid, and this is related to the amount of fluid which has been set in motion at any specified time t, as shown in Fig. 6.1. Because of the fluid's compressibility and inertia, it takes time to get it into motion; a wavefront of starting will travel away from the piston, reaching point B at time t. Suppose at time t the wavefront has traveled a distance ct, where c is the velocity of the wave. Then all the fluid to the right of this front, at B, is in its original equilibrium condition, at rest with density ρ. The fluid in the region D to B, in motion with the velocity u of the piston, is compressed to a density $\rho + \delta$, and thus occupies a volume $\rho/(\rho + \delta)$ times its original volume $Al = Act$. The difference between this volume and the original volume Act is, of course, the volume Aut, swept out by the piston in time t, so that

$$Act - A\frac{\rho ct}{\rho + \delta} = Aut$$

or

$$\frac{u}{c} = 1 - \frac{\rho}{\rho + \delta} = \frac{\delta}{\rho + \delta} \simeq \frac{\delta}{\rho} \tag{6.1.6}$$

where the last formula is valid only as long as the density change δ, caused by the motion, is small compared with the equilibrium density ρ.

In traveling the distance ut, the piston has done an amount of work $A(P + p)ut$, which must have gone into additional energy of the fluid. Part of this energy will be kinetic energy, one-half of the mass of the fluid in motion, $\frac{1}{2}A\rho ct$, times the square of its velocity, u^2. The other part is potential energy of compression, the work done in compressing the fluid from volume Act to volume $Act[\rho/(\rho + \delta)]$. This work is the integral of the pressure $P + p$ times the change in volume dV, integrated between these limits. But from Eq. (6.1.1), or its equivalent for adiabatic compression, $dV = V\kappa\, dp$, where, as long as p is small compared with P, V can be set equal to Act and κ can be considered constant. Thus the potential energy is

$$\int (P + p)V\kappa\, dp = PV\kappa p + \tfrac{1}{2}V\kappa p^2 \simeq Aut\left(P + \frac{u}{2\kappa c}\right)$$

where, to obtain the last form, we have set $p = \delta/\rho\kappa = u/\kappa c$, from Eqs. (6.1.5) and (6.1.6). Thus the equation for energy balance becomes

$$\text{Force} \times \text{distance} = \text{potential plus kinetic energy}$$

$$Aut(P + p) = AutP + Aut\frac{u}{2\kappa c} + Aut\frac{2}{\rho cu}$$

The first terms on both sides cancel. In fact, the work $APut$ should not be charged to energy of the sound wave, since it is the work done pushing against the equilibrium pressure; if there had been fluid pressure P on the

left-hand surface of the piston, it would have provided the AP part of the force, and only force Ap would have been required to move the piston. In fact, from now on, we shall use the expressions

$$\begin{aligned} \tfrac{1}{2}\kappa p^2 &= \text{potential energy density} \\ \tfrac{1}{2}\rho u^2 &= \text{kinetic energy density} \end{aligned} \tag{6.1.7}$$

for the energies per unit volume of the sound, correct to second order in the small quantities p and u.

Returning to the energy-balance equation, which can be reduced to $p = (u/2\kappa c) + (\rho c u/2)$, we can again set $p = u/\kappa c$ and obtain $u/\kappa c = (u/2\kappa c) + (\rho c u/2)$, which in turn becomes

$$c^2 = \frac{1}{\rho\kappa} \tag{6.1.8}$$

What we have shown by this manipulation is that the mass of the fluid, represented by the equilibrium density ρ, and its elasticity, represented by κ, so combine that it takes time for the motion of one part of the fluid to be transmitted to another part, the speed of transmission being equal to $\sqrt{1/\rho\kappa}$ as long as the fluid motion is small enough so that the excess pressure p is small compared with the equilibrium pressure P.

The impulse-momentum relationship could also be used to compute c. The excess force Ap contributes an impulse Apt to the fluid in time t, which must equal the momentum contributed to the fluid newly set into motion in time t. The acoustic momentum density of the fluid is $u\rho$, so that

$$Apt = (u\rho)(Act) \simeq Ac^2\,\delta t$$

from Eq. (6.1.6). The resulting equation, $p \simeq c^2\delta$, is identical with either Eq. (6.1.2) or (6.1.3), depending on whether the compression is isothermal or adiabatic, if c^2 is given by Eq. (6.1.8).

It is obvious that the wave velocity c will be independent of the amplitude and shape of the sound wave only as long as the sound pressure p is small compared with the equilibrium pressure P. This must be true in order for Eq. (6.1.6) to hold, and also in order to consider κ to be independent of p. For fluids like water, where κ is small enough so that δ/ρ is small even though p/P is not small, another limitation enters. For example, if the piston were pushed to the left, a wave of rarefaction would be sent along the tube, and if u were large enough, the p computed from the formulas might be equal to or larger than P; the piston would be withdrawn rapidly enough to produce a vacuum behind it. When this happens the fluid loses contact with the piston, leaving a large bubble of vapor in the interspace, and it is obvious that our equations no longer hold. Long before this happens the value of κ begins to change, so that κ cannot be considered independent of p unless $|p|$ is small compared with P.

Fluid displacement and velocity

Our next task is to work out an approximate equation of motion for small-amplitude sound waves in a fluid. But before we do this, we must develop a notation which will adequately describe the motion. As a matter of fact, we shall develop two notations, one of which will be generally employed; but the other will be occasionally useful.

The most straightforward way, the extension of the notation we have used for the chain of masses and springs, is to label each portion of the fluid (each molecule in it, so to speak) by giving its position at $t = 0$, and then to give the subsequent position of each portion as a function of time and of its initial position. In this notation, called the *Lagrange* description, we follow the fluid in its motion. In one-dimensional motion, the portion of fluid initially at x_0 has position X at time t, where X is a function of t and of x_0, as shown in Fig. 6.2. All other quantities similarly follow the motion of the fluid. The density $\rho(x_0,t)$ is the density of the portion of the fluid which was initially at x_0 and which is at X at time t.

This notation gives us a good insight into what is going on inside the fluid as it moves about. For example, if the time rate of change $d\rho/dt$ of this density is less than zero, we can be sure the fluid is expanding. We shall reserve the *total derivative* notation, d/dt, for the time rate of change in the Lagrange description, the change in a particular portion of the fluid as it moves along. The derivative du/dt, where $u = dX/dt$ is the fluid velocity, is the acceleration of the element of fluid which was originally at x_0, and is thus equal to the force acting on the element, divided by its mass. However, the notation also has its disadvantages, for the coordinate system moves with the fluid, and it is not easy to determine what the fluid is doing at some specified point in space at a given time.

So the alternative notation, the *Euler description*, sets up a coordinate system (x, in one dimension) fixed in space and describes the properties of the fluid which happens to be at a given point at a given time. Therefore, in this notation, the various quantities are functions of x and of t, for one dimension, as shown in the lower half of Fig. 6.2. These are quantities, such as density, velocity, etc., measured at a *fixed point in space* at a given

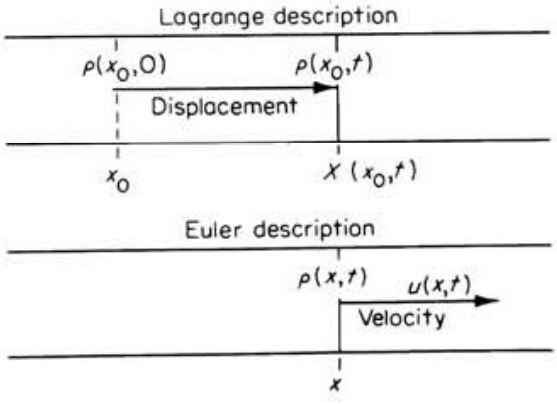


FIGURE 6.2
Alternative ways of describing the motion of a fluid.

time, and therefore corresponding to *different portions* of the fluid, as the fluid streams past the point. This is the description we shall be using most of the time, with ρ and u given as explicit functions of x and t; by it we can tell what is happening where, without having to go back to find which part of the fluid happens to be there at time t. In particular, the Euler description enables us to compute easily the spatial variation of a given quantity at time t, the partial derivative $\partial\rho/\partial x$ being the rate of change of density at x, as might be measured from an instantaneous photograph of the density at time t.

