Both effective density and effective wave velocity are complex quantities, because of the frictional term in Φ. At very high frequencies both are real; $\rho_w \to \rho_p$ is somewhat larger than ρ, the density of the fluid in the open; and $c_w \to c_p$ is usually a little smaller than the wave velocity $c = \sqrt{1/\rho\kappa}$ in the open. Thus a plane wave of frequency $\omega/2\pi$, traveling in the positive x direction, is represented by the expression

$$\Psi = Ae^{i(\omega/c_e)x - i\omega t} = Ae^{i(k_e x - \omega t) - \gamma_e x}$$

$$k_e = \omega\sqrt{\rho_p\kappa_p\Omega}\left[\tfrac{1}{2}\sqrt{1+\left(\frac{\Phi}{\rho_p\omega}\right)^2}+\tfrac{1}{2}\right]^{\frac{1}{2}} \to \begin{cases}\omega\sqrt{\rho_p\kappa_p\Omega} & \omega \text{ large}\\ \sqrt{\omega\kappa_p\Omega\Phi} & \omega \text{ small}\end{cases}$$

$$\gamma_e = \omega\sqrt{\rho_p\kappa_p\Omega}\left[\tfrac{1}{2}\sqrt{1+\left(\frac{\Phi}{\rho_p\omega}\right)^2}-\tfrac{1}{2}\right]^{\frac{1}{2}} \to \begin{cases}\tfrac{1}{2}\Phi\sqrt{\dfrac{\kappa_p\Omega}{\rho_p}} & \omega \text{ large}\\ \sqrt{\omega\kappa_p\Omega\Phi} & \omega \text{ small}\end{cases} \qquad (6.2.26)$$

The phase velocity ω/k_e is, roughly, equal to $1/\sqrt{\rho_p\kappa_p\Omega} = c_p$ when $\omega \gg \Phi/\rho_p$, somewhat smaller than the velocity c in the open for most porous materials. The wave amplitude in the porous medium is attenuated by a factor $1/e$ in a distance $1/\gamma_e \simeq 2\rho_p c_p/\Phi$ when Φ is small compared with $\rho_p\omega$. For sound-absorbing materials this distance is of the order of a centimeter or less, for frequencies above 2,000 cps.

Of course, this procedure of representing the effect of the pores in terms of an effective density and wave velocity is possible only for simple-harmonic waves. Its usefulness can be extended by means of the Fourier transform, however. Any waveform can be expressed, by this means, as a sum of simple-harmonic components, each of which will behave as though the fluid had density and wave velocity as given in Eqs. (6.2.26). We return to this in the next section.

Energy and momentum

In a porous medium energy is lost to frictional forces, so that the equation of continuity between energy density w and energy flow $\mathbf{I}$ must take this into account. If the amplitude of the velocity potential Ψ is given in rms terms, the time average of the energy density in the sound wave would be, according to Eq. (6.1.7),

$$w = \tfrac{1}{2}\kappa_p|p|^2 + \tfrac{1}{2}\rho_p|\mathbf{u}|^2 = \tfrac{1}{2}\kappa_p|\rho_p\Psi_t + \Phi\Psi|^2 + \tfrac{1}{2}\rho_p \operatorname{grad}\Psi^* \cdot \operatorname{grad}\Psi$$

and the intensity vector would be

$$\mathbf{I} = \tfrac{1}{2}p^*\mathbf{u} + \tfrac{1}{2}\mathbf{u}^*p = -\tfrac{1}{2}[(\rho_p\Psi_t^* + \Phi\Psi^*)\operatorname{grad}\Psi + (\rho_p\Psi_t + \Phi\Psi)\operatorname{grad}\Psi^*]$$

The mean rate of loss of energy per unit volume should be equal to the flow resistance times the mean square of the velocity, $\Phi|\mathbf{u}|^2$, so that the equation

of continuity for the energy should be

$$\frac{\partial w}{\partial t} + \operatorname{div} \mathbf{I} = -\Phi\,|\mathbf{u}|^2 = -\Phi \operatorname{grad} \Psi \cdot \operatorname{grad} \Psi^* \tag{6.2.27}$$

If we insert the expressions for w and $\mathbf{I}$ into the left-hand side of this equation, carry out the algebra, and use Eq. (6.2.23), we find that it does indeed equal the right-hand side.

The dissipative term $\kappa_p \Omega \Phi (\partial \Psi / \partial t)$ in Eq. (6.2.23) impedes our use of the Lagrange-Hamilton formalism, however. The only way in which we can use it is to combine the solution Ψ of Eq. (6.2.23) with a solution Ψ^a of an *adjoint* system, having negative flow resistance, and thus having a complementary rate of energy *gain* per unit volume. We set the combined Lagrange density to be

$$L = \tfrac{1}{2}\kappa_p \rho_p{}^2 \Omega \frac{\partial \Psi^a}{\partial t}\frac{\partial \Psi}{\partial t} - \tfrac{1}{2}\rho_p \operatorname{grad} \Psi^a \cdot \operatorname{grad} \Psi + \tfrac{1}{4}\rho_p \kappa_p \Phi \Omega \left(\frac{\partial \Psi^a}{\partial t}\Psi - \Psi^a \frac{\partial \Psi}{\partial t} \right) \tag{6.2.28}$$

The first two terms of this expression are similar to those of Eq. (6.2.19), except that, instead of the complex conjugate Ψ^*, the adjoint potential Ψ^a appears. The last term ensures that Ψ^a satisfies a different equation from Ψ. Applying the variational procedure to the integral of L [see the discussion of Eq. (5.2.5)], the variation of Ψ^a results in an equation like the first of Eqs. (6.2.20), except that Ψ^a occurs instead of Ψ^*. This, when applied to Eq. (6.2.28), results in the damped-wave equation (6.2.24).

On the other hand, variation of Ψ produces the second of Eqs. (6.2.20), which becomes

$$\nabla^2 \Psi^a = \kappa_p \rho_p \Omega \frac{\partial^2 \Psi^a}{\partial t^2} - \kappa_p \Phi \Omega \frac{\partial \Psi^a}{\partial t} \tag{6.2.29}$$

when the L of Eq. (6.2.28) is used. This has the first-derivative time term with a sign opposite to that of Eq. (6.2.23), thus representing energy accretion rather than dissipation.

The corresponding energy-momentum tensor has elements

$$\begin{aligned} W_{tt} &= \Psi_t{}^a \left(\frac{\partial L}{\partial \Psi_t{}^a} \right) + \Psi_t \left(\frac{\partial L}{\partial \Psi_t} \right) - L = w_0 \\ W_{tx} &= \Psi_t{}^a \left(\frac{\partial L}{\partial \Psi_x{}^a} \right) + \Psi_t \left(\frac{\partial L}{\partial \Psi_x} \right) = I_{0x} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{6.2.30}$$

with the same general form as those of Eqs. (6.2.21), with Ψ^a substituted for Ψ^*. Since Ψ^a represents an adjoint system, however, which gains energy as rapidly as Ψ loses it, the quantities ϵ_0, I_0, etc., are *constant in time* and satisfy nondissipative equations of continuity $(\partial \epsilon_0 / \partial t) + \operatorname{div} \mathbf{I}_0 = 0$, etc. Thus, only by introducing an adjoint system can we retain the Lagrange-Hamilton formalism when friction enters the picture.

6.3 WAVES IN AN INFINITE MEDIUM

We have now sketched the general properties of wave motion of small amplitude in a uniform medium. We have seen that, as long as the fluid is uniform and in equilibrium, if viscosity and thermal conduction can be neglected and if acoustic pressure p is small compared with equilibrium pressure P, then p, $\mathbf{u}$, etc., will satisfy a linear wave equation of the form of Eqs. (6.2.8) or of Eq. (6.2.24). For the next several chapters we shall work out the implications of these equations.

There are several reasons for spending so much time on so specialized a model of acoustic behavior. In the first place, it turns out to represent many acoustic phenomena surprisingly well. Sounds of ordinary intensity involve acoustic pressures of 10^{-4} atm or smaller (by definition $P = 1$ atm under standard conditions). In many situations of acoustic importance (such as the interior of an auditorium during a concert), we strive to keep the air quiescent and uniform, except for the sound. Solutions of the first-order equations (6.2.8) represent quite accurately much of the acoustic phenomena we are accustomed to. When we come to deal with the products of modern technology, however, such as jet aircraft or atom-bomb explosions, it becomes imperative that we study higher-order effects and propagation in a moving medium.

In the second place, the first-order wave equation is very much easier to handle mathematically than is the equation appropriate for high-intensity sounds or for a viscous or a moving medium. Equations (6.2.4), (6.2.8), and (6.2.24) are linear in the dependent variable, and we have commented in earlier chapters on the great advantages of linear equations. In fact, even when we come later to study the effects of motion of the fluid medium, we shall usually begin by studying linearized approximations to the correct equations, in order to obtain analytic solutions. Nonlinear equations usually must be solved numerically, a separate solution being worked out for each different situation of interest. Analytic solutions, though approximate, provide a quick survey of the general features of the phenomena.