Comparison of Lagrange and Euler descriptions

We should note that the displacement of a given portion of the fluid at time t ($X - x_0$ in the Lagrange description) is not usually specified in the Euler notation; in fact, there is no simple way to write this quantity. The Euler description starts with the fluid velocity $u(x,t)$, at point x at time t, which equals the quantity dX/dt for that portion of the fluid for which $X(x_0,t) = x$ at time t, in the Lagrange description, and also deals with the other properties of the fluid, such as density $\rho(x,t)$, temperature $T(x,t)$, and so on. To show how the two descriptions differ, consider the situation pictured in Fig. 6.3. We have assumed that a one-dimensional sinusoidal wave, of frequency $\omega/2\pi$, is traveling in the positive x direction with velocity c, so that the wavelength is $\lambda = 2\pi/k = 2\pi c/\omega$. We also assume that the amplitude of motion of the fluid is *not* small compared with the wavelength, so that the effects will be easily seen.

By the Lagrange description the displacement from its equilibrium position of that portion of the fluid which was at x_0 before the wave came along is a traveling sinusoidal wave,

$$X - x_0 = A \sin(\omega t - kx_0) = -A \sin[k(x_0 - ct)]$$

In accordance with the Lagrange description, X is a function of t and of the parameter x_0, which describes where the fluid was before the wave started, not where it is while the wave is present. The velocity of the portion of the fluid, which is at X at time t, is

$$u(x_0,t) = \frac{dX}{dt} = A\omega \cos(\omega t - kx_0)$$

The various curves in Fig. 6.3*a* are plots of X as a function of t for the various values of x_0 which are marked on the right-hand end of each curve. We see that the fluid, in the region of $x = 0$, is compressed at $t = 0$, when the fluid is moving in the direction of the wave motion, and is correspondingly rarefied at $t = 10$, when the motion is in the opposite direction. We have chosen the scales of x and t so that the dashed lines, which indicate the motion of the wavefronts, are inclined at 45° to the t axis. Parenthetically, if the amplitude of motion in an actual sound wave were as large as this, the

displacement would not be sinusoidal. This does not invalidate the point of our discussion, however, as will be appreciated later.

To see the difference between the Lagrange and the Euler descriptions, let us try to answer the following question: What is the displacement $X - x_0$ of the portion of fluid which happens to be at point x at time t?

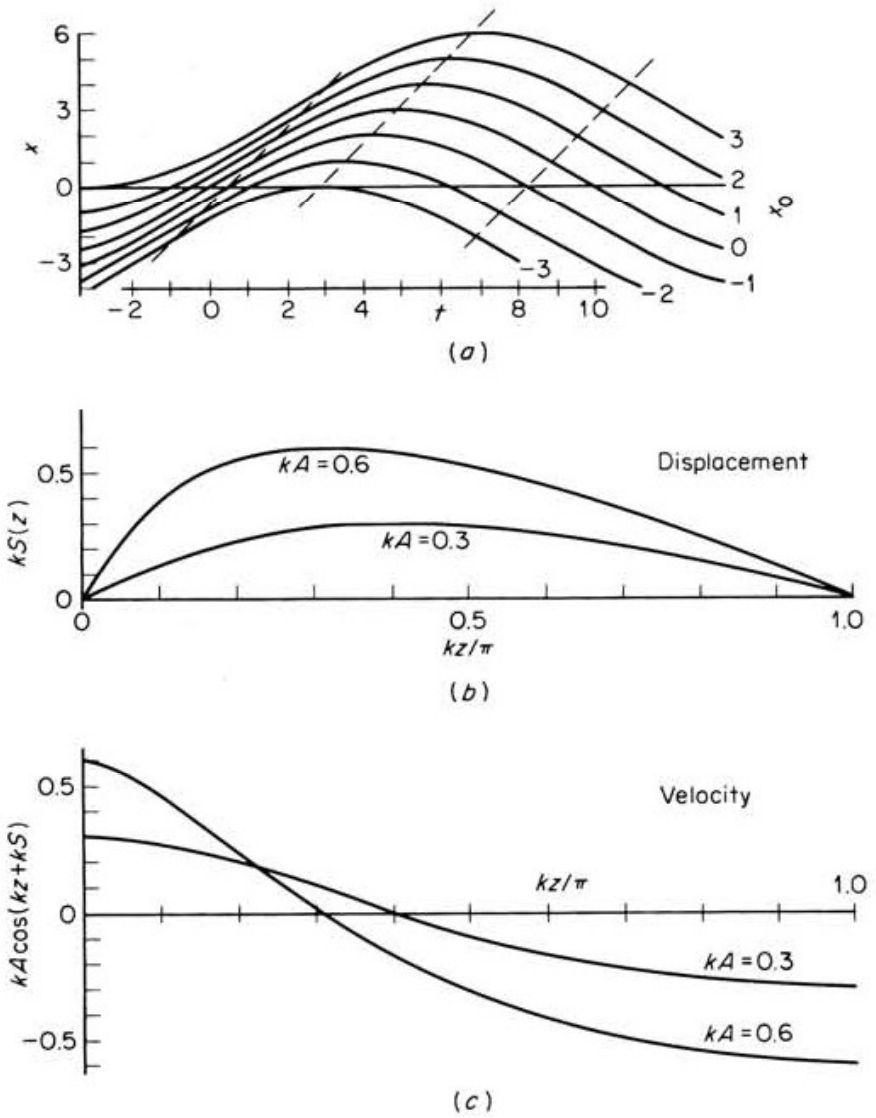


FIGURE 6.3
Comparison between Lagrange and Euler descriptions of one-dimensional fluid motion.

To answer this we must change our description from initial position x_0 to the final position x at time t. For example, at $t = 0$ the fluid which was at $x_0 = 0$ at equilibrium (i.e., the fluid following curve $x_0 = 0$) is back at $x = 0$. But at $t = 1$ the fluid following curve $x_0 = -2$ (i.e., the fluid which was at $x_0 = -2$ before the wave motion) is at $x = 0$, and at $t = 3$ the fluid of $x_0 = -3$ is at $x = 0$. To translate what happens to the fluid following each curve into what fluid happens to be at x, we must change from x_0, t to x, t.

Since

$$x = (X - x_0) + x_0 = x_0 - A \sin [k(x_0 - ct)]$$

we shall write $x_0 - ct = -z_0$ and $x - ct = -z$, so that $z = z_0 - A \sin (kz_0)$.

To find z_0 as a function of z, we write $A \sin (kz_0) = S(z)$ and see that the function S must satisfy the implicit equation

$$S(z) = A \sin \{k[z + S(z)]\}$$

which can be solved with little difficulty by successive approximations. Some solutions are plotted in Fig. 6.3*b* for $kA = 0.3$ and $kA = 0.6$. Function $S(ct - x) = -S(x - ct) = -A \sin [k(x_0 - ct)] = A \sin (kz_0)$ gives the displacement $X - x_0$ from equilibrium of that portion of the fluid which happens to be at point x at time t. In particular, $S(ct)$ is the displacement which has occurred to that portion of the fluid which is at the origin at time t. It may be obtained from curves *a* by plotting, against t, the negative of x_0 (that is, $x - x_0$ for $x = 0$) for the curve which crosses the $x = 0$ axis at time t. We see that this displacement is far from being a sine wave. During the first part of the cycle, when the fluid is compressed, S rises rapidly to its maximum, then goes more slowly back to zero during the part of the cycle when the motion is opposed to the direction of propagation and the fluid is rarefied.

As pointed out, the velocity of the portion of the fluid which was originally at x_0 is given by a simple cosine function in terms of x_0 and t. But the Eulerian velocity $u(x,t)$, the velocity of the part of the fluid which is at x at time t, is

$$dX/dt = kcA \cos \{k[z + S(z)]\} = \omega A \cos \{k[ct - x + S(ct - x)]\}$$

which is plotted as a function of kz in Fig. 6.3*c*, for $kA = 0.3$ and 0.6. We see that it is far from being a simple cosine function of t. But also, it is *not* the time derivative $\partial S/\partial t$ of the implicit function S. If we were to forget the difference between the Lagrange and the Euler descriptions, and try to find the displacement X of some part of the fluid by integrating $u(x,t)$ with respect to t, Fig. 6.3*c* shows that we should become erroneously convinced that the fluid had a negative mean displacement (since u is negative for a longer time than it is positive). Of course, the curves of Fig. 6.3*a* show that there is no such average drift; the mistake would come from the assumption that $u(x,t)$ is the velocity of a given part of the fluid, instead of the velocity of whichever fluid happens to be at x at time t.