In this chapter we shall concentrate on the behavior of plane-wave solutions of the simple wave equation. Later chapters will take up cylindrical and spherical waves, in which the peculiarities of two- and three-dimensional motion become apparent.

Traveling waves

We start by considering the propagation of a plane wave in a uniform nonviscous medium of infinite extent, at rest in the absence of sound. We choose the x axis to point in the direction of propagation, in which case the sound pressure p and all the other characteristics of the wave are functions of x and t only, and the fluid velocity $\mathbf{u}$ and all other vectors related to the wave are pointed in the x direction. The equation of motion which the

sound pressure must satisfy is the simple wave equation (6.2.4), which we repeat here:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \qquad \frac{1}{c^2} = \rho\kappa \tag{6.3.1}$$

where ρ is the equilibrium density of the medium, and κ is the adiabatic compressibility of the fluid, at equilibrium, as defined in Eqs. (6.1.3) and (6.1.5). As explained earlier, the change in density δ and in temperature τ of the fluid, and the fluid velocity u, caused by the wave, are in the first order linearly related to p, according to the equations

$$\delta = \rho\kappa p \qquad \tau = (\gamma - 1)\frac{\kappa}{\beta}p$$

and

$$\frac{\partial p}{\partial x} = -\rho\frac{\partial u}{\partial t} \qquad \kappa\frac{\partial p}{\partial t} = -\frac{\partial u}{\partial x} \tag{6.3.2}$$

Therefore δ, τ, and u also are solutions of Eq. (6.3.1).

We have already seen, in Sec. (4.1), that a general solution of the wave equation (6.3.1), traveling in the positive x direction, is given by the function $p(x - ct)$. This represents a wave of constant form $p(z)$, traveling with constant speed c (which equals 331 m per sec for air at 273°K and equals 1,500 m per sec for water at the same temperature). In the case of the string, the displacement is transverse to the direction of wave propagation; in the case of sound waves, the fluid motion is in the direction of propagation. By referring back to Eqs. (6.2.11), (6.2.13), and (6.2.15), we find we can write out all the important quantities characterizing the wave motion in terms of the velocity potential $\Psi(x - ct)$, where

$$\Psi(z) = -\frac{1}{\rho c}\int_0^z p(u)\,du \qquad \Psi'(z) = \frac{d\Psi}{dz}$$

$$u = -\Psi'(x - ct) \qquad p(x - ct) = -\rho c\Psi'(x - ct) = \rho c u(x - ct)$$

$$\text{Mass flux} = \rho u = \text{fluid momentum density} = \frac{p}{c}$$

$$\text{Fluid momentum flux} = \rho\,|u|^2 = \frac{|p|^2}{\rho c^2} = w \tag{6.3.3}$$

$$\text{Wave energy density } w = \tfrac{1}{2}\rho\,|u|^2 + \tfrac{1}{2}\kappa\,|p|^2 = \frac{|p|^2}{\rho c^2}$$

$$\text{Sound intensity } I_x = \tfrac{1}{2}(p^*u + pu^*) = \rho c\,|u|^2 = \frac{|p|^2}{\rho c} = cw$$

to the first order in the small quantities p, u, if the result is nonzero; otherwise, to the second order. We are here using complex quantities for Ψ, p, u, and τ; the amplitudes of the first-order quantities are rms, and the second-order

quantities are time averages. We thus are assuming that the variables can be expressed as sums of simple-harmonic terms of the type $A_n e^{-i\omega_n t}$ and that Eq. (6.2.19) can express the system's Lagrange density.

It should be pointed out again that we can neglect the difference between the Lagrange and the Euler descriptions, discussed in connection with Fig. 6.3, only to the first order. For example, if we tried to improve the accuracy of the mass-flux term by using $(\rho + \delta)u = \rho u + (\rho/c)u^2$, we might be tempted to conclude that, even if u were purely sinusoidal, there was a net forward flow of fluid, corresponding to the u^2 term. This would be wrong, of course, because one cannot compute displacement by integrating the Eulerian velocity or flux. In fact, on page 238, we showed that the fluid displacement is, to third order, equal to the integral of $u + (u^2/c)$, not the integral of u. When we come to discuss higher-order effects, we shall find it necessary to start afresh, keeping in mind the distinction between Euler and Lagrange descriptions at each step. Here we can neglect the distinction *as long as we adhere to the degree of approximation* given in Eqs. (6.3.3) *as written.*

The relationships given in Eqs. (6.3.3) are simple enough; they are the bases of most acoustical measurements. In a plane wave, pressure is in phase with fluid velocity; the region which is moving most rapidly in the direction of propagation is the region of greatest sound pressure (this is apparent from examination of Fig. 6.3*a*). The ratio between acoustic pressure p and fluid velocity u, $\rho c = \sqrt{\rho/\kappa}$ (which equals 429 newton-sec per cu m for air at 0°C and 1 atm pressure and equals 1.5×10^6 newton-sec per cu m for water at 0°C), is called the *characteristic acoustic impedance* of the medium.