To be quantitative about the difference between fluid displacement and the integral of $u(x,t)$ and to generalize the equations for S, we assume that the displacement $X - x_0$ of the fluid originally at x_0 is some function $F(ct - x_0)$. Then the displacement $D(ct - x)$ of that portion of the fluid which happens to be at point x at time t must satisfy the equation

$$D(ct - x) = F[ct - x + D(ct - x)]$$

The actual velocity $u(x,t)$ of that portion of the fluid which is at x at time t is, of course, *not* $\partial D/\partial t$ (holding x constant), but is $\partial F/\partial t$ (holding x_0 constant) $= cF'(ct - x + D)$, where F' is the derivative of F with respect to its argument. Since

$$\frac{\partial D}{\partial t} = cD'(ct - x) = c[1 + D'(ct - x)]F'(ct - x + D)$$

the relationship between u and D' is

$$u(x,t) = cF' = \frac{cD'(ct - x)}{1 + D'(ct - x)} = c[D' - (D')^2 + (D')^3 \cdots]$$

the series expansion converging if D' is smaller than unity (i.e., if the fluid velocity is everywhere smaller than c).

If F is a periodic function [as, for example, $F(u) = A \sin(ku)$], function D also is periodic, and the time average of all odd powers of D' will be zero. In this case the time integral of the Eulerian velocity, over a long time interval,

$$\int_0^T u(x,t)\,dt \to -cT\left\langle \frac{(D')^2}{1 - (D')^2} \right\rangle_{\text{avg}} \qquad \text{for } T \to \infty$$

is *negative*, roughly equal to $-cT$ times the time average of u^2/c^2 when $u \ll c$, even though the actual fluid displacement

$$\int_0^T D'c\,dt = c\int_0^T [1 + D'(u)]F'[u + D(u)]\,du$$

is zero when F and D are periodic functions and T is an integral number of periods. Thus *only* when the second-order term $(u/c)^2$ can be neglected compared with u/c can we assume that the integral of $u(x,t)$ is equal to the fluid displacement.

The main conclusion to be drawn from this rather long digression is that the time derivative of the displacement of some part of the fluid differs from $u(x,t)$ by an amount which is small only for small amplitudes. The difference between the Euler and the Lagrange descriptions can be neglected only when we can neglect the squares of u, p, and δ; even in this case there is a difference in inhomogeneous fluids.

The time derivatives

The crucial difference between the two notations comes in the time rates of change. In the Lagrange notation the *total derivative* df/dt of some property f of the fluid is the time rate of change of f in a portion of fluid, *as it moves through* point x at time t. In contrast, in the Euler notation, the *partial derivative* $\partial f/\partial t$ (we shall always use the partial notation for the Eulerian time derivative) is the change of f *at the fixed point* x, as the fluid streams past. Thus $\partial f/\partial t$ includes not only the time rate of change of f in

the moving fluid, but also the change in f at x because the fluid is moving past, if f differs from point to point in the fluid.

The chief advantage of the Lagrange description lies in the time derivatives; the total derivative $d\rho/dt$ measures whether the fluid is expanding or not as it moves; du/dt is the true acceleration of a portion of the fluid, and thus is proportional to the net force acting on the fluid at x, t. But the Euler description has many other advantages; so it will be useful to find a measure of the difference between the two time derivatives.

To find the total derivative df/dt of some property of the fluid, in terms of the partial derivatives at t, x, we compare the value of f at x, t with its value at time $t + dt$ *at the point* $x + u\,dt$ *to which the fluid has moved in time* dt. In accord with the definitions of derivatives, this is

$$\begin{aligned}\frac{df}{dt}\,dt &= f(x + u\,dt, t + dt) - f(x,t)\\ &= f(x,t) + \frac{\partial f}{\partial x}\,u\,dt + \frac{\partial f}{\partial t}\,dt - f(x,t)\end{aligned}$$

or

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} \tag{6.1.9}$$

The actual change in f, as the fluid moves past x with velocity u, is the change in f with time at the fixed point x, plus the spatial change in f times its velocity u of flow past x. Thus the true acceleration of the fluid at x at time t is given, in terms of the Eulerian velocity function $u(x,t)$, by

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \tag{6.1.10}$$

We see that the correction term $u(\partial u/\partial x)$ is second order in the fluid velocity, and therefore, if u is small (as it often is for sound waves), $du/dt \simeq \partial u/\partial t$ to the first order of approximation. The distinction cannot be forgotten, however, because the sound energy and intensity are second-order expressions, and the correction term may have to be included. Also, when the fluid is moving as a whole with respect to the observer, its velocity may not be small; so the second terms in Eqs. (6.1.9) and (6.1.10) may be as large as the first terms. We should note that Eq. (6.1.10) is really just a restatement of the discussion on page 238.

The equation of continuity

In addition to these questions of notation, we must also take into account the fact that what happens in one part of a fluid affects what happens in other parts. For example, we must express in equation form the fact that the net flow across a closed surface in the fluid produces a change in the properties of the fluid inside the surface. Suppose f is some property like entropy density or electric charge density that can be created in the

fluid and, once created, can be carried along with the fluid as it moves. For one-dimensional situations, where both f and u depend only on x and t, the *flux* of f is the total amount of f that passes per second in the positive x direction across a unit cross-sectional area perpendicular to x. If f travels with the fluid, this flux $J(x,t)$ must be equal to the density f times the fluid velocity u at x and t.

Now consider the slice of fluid between a plane surface of unit area, normal to the x axis, at $x + dx$, and a parallel surface at x. The flux $J(x + dx)$ represents a loss of f from the region, and the flux $J(x)$ into the region represents a gain. The difference must equal the rate of loss by outflow of f from the region. If, in addition, f is created at a rate $Q(x,t)$ per unit volume of fluid, the net rate of change of f within the region is

$$\frac{\partial f}{\partial t}\,dx = Q\,dx - J(x + dx) + J(x) = Q\,dx - \frac{\partial J}{\partial x}\,dx$$

where partial derivatives are used because we are talking about a region fixed in space. Therefore

$$\frac{\partial f}{\partial t} = Q - \frac{\partial fu}{\partial x} = Q - f\frac{\partial u}{\partial x} - u\frac{\partial f}{\partial x} \tag{6.1.11}$$

This equation, which states that any increase in f in a region must have been brought there by fluid flow or else by specific creation there, as represented by Q, is called the *equation of continuity*. If we wish to find the net gain in f in a region which moves with the fluid, we substitute from Eq. (6.1.9),

$$\begin{aligned}\frac{df}{dt} &= Q - u\frac{\partial f}{\partial x} - f\frac{\partial u}{\partial x} + u\frac{\partial f}{\partial x}\\ &= Q(x,t) - f\frac{\partial u}{\partial x}\end{aligned} \tag{6.1.12}$$

If the fluid flow is the same throughout the fluid, $\partial u/\partial x = 0$ and Q equals the total derivative df/dt, as might be expected.

In particular, if f is the mass density of fluid ρ and Q is the rate of creation of fluid (sometimes this is possible), then $(\partial\rho/\partial t) + (\partial\rho u/\partial x) = Q$, or if we write the density as $\rho + \delta(x,t)$, where ρ is the constant equilibrium density and δ is the variable part,

$$Q = \frac{\partial\delta}{\partial t} + u\frac{\partial\delta}{\partial x} + (\rho + \delta)\frac{\partial u}{\partial x} = \frac{d\delta}{dt} + (\rho + \delta)\frac{\partial u}{\partial x} \tag{6.1.13}$$

Even if $Q = 0$, the density of a given portion of fluid can change with time if the fluid velocity changes with x. This equation is valid only for homogeneous fluids.

It is not difficult to generalize these equations to three dimensions if we use vector notation. Property f in the Euler description is a function of position, denoted by vector $\mathbf{r}$, with components x, y, and z, and fluid velocity at $\mathbf{r}$, t is a vector $\mathbf{u}$, with components u_x, u_y, and u_z. The flux of f is vector $\mathbf{J} = f\mathbf{u}$. Then Eq. (6.1.9), relating the total time derivative of $f(\mathbf{r},t)$ and its partial time derivative, is

$$\frac{df}{dt}\,dt = f(\mathbf{r} + \mathbf{u}\,dt, t + dt) - f(\mathbf{r},t)$$

$$= \left(\frac{\partial f}{\partial t} + u_x\frac{\partial f}{\partial x} + u_y\frac{\partial f}{\partial y} + u_z\frac{\partial f}{\partial z}\right)dt$$

or

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{u}\cdot\text{grad})f \tag{6.1.14}$$

where the operator

$$\mathbf{u}\cdot\text{grad} \equiv \mathbf{u}\cdot\boldsymbol{\nabla} = u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$$

measures the rate of change of f at $\mathbf{r}$, t, caused by the fluid flow; it can be used on either a scalar or a vector.

To obtain the three-dimensional equation of continuity we first show that the net flux out of volume element $dx\,dy\,dz$ is

$$dy\,dz\left(\frac{\partial J_x}{\partial x}\right)dx + dx\,dz\left(\frac{\partial J_y}{\partial y}\right)dy + dx\,dy\left(\frac{\partial J_z}{\partial z}\right)dz = \text{div}\,\mathbf{J}\,dx\,dy\,dz$$

where

$$\text{div}\,\mathbf{J} = \boldsymbol{\nabla}\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

The equation of continuity is therefore

$$\frac{\partial f}{\partial t} \equiv Q(r,t) - \text{div}\,\mathbf{J} = Q - f\,\text{div}\,\mathbf{u} - \mathbf{u}\cdot\text{grad}\,f \tag{6.1.15}$$

or

$$\frac{df}{dt} = Q - f\,\text{div}\,\mathbf{u}$$

where $\text{grad}\,f \equiv \boldsymbol{\nabla}f$ is a vector with components $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

6.2 WAVE MOTION, ENERGY, AND MOMENTUM

We now are ready to work out the equation of acoustic wave motion which is valid when the wave amplitude is small. The more accurate form of the equation will be considered later, when we discuss large-amplitude waves and waves in moving media. We start here with a fluid that, in the absence of sound, has uniform density ρ, pressure P, and temperature T and is everywhere at rest. We assume in this section that it is nonviscous and has zero heat conductivity, so that the only energy involved in the acoustic motion is mechanical, and the only forces are those of compressive elasticity,

measured by the compressibility κ (the adiabatic compressibility is the appropriate one to use in the case of zero heat conduction, but we need not continue to write the subscript s).

In the presence of a sound wave the pressure becomes $P + p(x,t)$ if the wave is one-dimensional, the wavefronts being planes parallel to the yz plane. The density becomes $\rho + \delta(x,t)$, and the temperature becomes $T + \tau(x,t)$ over the whole of the plane a distance x from the yz plane. Waves of this sort are appropriately called *plane waves*. As mentioned previously, the change in pressure $p(x,t)$ is called the acoustic, or sound pressure. It is responsible for the motion of the fluid, and the motion is in turn responsible for the change in density, $\delta(x,t) \simeq \rho\kappa p(x,t)$.

The one-dimensional equation

The net force on the mass $(\rho + \delta)\,dx$ of fluid in the volume enclosed between two plane surfaces, each of unit area, one at x and the other at $x + dx$, is $p(x) - p(x + dx)$, which must equal the mass of the fluid times its acceleration,

$$p(x) - p(x + dx) = -\frac{\partial p}{\partial x}\,dx = \frac{du}{dt}(\rho + \delta)\,dx$$

or

$$\frac{\partial p}{\partial x} = -(\rho + \delta)\left(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x}\right) \tag{6.2.1}$$

from Eq. (6.1.10), changing from the total to the partial time derivative of u, when u depends only on x and t.

Also, from Eq. (6.1.13), if the source function Q is zero (if no new fluid is introduced at x), the equation of continuity for the density $\rho + \delta$ is

$$\frac{\partial \delta}{\partial t} = -(\rho + \delta)\frac{\partial u}{\partial x} - u\frac{\partial \delta}{\partial x}$$

which becomes

$$\rho\kappa\frac{\partial p}{\partial t} = -\rho(1 + \kappa p)\frac{\partial u}{\partial x} - \rho\kappa u\frac{\partial p}{\partial x} \tag{6.2.2}$$

if we can use the approximate formula $\delta = \rho\kappa p$ relating density and pressure change. Eliminating u between these two equations will produce the equation of wave motion.

We have been assuming that δ and p are small compared with ρ and P, and also that u is small; thus there are several hierarchies of smallness in these two equations. In both equations the term on the left is first order in the small quantities u, δ, and p. In Eq. (6.2.1), $\rho(\partial u/\partial t)$ is first order, $\delta(\partial u/\partial t)$ and $\rho(\partial u^2/\partial x)$ are second order, and $\delta(\partial u^2/\partial x)$ is third order. In Eq. (6.2.2) $\rho(\partial u/\partial x)$ is the first-order term on the right and $-\rho\kappa p(\partial u/\partial x)$ and $-\rho\kappa u(\partial p/\partial x)$ are second-order terms. [Actually, there is a second-order term on the left of Eq. (6.2.2) involving $\partial\kappa/\partial P$, but this term is usually much smaller than the other second-order terms and is usually neglected.]

The first-order equation of motion is obtained by neglecting higher-order terms in both equations, leaving

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial u}{\partial t} \qquad \kappa \frac{\partial p}{\partial t} = -\frac{\partial u}{\partial x} \tag{6.2.3}$$

The first of these states that a pressure gradient produces an acceleration of the fluid; the second states that a velocity gradient produces a compression of the fluid.

Differentiating the first by x and the second by t enables us to eliminate u, thus obtaining the first-order equation of acoustic motion,

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \qquad c^2 = \frac{1}{\kappa\rho} \tag{6.2.4}$$

which is, of course, the familiar wave equation, with wave velocity inversely proportional to the square root of the density ρ times the compressibility κ of the fluid in equilibrium, as predicted by Eq. (6.1.8).

The extension of these equations to three-dimensional motion is not difficult. The equation relating fluid acceleration and pressure gradient becomes

$$(\rho + \delta)\frac{d\mathbf{u}}{dt} \equiv (\rho + \delta)\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\mathbf{u}\right] = -\text{grad}\, p \tag{6.2.5}$$

where the velocity $\mathbf{u}$ is now a vector, and we have used Eq. (6.1.4) to change from total to partial time derivative. The equation of continuity (6.1.15) for mass flow becomes

$$\begin{aligned}\frac{\partial \delta}{\partial t} \equiv \kappa\rho \frac{\partial p}{\partial t} &= -\text{div}\,[(\rho + \delta)\mathbf{u}] \equiv -(\rho + \delta)\,\text{div}\,\mathbf{u} - \mathbf{u} \cdot \text{grad}\,\delta \\ &= -\rho(1 + \kappa p)\,\text{div}\,\mathbf{u} - \kappa\rho\mathbf{u} \cdot \text{grad}\, p\end{aligned} \tag{6.2.6}$$

where we have again assumed that κ is independent of p.

To first order in the small quantities p, $\mathbf{u}$, we then have

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\text{grad}\, p \qquad \kappa \frac{\partial p}{\partial t} = -\text{div}\,\mathbf{u} \tag{6.2.7}$$

Taking the divergence of the first equation and using the second to eliminate $\mathbf{u}$ results in

$$\text{div grad}\, p \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)p \equiv \nabla^2 p = \kappa\rho \frac{\partial^2 p}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{6.2.8}$$

which is the three-dimensional form of the wave equation. The symbol ∇^2, the three-dimensional counterpart to the operator of Eq. (5.2.3), is again called the Laplacian. We shall discuss its various forms, for different coordinate systems, in a later chapter.

For future reference, though we shall not use the equations in this chapter, we write down the equation of motion to the second order in the small quantities $\mathbf{u}$, p (though still not including the effects of viscosity, thermal conduction, or change of compressibility with pressure). To this order it is more convenient to express both p and u in terms of a scalar function Ψ, which we call the *velocity potential*, since $\mathbf{u}$ is obtained by taking the gradient of Ψ, $\mathbf{u} = -\text{grad}\,\Psi$. Then, if we set, to second order,

$$p = \rho\frac{\partial\Psi}{\partial t} - \tfrac{1}{2}\rho\left[|\text{grad}\,\Psi|^2 - \kappa\rho\left(\frac{\partial\Psi}{\partial t}\right)^2\right]$$

the equation of motion for Ψ, to second order, is

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial}{\partial t}\left[|\text{grad}\,\Psi|^2 - \kappa\rho\left(\frac{\partial\Psi}{\partial t}\right)^2\right] = \frac{1}{\kappa\rho}\nabla^2\Psi \qquad (6.2.9)$$

assuming that κ is a constant. Thus, even when viscosity and thermal conduction are neglected, second-order terms modify the equation for simple wave motion. We shall see later what effect this correction term has on the solutions.