The analogy with an electrical transmission line is fairly obvious; pressure corresponds to voltage, and u to current; furthermore, the intensity I, the mean power transmitted in joules per second per unit area, is equal to the impedance ρc times the square of u. Both sound pressure p and quantity $\rho c^2 = 1/\kappa$ have the dimensions of energy density. Sound pressures p are usually smaller than 1 dyne per sq cm = 0.1 newton per sq m or joule per cu m, whereas ρc^2 for air at 0°C and 1 atm is 1.42×10^5 joules per cu m. The energy density w of the wave is thus the pressure times the ratio of the pressure to this standard energy density, usually much smaller than 10^{-5} times the value of p itself.

Reflection from a locally reacting surface

Suppose this small-amplitude plane wave is incident on a plane surface, such as a wall. In general, some of the energy is reflected, and some is absorbed by the surface. For convenience, we take the surface to be the xz plane, with the fluid occupying the half-space above this plane, for positive values of y. The incident plane wave will have its direction of propagation in the xy plane, inclined at an angle ϑ_i to the y axis, as shown in Fig. 6.4

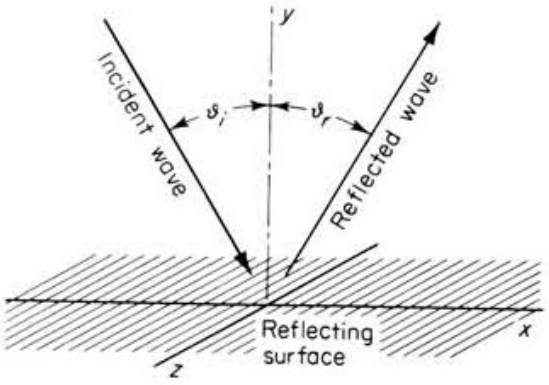

FIGURE 6.4
Reflection from a plane surface in the xz plane.

(ϑ_i is the *angle of incidence*). A rotation of axes of the expressions (6.3.3) shows that we can write them in terms of the velocity potential $\Psi_i(Z_i)$, with $Z_i = x \sin \vartheta_i - y \cos \vartheta_i - ct$.

$$p_i(x,y) = -\rho c \Psi_i'(Z_i) \qquad u_{ix} = -\sin \vartheta_i \Psi_i'(Z_i)$$

$$u_{iy}(x,y) = \cos \vartheta_i \Psi_i'(Z_i) \qquad w = \frac{|p_i|^2}{\rho c^2}$$

$$I_x = \frac{|p_i|^2}{\rho c} \sin \vartheta_i \qquad I_y = -\frac{|p_i|^2}{\rho c} \cos \vartheta_i$$

A possible reflected wave would be represented by the velocity potential $\Psi_r(x \sin \vartheta_r + y \cos \vartheta_r - ct)$ moving away from the reflecting surface at an angle ϑ_r to the y axis (the *angle of reflection*). Resulting pressures, velocities, etc., are correspondingly propagated upward at an angle ϑ_r, which is to be determined. The relationship between the functions Ψ_i and Ψ_r must also be found.

To make this determination, the acoustical nature of the reflecting surface must be specified. The acoustic pressure p acts on the surface and tends to make it move, or else tends to force more fluid into the pores of the surface. If any fluid motion normal to the surface is possible, there will be wave motion in the material forming the surface. Thus the motion of the surface at one point will be related to motion at another point of the surface, the relationship being determined by the wave motion inside the material, as well as by the incident and reflected waves.

In certain limiting cases, however, the various parts of the surface are not strongly coupled together, and we can consider that the motion, normal to the surface, of one portion of the surface is dependent only on the acoustic pressure incident on that portion and independent of the motion of any other part of the area. When this is the case, we say that the surface is one of *local reaction*; the consequent interaction of such a surface with incident and reflected waves is particularly simple. In the next subsection we shall work out the circumstances under which a surface may have local reaction; at present we assume it has.