For small motions of the fluid, however, such that the acoustic pressure p is small compared with the equilibrium pressure P, the motion, with its accompanying small changes of pressure, propagates as simple waves with velocity $c = \sqrt{1/\rho\kappa}$. From Eq. (6.1.5), if the fluid is a perfect gas, $c = \sqrt{\gamma RT/M}$, where T is the absolute temperature of the gas at equilibrium, M its molecular weight, γ the ratio between its specific heats, and R the universal gas constant. In this case the sound velocity is proportional to the square root of T, being independent of the gas pressure or density (as long as the first-order equation is valid). Also, in the case of the perfect gas, the temperature change τ accompanying the wave pressure p is $\tau = [(\gamma - 1)M/\gamma R\rho]p = (\gamma - 1)(T/\rho c^2)p = (\gamma - 1)(u/c)T$, a quantity much smaller than T, since p is much smaller than ρc^2 if Eq. (6.2.8) is valid.

Fluid momentum and energy

It is interesting, and will also be useful later, to demonstrate that the equation of motion (6.2.5) can be written as an equation of continuity for fluid momentum. The quantity $\rho(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)$ is the momentum density of the fluid at point $\mathbf{r}$ and time t. According to Eq. (6.1.15), any change in this density must be produced by the divergence of the flux of momentum (analogous to the div $\mathbf{J}$ term) or else by the introduction of "new" momentum at $\mathbf{r}$, t (analogous to the Q term). Since momentum density is a vector, there must be three equations of continuity, one for each component of $\rho\mathbf{u}$, and therefore three flux vectors, J_x, J_y, J_z, and three "source" terms, Q_x, Q_y, Q_z.

A "source" of momentum must be a net force on a unit volume of the fluid; a force causes a change in momentum. Considering the amount of fluid inside an element of volume $dx\,dy\,dz$ at the point $\mathbf{r} = (x,y,z)$, the force on this amount, transmitted from outside across the face $dy\,dz$ at x, y, z, can be written $P_x(x,y,z,t)\,dy\,dz$, where the vector $\mathbf{P}_x$ is called the x component of the fluid *stress*. In a nonviscous fluid, of the sort considered in this section, $\mathbf{P}_x$ is simply the pressure, a vector in the x direction, of magnitude equal to the pressure $P(x,y,z,t)$ at $\mathbf{r}$, t. Later we shall show that when viscosity is taken into account, $\mathbf{P}_x$ is not pointed in the x direction, but has components P_{xy} and P_{xz} as well.

The force across the $dy\,dz$ face of the element farthest from the origin, pushing back on the interior of $dx\,dy\,dz$, is then $-\mathbf{P}_x(x + dx, y, z, t)\,dy\,dz = -\mathbf{P}_x(x,y,z,t)\,dy\,dz - (\partial\mathbf{P}_x/\partial x)\,dx\,dy\,dz$; so the net force on the fluid in the element, transmitted across the faces perpendicular to the x axis, is $-(\partial\mathbf{P}_x/\partial x)\,dx\,dy\,dz$. For a nonviscous fluid this force is $-(\partial P/\partial x)\,dx\,dy\,dz$, pointed in the x direction, where P is the pressure. Similarly, the net stress force transmitted across the $dx\,dz$ faces, perpendicular to the y axis, is $-(\partial\mathbf{P}_y/\partial y)\,dx\,dy\,dz$, and that across the $dx\,dy$ faces is $-(\partial\mathbf{P}_z/\partial z)\,dx\,dy\,dz$. For a nonviscous fluid, these forces are $-(\partial P/\partial y)\,dx\,dy\,dz$, pointed in the y direction, and $-(\partial P/\partial z)\,dx\,dy\,dz$, pointed in the z direction, respectively. The total stress force is thus $-\text{grad}\,(P)\,dx\,dy\,dz$ for the nonviscous case, and $-[(\partial\mathbf{P}_x/\partial x) + (\partial\mathbf{P}_y/\partial y) + (\partial\mathbf{P}_z/\partial z)]\,dx\,dy\,dz$, with x component

$$-\left(\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{yx}}{\partial y} + \frac{\partial P_{zx}}{\partial z}\right) dx\,dy\,dz \quad \cdots$$

when viscosity is included.

In addition, there may be an external force $\mathbf{F}(\mathbf{r},t)$, such as the force of gravity, acting on each unit volume of the fluid. This force on the fluid inside the elementary volume is $\mathbf{F}\,dx\,dy\,dz$, with x component $F_x(\mathbf{r},t)\,dx\,dy\,dz$, etc. Therefore the momentum "source" term for the x component of the momentum conservation equation is

$$Q_x\,dx\,dy\,dz = \left(F_x - \frac{\partial P_{xx}}{\partial x} - \frac{\partial P_{yx}}{\partial y} - \frac{\partial P_{zx}}{\partial z}\right) dx\,dy\,dz$$

The momentum flux $\mathbf{J}_x$ in the x direction is the momentum density $\rho\mathbf{u}$ times the x component of the fluid velocity, u_x. Thus $\mathbf{J}_x$ has components $\rho u_x u_x$, $\rho u_y u_x$, and $\rho u_z u_x$; similarly for the other fluxes. We thus see that the total stress and the total flux each have nine components rather than three; they constitute what are called *tensors*, which can be written in matrix form,

$$\mathfrak{P} \equiv \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix} \qquad \mathfrak{J} \equiv \begin{pmatrix} \rho u_x u_x & \rho u_x u_y & \rho u_x u_z \\ \rho u_y u_x & \rho u_y u_y & \rho u_y u_z \\ \rho u_z u_x & \rho u_z u_y & \rho u_z u_z \end{pmatrix} \tag{6.2.10}$$

instead of as trios of vectors, $\mathbf{P}_x$, $\mathbf{P}_y$, $\mathbf{P}_z$ and $\mathbf{J}_x$, $\mathbf{J}_y$, $\mathbf{J}_z$. We note that, for

nonviscous fluids, the stress tensor has the particularly simple form

$$\mathfrak{P} = \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}$$

The flux of x momentum $\mathbf{J}_x(x,y,z,t)$ may bring in to the volume element more momentum than it takes out over the other face. The net amount left, via the $dy\,dz$ faces, is $J_{xx}(x,y,z,t)\,dy\,dz - J_{xx}(x+dx, y, z, t)\,dy\,dz = -(\partial J_{xx}/\partial x)\,dx\,dy\,dz$; and the total amount, from all faces, of x momentum is

$$-\left(\frac{\partial J_{xx}}{\partial x} + \frac{\partial J_{xy}}{\partial y} + \frac{\partial J_{xz}}{\partial z}\right) dx\,dy\,dz = -\text{div}\,\mathbf{J}_x\,dx\,dy\,dz$$

In addition, there is the amount $Q_x\,dx\,dy\,dz$ "created" per second by the external force and the internal stresses. Therefore the net gain in x momentum in the element per second is

$$\frac{\partial}{\partial t}(\rho u_x)\,dx\,dy\,dz = (Q_x - \text{div}\,\mathbf{J}_x)\,dx\,dy\,dz$$

with similar equations for the y and z components. The equations of continuity for the fluid momentum are thus

$$\frac{\partial}{\partial t}(\rho u_x) = F_x - \frac{\partial}{\partial x}(P_{xx} + \rho u_x u_x) - \frac{\partial}{\partial y}(P_{yx} + \rho u_x u_y) - \frac{\partial}{\partial z}(P_{zx} + \rho u_x u_z)$$

and similarly for the y and z components. These can be combined symbolically into a single equation of the general form of Eq. (6.1.15),

$$\frac{\partial \mathbf{f}}{\partial t} + \boldsymbol{\nabla} \cdot \mathfrak{T} = \mathbf{F} \tag{6.2.11}$$

where, in the present case, the vector density $\mathbf{f}$ is $\rho\mathbf{u}$, the momentum per unit volume of fluid, the tensor flux $\mathfrak{T}$ is the sum $\mathfrak{P} + \mathfrak{J}$ of the stress tensor $\mathfrak{P}$ and the momentum flux tensor $\mathfrak{J}$, and the source vector $\mathbf{F}$ is the external force per unit volume, such as that of gravity, or possible electric forces.

For nonviscous fluids the tensor $\mathfrak{P}$ is the simple one, with pressure P on the main diagonal and zeros elsewhere. Going through the components, we see that $\boldsymbol{\nabla} \cdot \mathfrak{P}$ then becomes simply grad P. To identify these equations with the equation of motion (6.2.5), we perform the differentiations indicated by $\boldsymbol{\nabla} \cdot \mathfrak{J}$. For the x components,

$$\begin{aligned}\text{div}\,\mathbf{J}_x = u_x \frac{\partial}{\partial x}\rho u_x + u_x \frac{\partial}{\partial y}\rho u_y + u_x \frac{\partial}{\partial z}\rho u_z \\ + \rho u_x \frac{\partial}{\partial x} u_x + \rho u_y \frac{\partial}{\partial y} u_x + \rho u_z \frac{\partial}{\partial z} u_x \\ = u_x\,\text{div}\,(\rho\mathbf{u}) + \rho(\mathbf{u} \cdot \text{grad})u_x\end{aligned}$$

and for the whole vector we can write

$$\nabla \cdot \mathfrak{J} = \mathbf{u} \operatorname{div}(\rho\mathbf{u}) + \rho(\mathbf{u} \cdot \operatorname{grad})\mathbf{u}$$

Finally, we use the equation of continuity (6.1.15) for the scalar density, $\partial\rho/\partial t = -\operatorname{div}(\rho\mathbf{u})$, to change the term $\partial\rho\mathbf{u}/\partial t$ into $\rho(\partial\mathbf{u}/\partial t) - \mathbf{u}\operatorname{div}(\rho\mathbf{u})$ and thus obtain (for $\mathbf{F} = 0$)

$$\rho\frac{\partial\mathbf{u}}{\partial t} - \mathbf{u}\operatorname{div}(\rho\mathbf{u}) = -\operatorname{grad} P - \mathbf{u}\operatorname{div}(\rho\mathbf{u}) - \rho(\mathbf{u} \cdot \operatorname{grad})\mathbf{u}$$

which does indeed correspond to the equation of motion (6.2.5) if we cancel the divergence terms and substitute $\rho + \delta$ for ρ and $P + p$ for P, in accordance with our present notation.

We also could work out an equation of continuity for the energy density of the fluid which would include the kinetic energy density $\frac{1}{2}\rho\,|\mathbf{u}|^2$ as well as the internal (heat) energy of the fluid. This will be done in a later section, when we take up the effects of heat conduction and other internal energy changes. Here we go on to discuss the momentum and energy of the wave motion, which differ from the momentum and energy of the fluid, just as the longitudinal energy and momentum of wave motion on the string differ from the string's transverse energy and momentum, as noted in connection with Eq. (4.1.12). In the present case, however, both sets are longitudinal.

Bernoulli's equation

We shall have occasion, several times, to refer to Bernoulli's equation, which is a special form of the momentum equation (6.1.15) for a nonviscous fluid,

$$\rho\frac{d\mathbf{u}}{dt} \equiv \rho\frac{\partial\mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\operatorname{grad} P + \mathbf{F}$$

For isentropic flow the pressure is a function of density alone; so we can express $(1/\rho)\operatorname{grad} P$ as

$$\frac{1}{\rho}\frac{dP}{d\rho}\operatorname{grad}\rho = \operatorname{grad}\int\frac{1}{\rho}\frac{dP}{d\rho}\,d\rho = \operatorname{grad}\int\frac{dP}{\rho}$$

Introducing the velocity potential Ψ ($\mathbf{u} = -\operatorname{grad}\Psi$) and the vector relationship

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \operatorname{grad}\frac{u^2}{2} - (\mathbf{u} \times \operatorname{curl}\mathbf{u})$$

the momentum equation becomes

$$\operatorname{grad}\left(\frac{u^2}{2} + V + \int\frac{dP}{\rho} - \frac{\partial\Psi}{\partial t}\right) = \mathbf{u} \times \operatorname{curl}\mathbf{u}$$

where we have assumed that the external force $\mathbf{F}$ is minus the gradient of a potential V.

We now multiply both sides of this equation by $\mathbf{u}$ (scalar product). Since $\mathbf{u} \times \mathbf{A}$ is perpendicular to $\mathbf{u}$ for any vector $\mathbf{A}$, we have

$$\mathbf{u} \cdot \text{grad} \left(-\frac{\partial \Psi}{\partial t} + \frac{u^2}{2} + V + \int \frac{dP}{\rho} \right) = 0$$

where $\mathbf{u} \cdot \text{grad} = u(\partial/\partial n)$ is the derivative in the direction of $\mathbf{u}$, that is, along the streamline defined by $\mathbf{u}$. Integrating the quantity along a streamline then results in Bernoulli's equation

$$-\frac{\partial \Psi}{\partial t} + \frac{u^2}{2} + V + \int \frac{dP}{\rho} = C(t) \tag{6.2.12}$$

where constant C may be a function of time.

For time-independent flow this reduces to the well-known equation $\frac{1}{2}u^2 + V + \int (dP/\rho) = \text{constant}$, related to the constancy of energy density. When $P/P_0 = (\rho/\rho_0)^\gamma$ and $c^2 = \gamma P/\rho$, this becomes

$$\frac{c^2}{\gamma - 1} + \tfrac{1}{2}u^2 + V = \text{constant} = V + \tfrac{1}{2}u^2 + \frac{\gamma}{\gamma - 1}\frac{P}{\rho}$$

The Lagrange density

To find the momentum and energy flux specifically connected with the wave motion, we first work out the Lagrange density [Eq. (4.1.8)] for sound waves in a fluid. As noted in this section, we are neglecting viscosity and heat conduction for the time being. Therefore the basic equations, to first order, are

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\text{grad}\, p \qquad \kappa \frac{\partial p}{\partial t} = -\text{div}\, \mathbf{u}$$

as before. According to Eq. (6.1.7), the Lagrange density, the difference between the kinetic and potential energy densities, is $\frac{1}{2}\rho\,|\mathbf{u}|^2 - \frac{1}{2}\kappa p^2$. To apply the machinery of the Hamilton principle to this, we must express $\mathbf{u}$ and p in terms of some common function of $\mathbf{r}$ and t. If we use p itself, there is no expression for $\mathbf{u}$ in terms of derivatives of p. However, if we let p equal ρ times the time derivative of some scalar function Ψ, we see, from the first of Eqs. (6.2.7), that $\mathbf{u}$ is minus the gradient of Ψ; so Ψ, the velocity potential mentioned earlier, is a possible function with which to express L.

$$\begin{aligned} p &= \rho \frac{\partial \Psi}{\partial t} \qquad \mathbf{u} = -\text{grad}\, \Psi \\ L &= -\tfrac{1}{2}\rho \left[\left(\frac{\partial \Psi}{\partial x}\right)^2 + \left(\frac{\partial \Psi}{\partial y}\right)^2 + \left(\frac{\partial \Psi}{\partial z}\right)^2 - \rho\kappa \left(\frac{\partial \Psi}{\partial t}\right)^2 \right] \end{aligned} \tag{6.2.13}$$

This formalism has the somewhat anomalous property that the kinetic-energy term has the space derivatives of Ψ, and the potential-energy term has the time derivative, thus forcing us to change the sign of L. We could, of

course, have taken $\mathbf{u}$ to be the time derivative of some vector $\mathbf{q}$. It is more effort, however, to deal with the three components of a vector than with the one function Ψ. In addition, we might be tempted to call $\mathbf{q}$ the displacement of the fluid (forgetting the difference between the Euler and the Lagrange description) and thus draw erroneous conclusions regarding actual fluid displacement.

So we shall use the L of Eq. (6.2.13). The result of using Hamilton's principle [see the discussion of Eq. (5.2.6)] is the Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Psi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Psi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \Psi_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \Psi_z}\right) - \frac{\partial L}{\partial \Psi} = 0 \quad (6.2.14)$$

where the subscripts indicate partial differentiation. Applying this to the L of Eq. (6.2.13), we obtain

$$\rho^2\kappa\left(\frac{\partial^2\Psi}{\partial t^2}\right) - \rho\,\nabla^2\Psi = 0$$

the wave equation, to first order.

We could use the Lagrange density

$$L = \tfrac{1}{2}\rho\left[1 + \kappa\rho\left(\frac{\partial\Psi}{\partial t}\right)\right]\left[\kappa\rho\left(\frac{\partial\Psi}{\partial t}\right)^2 - \left(\frac{\partial\Psi}{\partial x}\right)^2 - \left(\frac{\partial\Psi}{\partial y}\right)^2 - \left(\frac{\partial\Psi}{\partial z}\right)^2\right]$$

correct to third order (but omitting viscosity and heat conduction) to obtain the second-order wave equation (6.2.9). It is interesting to see that the second-order correction term in this equation is itself proportional to the first-order Lagrange density of Eq. (6.2.13).