The motion of the surface (or the motion of the fluid into the surface pores) at a given point is thus determined (we assume) by the acoustic pressure at that point. For small amplitudes the relationship is linear, and the velocity u_y normal to the surface is proportional to the pressure. Motion of the fluid tangential to the surface will not affect the surface motion; we are assuming at present that viscosity is negligible. The ratio between pressure and normal fluid velocity at a point on the surface is called the *acoustic impedance* of the surface. This impedance may depend on the frequency of the incident wave; it may be complex, in which case the normal velocity is not in phase with the pressure. But if the surface is one of local reaction, the impedance will not depend on the spatial distribution of the wave causing the pressure.

We usually symbolize this acoustic impedance by the letter z. Its natural units, as we shall soon see, are the characteristic impedance ρc of the medium. For convenience, we define the following terms:

$$\zeta = \frac{z}{\rho c} = \theta - i\chi \qquad \text{and} \qquad \beta = \frac{\rho c}{z} = \xi - i\sigma$$
$$\xi = \frac{\theta}{\theta^2 + \chi^2} \qquad \sigma = -\frac{\chi}{\theta^2 + \chi^2} \tag{6.3.4}$$

The ratio ζ between the acoustic impedance z of the locally reacting surface and ρc will be called the *specific acoustic impedance* of the surface. Its real part is the specific acoustic resistance θ; its reciprocal β is the specific acoustic admittance; and so on. As will be demonstrated, the interaction between sound waves and a surface of local reaction is completely defined by specifying the specific acoustic impedance (or its admittance) as a function of frequency.

For the moment let us assume that the impedance z of the surface is independent of frequency; that the ratio $(p/-u_y)$ at the surface is the same no matter what the shape of the wave. In this case we must adjust the reflected wave $\Psi_r(x \sin \vartheta_r + y \cos \vartheta_r - ct)$ so that, together with the incident wave $\Psi_i(x \sin \vartheta_i - y \cos \vartheta_i - ct)$, it has a pressure which is just z times the combined normal velocity u_y. At $y = 0$ the combined pressure is

$$-\rho c[\Psi_i'(x \sin \vartheta_i - ct) + \Psi_r'(x \sin \vartheta_r - ct)]$$

We certainly shall not be able to match $(p/-u_y)$ to z at every point unless Ψ_r' bears the same relation to Ψ_i' at every point. Therefore ϑ_r, the angle of reflection, must equal ϑ_i, the angle of incidence ($\vartheta_r = \vartheta_i = \vartheta$), and Ψ_r' must be a constant C_r times Ψ_i'. We thus have the combined pressure and normal velocity at $y = 0$.

$$p(x,0) = -\rho c(1 + C_r)\Psi_i'(x \sin \vartheta - ct)$$
$$u_y(x,0) = \cos \theta(1 - C_r)\Psi_i'(x \sin \vartheta - ct)$$

Note that the velocity of the wave *along the surface* is $c/\sin \vartheta$, greater than the velocity of propagation c. The ratio C_r between incident and reflected pressure waves is called the *reflection coefficient*.

We next must adjust C_r so that $(p/-u_y)$ equals z, the impedance of the surface. Going through the algebra, we find that

$$C_r = \frac{z \cos \vartheta - \rho c}{z \cos \vartheta + \rho c} \tag{6.3.5}$$

For this kind of surface, therefore, the reflection coefficient depends on the ratio between the surface impedance z and the characteristic impedance ρc of the fluid, and also on the cosine of the angle of incidence (and of reflection).

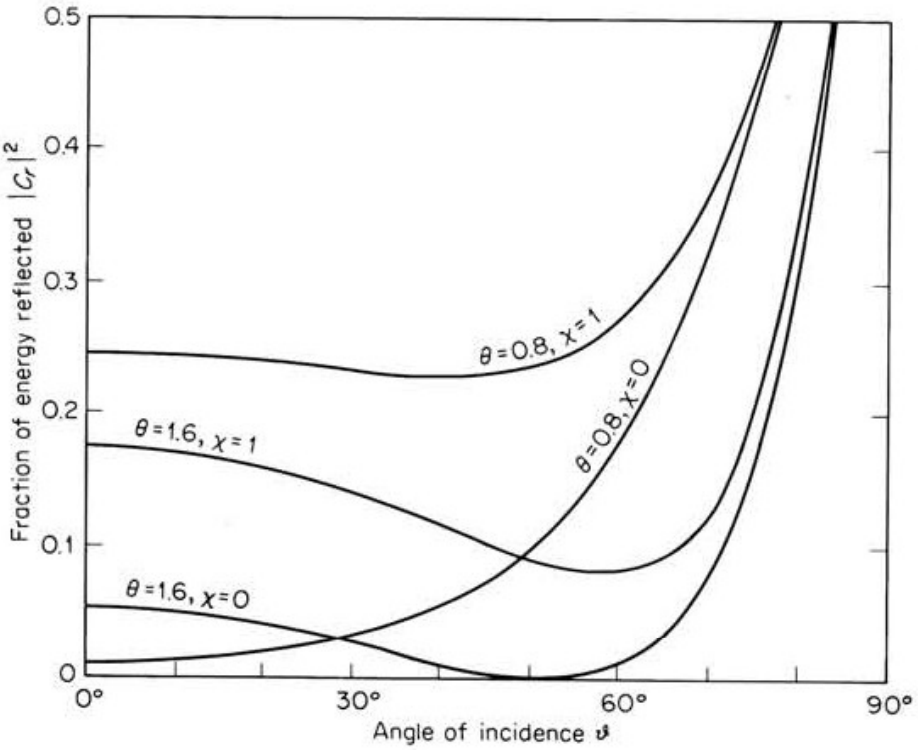

FIGURE 6.5
Fraction of energy reflected from a locally reacting surface of point impedance $z = \rho c(\theta - i\chi)$, for different values of resistance parameter θ and reactance parameter χ, as function of angle of incidence ϑ.

We note that if z is purely imaginary (reactive), then the magnitude of C_r is unity, the reflected wave differs only in phase angle from the incident wave, the reflected intensity $\rho c\,|C_r|^2\,|\Psi_i'|^2$ is equal to the incident intensity $\rho c\,|\Psi_i'|^2$, and no energy is absorbed by the surface. On the other hand, if z is real (resistive), then the reflection coefficient C_r is smaller than unity, the reflected intensity is smaller than the incident intensity, and the surface absorbs the difference. Since the waves are at an angle ϑ to the normal to the surface, a unit-area portion of the wave is spread over an area $1/\cos\vartheta$ of the surface, so that the energy absorbed per unit area of surface is

$$\rho c(1 - |C_r|^2)\,|\Psi_i'|^2 \cos\vartheta = \rho^2 c^2 \cos^2\vartheta \frac{4z}{(z\cos\vartheta + \rho c)^2}\,|\Psi_i'|^2 \tag{6.3.6}$$

when z is real. If z is also larger than ρc, there is an angle of incidence, $\vartheta = \cos^{-1}(\rho c/z)$, at which no wave is reflected; all the incident energy is absorbed. At this angle the component of the wall resistance, $z\cos\theta$, just equals the characteristic impedance ρc of the medium. Figure 6.5 shows curves of $|C_r|^2$ as functions of θ for a few values of $z/\rho c = \theta - i\chi$.

Frequency-dependent surface impedance

If the surface impedance z is a function of frequency, though still a point function, independent of the spatial distribution of the wave, we can start by calculating the reflection of a simple-harmonic wave of frequency $\omega/2\pi$. By procedures similar to those which produced Eq. (6.3.5), we find that the appropriate combination for an incident wave of rms amplitude A_i is

$$\Psi(x,y;\omega,t) = A_i e^{-i\omega T_i} + A_i C_r e^{-i\omega T_r}$$

where $$C_r = \frac{z(\omega)\cos\vartheta - \rho c}{z(\omega)\cos\vartheta + \rho c} \qquad T_i = t - \frac{1}{c}(x\sin\vartheta - y\cos\vartheta)$$

$$T_r = t - \frac{1}{c}(x\sin\vartheta + y\cos\vartheta)$$

with the incident wave going in the direction of the unit vector

$$\mathbf{a}_i = \mathbf{a}_x \sin\vartheta - \mathbf{a}_y \cos\vartheta$$

and a reflected plane wave, of amplitude $A_i C_r$, going in the direction of the vector $\mathbf{a}_r = \mathbf{a}_x \sin\vartheta + \mathbf{a}_y \cos\vartheta$, where $\mathbf{a}_x$, $\mathbf{a}_y$ are unit vectors in the x and y directions, respectively. The pressure, fluid velocity, mean energy density, and intensity for this wave combination are

$$\begin{aligned}
p &= \rho\frac{\partial\Psi}{\partial t} = B(\omega)(e^{-i\omega T_i} + C_r e^{-i\omega T_r}) \qquad B = -i\omega\rho A_i \\
\mathbf{u} &= -\operatorname{grad}\Psi = \frac{B}{\rho c}(\mathbf{a}_i e^{-i\omega T_i} + C_r \mathbf{a}_r e^{-i\omega T_r}) \\
w &= \tfrac{1}{2}\rho\,|u|^2 + \tfrac{1}{2}\kappa\,|p|^2 = \frac{1}{\rho c^2}|B|^2(1 + |C_r|^2) \\
\mathbf{I} &= \tfrac{1}{2}(p^*\mathbf{u} + p\mathbf{u}^*) = \frac{1}{\rho c}|B|^2(\mathbf{a}_i + |C_r|^2\,\mathbf{a}_r) \qquad c^2 = \frac{1}{\rho\kappa}
\end{aligned} \tag{6.3.7}$$