As was shown in Eqs. (4.1.9) for the string and Eqs. (5.2.8) for the membrane, the energy density $w = W_{tt}$ and the energy flow vector $\mathbf{I} = (W_{tx}, W_{ty}, W_{tz})$ of the wave motion are (using the L of Eq. 6.2.13)

$$W_{tt} = \Psi_t\left(\frac{\partial L}{\partial \Psi_t}\right) - L = \tfrac{1}{2}\rho\,|\text{grad}\,\Psi|^2 + \tfrac{1}{2}\kappa\rho^2\Psi_t^2 = \tfrac{1}{2}\rho\,|\mathbf{u}|^2 + \tfrac{1}{2}\kappa\rho^2 = w$$

$$W_{tx} = \Psi_t\left(\frac{\partial L}{\partial \Psi_x}\right) = -\rho\Psi_t\Psi_x = pu_x \qquad \cdots \qquad \mathbf{I} = p\mathbf{u} \quad (6.2.15)$$

These quantities satisfy the equation of continuity $(\partial w/\partial t) + \text{div}\,\mathbf{I} = 0$. Since, from Eq. (6.1.7), W_{tt} is the total energy density w of the wave motion, we can thus expect the vector $\mathbf{I}$ to be the flux of energy in the wave. Sound pressure times fluid velocity $\mathbf{u}$ is the power flux per unit area, generated by the wave motion; so vector $\mathbf{I}$ is called the *intensity* of the wave.

The wave momentum $\mathbf{M}$ (as contrasted with the fluid momentum $\rho\mathbf{u}$) has the three components

$$M_x = W_{xt} = \Psi_x\frac{\partial L}{\partial \Psi_t} = \kappa\rho^2\Psi_x\Psi_t = -\kappa\rho p u_x$$

or

$$\mathbf{M} = -\kappa\rho p\mathbf{u} \quad (6.2.16)$$

It is a second-order vector, proportional to the intensity **I**, but usually much smaller since $\kappa\rho$ is usually small. It also satisfies a vector equation of continuity (6.2.11), $(\partial\mathbf{M}/\partial t) + \nabla \cdot \mathfrak{W} = 0$ (the source term is zero), with the tensor

$$\mathfrak{W} = \begin{pmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{yx} & W_{yy} & W_{yz} \\ W_{zx} & W_{zy} & W_{zz} \end{pmatrix}$$

$$\begin{aligned} W_{xx} &= \Psi_x \frac{\partial L}{\partial \Psi_x} - L \\ &= \tfrac{1}{2}\rho(-u_x^2 + u_y^2 + u_x^2) - \tfrac{1}{2}\kappa p^2 \\ W_{xy} &= \Psi_x \frac{\partial L}{\partial \Psi_y} = -\rho u_x u_y \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{6.2.17}$$

being the *wave-stress tensor*, measuring the flux of momentum which accompanies the wave motion. Vector **M** has the dimensions of momentum per unit volume (or pressure divided by velocity); **M** multiplied by wave velocity, $c\mathbf{M} = (1/c)p\mathbf{u}$, is the *radiation* pressure exerted by the wave.

Complex notation

It is apparent by now that many calculations are more easily carried out if we use the complex quantity $(a + ib)e^{-i\omega t}$ rather than the real quantity $a\cos(\omega t) + b\sin(\omega t)$ to represent velocity or displacement. If we are to use complex quantities, however, we should learn how to use them to express such second-order quantities as kinetic energy, wave momentum, and the like. The question is one of interpretation. If we say that the displacement x of a simple oscillator is $(a + ib)e^{-i\omega t}$, do we mean that the measured displacement is the real part of this quantity, its imaginary part, or what? Furthermore, we have said that the potential energy of the oscillator is $\frac{1}{2}Kx^2$; does x in this case mean the real part of $(a + ib)e^{-i\omega t}$? If so, we must abandon the use of complex quantities when we discuss the second-order quantities.

Suppose we turn the representation around and define the second-order quantities in terms of the complex notation; what, then, is the meaning of the first-order quantities? Suppose we insist that the basic second-order quantities, the kinetic and potential energies, are always real quantities. This can be achieved by using the square of the magnitude of the complex quantity instead of the square of the quantity. As shown on page 9, the quantity $|C|^2 = C^*C = (a - ib)(a + ib) = a^2 + b^2$ is real and nonnegative. Let us try using a notation in which the potential energy is $\frac{1}{2}K|x|^2$ and the kinetic energy is $\frac{1}{2}m|v|^2$.

In our previous discussion we said that the notation $x = Ae^{-i\omega t} = (a + ib)e^{-i\omega t}$ meant that the measured displacement is the real part, $a\cos(\omega t) + b\sin(\omega t)$, representing simple-harmonic motion of amplitude $A = \sqrt{a^2 + b^2}$ and phase $\Phi = \tan^{-1}(b/a)$. In this case the potential energy is

$$\begin{aligned} &\tfrac{1}{2}K[a^2\cos^2(\omega t) + 2ab\cos(\omega t)\sin(\omega t) + b^2\sin^2(\omega t)] \\ &\qquad = \tfrac{1}{4}K(a^2 + b^2) + \tfrac{1}{4}K(a^2 - b^2)\cos(2\omega t) + \tfrac{1}{2}Kab\sin(2\omega t) \end{aligned}$$

which is a rather complicated combination of the real and imaginary parts of A and of $e^{-i\omega t}$. However, the *mean value* of the potential energy, averaged over time, is just $\frac{1}{4}K(a^2 + b^2) = \frac{1}{4}K\,|x|^2$, a much simpler expression. Likewise, the mean value of the kinetic energy is $\frac{1}{4}m(a^2\omega^2 + b^2\omega^2) = \frac{1}{4}m\,|dx/dt|^2$. This simplicity of the mean values of energy extends to cases where x is a combination of simple-harmonic motions. If

$$x = \sum A_n e^{-i\omega_n t} \tag{6.2.18}$$

then time average of $|x|^2$ is $\frac{1}{2}\sum |A_n|^2$.

Since we usually measure only the time average of the energy (or power) anyway, we may as well use a representation which will give us these average values directly, and interpret the velocities and displacements accordingly. Therefore we shall use the complex quantity x as the displacement, with the convention that the square of its magnitude (which enters into the energy expressions) is the time average of the square of the actual displacement. The magnitude of x, in this notation, will be called the *root-mean-square* (usually written rms) amplitude of x, and the kinetic and potential energies, averaged over time, are

$$\text{KE} = \tfrac{1}{2}m\left|\frac{dx}{dt}\right|^2 \qquad \text{PE} = \tfrac{1}{2}K\,|x|^2$$

Then if, later, we should wish to discuss the measured displacement or velocity, as a function of time, we can take $\sqrt{2}$ times the real part of x or $\sqrt{2}$ times the imaginary part of x, whichever is more convenient. The $\sqrt{2}$ factor enters to take care of the factor $\frac{1}{2}$ in Eq. (6.2.18), because the rms amplitude is only $1/\sqrt{2}$ times the maximum displacement in simple-harmonic motion.

The rms representation is particularly adaptable to the Lagrange-Hamilton approach to the equations of motion, because the basic integral of the Lagrangian [of Eq. (5.2.5), for example] is over time and space, so that time-average quantities can be used without changing the results in the slightest. Thus we can change over completely to the complex notation by using the square of the magnitude of the quantity in place of its square, if we remember that the resulting energies are time averages and the resulting amplitudes are rms amplitudes. In this notation, for example, the Lagrange density for small-amplitude motion of a compressible fluid is

$$L = \tfrac{1}{2}\kappa\rho^2 \frac{\partial \Psi}{\partial t}\frac{\partial \Psi^*}{\partial t} - \tfrac{1}{2}\rho\,\text{grad}\,\Psi^* \cdot \text{grad}\,\Psi \tag{6.2.19}$$

where now, in contrast to the Lagrangian of Eq. (6.2.13), Ψ is a complex quantity $\text{Re}\,\Psi + i\,\text{Im}\,\Psi$ and Ψ^* is its complex conjugate, $\text{Re}\,\Psi - i\,\text{Im}\,\Psi$. The amplitudes of Ψ and of all its derivatives are then rms amplitudes.

Since the real and imaginary parts of Ψ will be varied independently in the variational process (see page 158) to obtain the Lagrange equations of

motion, we can consider Ψ and Ψ^* as being varied independently. Therefore the variation of the integral of L gives rise to two equations, one for Ψ and one for Ψ^*.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Psi_t^*}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Psi_x^*}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \Psi_y^*}\right) + \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \Psi_z^*}\right) - \frac{\partial L}{\partial \Psi^*} = 0 \\
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Psi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Psi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \Psi_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \Psi_z}\right) - \frac{\partial L}{\partial \Psi} = 0
\end{aligned} \tag{6.2.20}$$

For the symmetric Lagrange density of Eq. (6.2.19) these two equations are identical; so in this case Ψ^* and Ψ satisfy the same equation, and thus can be complex conjugates of each other.

The forms of the energy-momentum matrix $\mathfrak{W}$, however, must be modified, since the second-order terms must be real. The modifications are fairly obvious, since W_{tt} must be

$$w = \tfrac{1}{2}\rho^2\kappa \left|\frac{\partial \Psi}{\partial t}\right|^2 + \tfrac{1}{2}\rho\,|\text{grad}\,\Psi|^2$$

the time average of the energy density. Therefore

$$\begin{aligned}
W_{tt} &= \Psi_t\left(\frac{\partial L}{\partial \Psi_t}\right) + \Psi_t^*\left(\frac{\partial L}{\partial \Psi_t^*}\right) - L = w \\
W_{xx} &= \Psi_x\left(\frac{\partial L}{\partial \Psi_x}\right) + \Psi_x^*\left(\frac{\partial L}{\partial \Psi_x^*}\right) - L \quad \cdots \\
W_{tx} &= \Psi_t\left(\frac{\partial L}{\partial \Psi_x}\right) + \Psi_t^*\left(\frac{\partial L}{\partial \Psi_x^*}\right) = \tfrac{1}{2}(p^*u_x + u_x^*p) = I_x \quad \cdots
\end{aligned} \tag{6.2.21}$$

Given that Ψ and Ψ^* satisfy Eqs. (6.2.20), one can show that the elements of $\mathfrak{W}$ satisfy the equations of continuity, similar to those of page 199.

$$\begin{aligned}
\frac{\partial W_{tt}}{\partial t} + \text{div}\,\mathbf{I} = 0 \qquad & \mathbf{I} = (W_{tx}, W_{ty}, W_{tz}) \\
\frac{\partial \mathbf{M}}{\partial t} + \boldsymbol{\nabla}\cdot\mathfrak{W} = 0 \qquad & \mathbf{M} = (W_{xt}, W_{yt}, W_{zt})
\end{aligned}$$

and $\mathfrak{W}$ is the wave-stress matrix of Eq. (6.2.17). Thus the interrelations between the second-order terms are the same as before; only now the quantities are time averages, not instantaneous values.

Propagation in porous media

In some cases of interest the fluid will be interpenetrating a porous solid, which resists the fluid's flow and also modifies the fluid's compressibility. Suppose that the pores in the material are interconnected in a random but isotropic way, so that the fluid can percolate through it with equal facility

in any direction, and so that the fraction of volume which is not occupied by the solid, which the fluid can occupy, is Ω, called the *porosity*. Also suppose that the solid, though porous, is perfectly rigid and incompressible. This is not a bad approximation for gases flowing through porous materials, though it would not be very good for liquids, such as water, flowing through sand, which is not rigid and not much less compressible than water.

The actual velocity of the fluid, as it passes through the pores, will not be constant in direction or amount. What is of practical interest is the mean volume of fluid passing across a unit area inside the medium, normal to the flow, per second, which we can call $\mathbf{u}(x,t)$, the mean velocity. At the inter-surface between the porous material and the outside, the normal component of $\mathbf{u}$ will thus be continuous. The fluid in the pores will have to be moving faster, normal to the surface, than the air outside, so that the volume of fluid leaving the outside per second is equal to the volume of fluid entering the pores per second, both being equal to the area of the surface times the normal component of $\mathbf{u}$.

The equation of continuity for $\mathbf{u}$ still holds for fluid flow in the pores if we consider a large enough volume so that irregularities of pore size average out, and if we consider the density $\rho + \delta$ of the fluid as being the mass of the fluid occupying the fraction Ω of the space available to the fluid. The equation of continuity becomes $\Omega(\partial\delta/\partial t) - \rho \operatorname{div} \mathbf{u} = 0$ to the first order. Also, to the first order, the relation between density change δ and acoustic pressure is $\delta = \kappa_p \rho p$, where κ_p is the effective compressibility of the fluid in the pores. At very high frequencies this would be the adiabatic compressibility κ_s of the fluid in the open; at low frequencies the near presence of the solid material would tend to hold the temperature constant, and it would equal the isothermal compressibility κ_T, larger than κ_s. For some acoustic materials (for air) the change-over occurs in the frequency range of greatest interest, so that κ_p must often be considered to be a function of ω. Thus one of the equations for sound waves in a porous material is $\kappa_p \Omega(\partial p/\partial t) + \operatorname{div} \mathbf{u} = 0$, or, for simple-harmonic waves,

$$i\omega\kappa_p \Omega p = \operatorname{div} \mathbf{u} \tag{6.2.22}$$

The other equation, giving the fluid acceleration in terms of the forces acting, must be modified to include the change in inertia of, and the frictional drag suffered by, the fluid as it moves through the pores. Some fibers of the material may move with the fluid, thus adding to its effective mass. Also, as will be indicated in Eq. (6.4.14), when fluid flows through a series of constrictions, its effective mass increases. Therefore, to express the inertial properties of the fluid in the pores, we use an effective density ρ_p, greater than the density ρ of the fluid in the open. In actual acoustic materials ρ_p is from 1.5 to 5 times ρ.

We express the frictional retardation to flow through the pores in terms of a *flow resistance* Φ, the pressure drop required to force a unit flow through

the material. Most porous materials used in sound absorption have flow resistances which lie between 0.2 and 2 atm-sec per sq m; put another way, the quantity $\Phi/\rho c$ for air in these materials lies between 50 and 500 m^{-1}. Thus the equation of motion is $\rho_p(\partial \mathbf{u}/\partial t) + \Phi \mathbf{u} + \text{grad } p = 0$, or, for simple-harmonic waves,

$$i\omega\rho_p\left[1 + i\frac{\Phi}{\rho_p\omega}\right]\mathbf{u} = \text{grad } p \tag{6.2.23}$$

for materials which are uniform and isotropic in their porous properties. We note that the effect of flow resistance is largest at low frequencies. For porous sound absorbers in air, $\Phi/\rho_p\omega$ lies between 0.5 and 10 for a frequency of 1,000 cps, in the intermediate range, neither very large nor very small.

To obtain the wave equation we define a velocity potential such that $\mathbf{u} = -\text{grad } \Psi$, as before. Then $p = \rho_p(\partial\Psi/\partial t) + \Phi\Psi$ and

$$\nabla^2\Psi = \kappa_p\rho_p\Omega\frac{\partial^2\Psi}{\partial t^2} + \kappa_p\Omega\Phi\frac{\partial\Psi}{\partial t} \tag{6.2.24}$$

Except for the last term, this is the usual wave equation with a wave velocity $c_p = \sqrt{1/\kappa_p\rho_p\Omega}$, instead of $c = \sqrt{1/\rho\kappa}$, the wave velocity in the open. Velocity c_p may be larger than c if Ω is small (material not very porous), but it usually is smaller than c because $\rho_p\Omega$ is usually larger than ρ. The last term in Eq. (6.2.24) represents the frictional loss, which is evidenced by either oscillations damped in time or attenuation in space of simple-harmonic waves.

Since κ_p and ρ_p are to some extent dependent on frequency, we can specialize Eq. (6.2.24) for motions having the time factor $e^{-i\omega t}$ and later utilize the Fourier transform to deal with nonharmonic motion. The basic equations (6.2.7) relating p and $\mathbf{u}$ for simple-harmonic motion can be written $(i\omega/\rho c^2)p = \text{div } \mathbf{u}$ and $i\omega\rho\mathbf{u} = \text{grad } p$. Equations (6.2.22) and (6.2.23) can be brought into this form by defining an effective fluid density ρ_e, compressibility κ_e, and wave velocity c_e, for the fluid in the pores, by the equations

$$\begin{aligned} \rho_e &= \rho_p\left(1 + \frac{i\Phi}{\rho_p\omega}\right) \qquad \kappa_e = \Omega\kappa_p \\ c_e &= \frac{1}{\sqrt{\rho_e\kappa_e}} = \frac{1}{\sqrt{\rho_w\kappa_p\Omega}} = c_p\left(1 + \frac{i\Phi}{\rho_p\omega}\right)^{-\frac{1}{2}} \end{aligned} \tag{6.2.25}$$

where $c_p = \sqrt{1/\rho_p\kappa_p\Omega}$. Therefore

$$\mathbf{u} = -\text{grad } \Psi \qquad p = i\omega\rho_e\Psi \qquad \frac{i\omega}{\rho_e c_e^2}p = \text{div } \mathbf{u}$$

and thus

$$\nabla^2\Psi + \left(\frac{\omega}{c_e}\right)^2\Psi = 0$$

which is the wave equation for simple-harmonic waves.