where $B(\omega)$ is the rms amplitude of the incident pressure wave, and BC_r that of the reflected pressure wave. The cross terms, with factors $e^{i\omega(T_i - T_r)}$, etc., in w and $\mathbf{I}$ vanish when averaged over time and space, leaving the mean energy density w as the sum of the energy densities of the two separate waves, and the mean energy flux $\mathbf{I}$ as the sum of the separate flows, each in their separate direction, as given by the unit vectors $\mathbf{a}_i$ and $\mathbf{a}_r$.

To take a typical example, let us assume that each point of the reflecting surface acts like a simple-harmonic oscillator, with effective mass m per unit area, mechanical resistance factor R, and stiffness $K = m\omega_0^2$ for motion

normal to the surface ($\omega_0/2\pi$ is the resonance frequency of the wall surface). The acoustic impedance z of the wall would then be $-i\omega m + R + im(\omega_0^2/\omega)$, or $(m/i\omega)[\omega^2 + i\omega(R/m) - \omega_0^2]$, according to Eq. (2.3.2), having a simple quadratic dependence on ω. For this case the ratio C_r, the reflection coefficient, and the square of its magnitude $|C_r|^2$, the ratio between reflected and incident intensities, are

$$C_r = \frac{(\omega^2 - \omega_0^2) + i\omega[(R/m) - (\rho c/m\cos\vartheta)]}{(\omega^2 - \omega_0^2) + i\omega[(R/m) + (\rho c/m\cos\vartheta)]}$$

$$= 1 - \frac{2i\omega(\rho c/m\cos\vartheta)}{(\omega + iq - \omega_0)(\omega + iq + \omega_0)} \qquad q = \frac{R}{2m} + \frac{\rho c}{2m\cos\vartheta} \tag{6.3.8}$$

$$|C|^2 = 1 - \frac{4\omega^2\rho cR\cos\vartheta}{(m\cos\vartheta)^2(\omega^2 - \omega_0^2)^2 + \omega^2(\rho c + R\cos\vartheta)^2}$$

For very low frequencies ($\omega \ll \omega_0$), the reflection coefficient C_r becomes unity for this wall, independent of θ; the incident pressure wave is reflected with no change in amplitude or phase; thus no energy is absorbed by the wall. This is also true for very high frequencies ($\omega \gg \omega_0$). At resonance ($\omega \simeq \omega_0$), however, C_r is smaller than unity and dependent on the angle of incidence θ. The reflected intensity at resonance is smallest when the angle of incidence is $\cos^{-1}(\rho c/R)$, if $R > \rho c$, or for normal incidence, if $R < \rho c$.

The power absorbed *per unit area* of the reflecting surface is

$$(|B|^2/\rho c)(1 - |C_r|^2)\cos\vartheta$$

which is the following fraction of the incident intensity $(|B|^2/\rho c)$:

$$\frac{4\omega^2\rho cR\cos^2\vartheta}{(m\cos\vartheta)^2(\omega^2 - \omega_0^2)^2 + \omega^2(\rho c + R\cos\vartheta)^2} \to \begin{cases} \dfrac{4\rho cR}{m^2\omega_0^2}\dfrac{\omega^2}{\omega_0^2} & \omega \ll \omega_0 \\ \dfrac{4\rho cR\cos^2\vartheta}{(\rho c + R\cos\vartheta)^2} & \omega = \omega_0 \\ \dfrac{4\rho cR}{m^2\omega^2} & \omega \gg \omega_0 \end{cases}$$

for this sort of reflecting surface. This fraction goes to zero as $\omega \to 0$ and as $\omega \to \infty$, being independent of the angle of incidence at these limiting frequencies. At resonance the absorbed power is strongly dependent on ϑ, being zero at grazing incidence and greatest at normal incidence.

Reflection of a pulse-wave

Having worked out the reflection of a simple-harmonic wave from a plane surface of local reaction, we can now call in the Fourier transform to compute the reflection of a wave of any form. From Eq. (1.3.25) a unit pulse-wave, traveling in the direction of $\mathbf{a}_i = \mathbf{a}_x\sin\vartheta - \mathbf{a}_y\cos\vartheta$, can be

built up out of combination of simple-harmonic waves,

$$\delta(T_i - t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t_0 - T_i)}\, d\omega$$

Comparing this with Eqs. (6.3.7), we see that if we set the amplitude $B(\omega)$ of the simple harmonic wave equal to $(1/2\pi)e^{i\omega t_0}$ and integrate the result over ω, we shall obtain

$$p_\delta(x,y;t,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [e^{i\omega(t_0 - T_i)} + C_r(\omega)e^{i\omega(t_0 - T_r)}]\, d\omega$$

$$= \delta(T_i - t_0) + \delta(T_r - t_0) + W(T_r - t_0)$$

$$W(T_r - t_0) = -\frac{\rho c}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega(t_0 - T_r)}\, d\omega}{z(\omega)\cos\vartheta + \rho c} \tag{6.3.9}$$

$$T_i = t - \frac{1}{c}(x\sin\vartheta - y\cos\vartheta) \qquad T_r = t - \frac{1}{c}(x\sin\vartheta + y\cos\vartheta)$$

representing a unit pulse-wave, incident on the reflecting plane at an angle of incidence ϑ, and a reflected wave, consisting of a unit pulse plus a "wake" W, represented by the integral, which is produced by the motion of the surface as it reacts to the pulse.

If, for example, the impedance is $z = -i\omega m + R + im(\omega_0^2/\omega)$, as in Eqs. (6.3.8), the integral becomes

$$W(T) = \frac{2\rho c}{2\pi i m\cos\vartheta}\int_{-\infty}^{\infty} \frac{\omega e^{i\omega T}\, d\omega}{(\omega + iq - \omega_0)(\omega + iq + \omega_0)} \qquad T = T_r - t_0$$

Referring to Eqs. (1.2.15) or Eq. (2.3.10), we see that

$$W(T_r - t_0) = \begin{cases} 0 & T < 0 \\ \dfrac{-2\rho c}{m\cos\vartheta}\exp\left[-\dfrac{T}{2m}\left(R + \dfrac{\rho c}{\cos\vartheta}\right)\right]\left[\cos(\omega_0 T)\right. & \\ \qquad \left. - \dfrac{1}{2m\omega_0}\left(R + \dfrac{\rho c}{\cos\vartheta}\right)\sin(\omega_0 T)\right] & T > 0 \end{cases} \tag{6.3.10}$$

which is a wave that damps out with time, the rate of damping being least for $\vartheta = 0$ and becoming infinite as $\vartheta \to \frac{1}{2}\pi$ (i.e., for grazing incidence). There is no wave when $T < 0$, that is, if t is less than $t_0 + (1/c)(x\sin\vartheta + y\cos\vartheta)$. At $T = 0$, along the reflected wavefront, $x\sin\vartheta + y\cos\vartheta = c(t - t_0)$, there is a unit pulse, immediately followed by the initially negative pressure of the "wake" W. As time goes on (T becomes larger than 0), W exhibits damped oscillations, dying out more rapidly, the smaller $\cos\vartheta$. As $\vartheta \to \frac{1}{2}\pi$ (grazing incidence), function W approaches the pulse function $-2\delta(T)$, which more than cancels the reflected pulse.

What happens is somewhat like the wave reflection illustrated in Fig. 4.9. The pressure pulse of the incident wave cannot move the mass load of the wall instantaneously; so the pulse is reflected unchanged. It has given an impulse to the surface, however; so the surface executes damped oscillation, first receding from the blow, which generates a pressure wave initially negative. The damping of the surface oscillations is proportional to the internal resistance R of the wall *plus* the radiative resistance $\rho c/\cos\vartheta$ engendered by the presence of the fluid medium.

Finally, as with Eq. (2.3.9), the complete solution for the reflection of an incident plane wave of arbitrary form $F(t)$, at angle of incidence ϑ, on a plane surface having point impedance $z(\omega)$, is

$$p = \int_{-\infty}^{\infty} F(t_0) p_\delta(x,y;t,t_0)\, dt_0 = F(T_i) + F(T_r) + \int_{-\infty}^{\infty} F(t_0) W(T_r - t_0)\, dt_0 \tag{6.3.11}$$

where the integral involving the function W represents the modification of the reflected wave because of the wall motion.

Reflection from a surface of extended reaction

Although many surfaces react to sound waves, at least approximately, as though each portion of the surface responded to local pressure without knowledge of motion elsewhere, many surfaces do not so react. When the surface behavior at one point depends on the behavior at neighboring points, so that the reaction is different for different incident waves, the surface can be called one of *extended reaction*. Such surfaces are of many sorts: ones which behave like membranes and ones with laminated structure, in which waves are propagated parallel to the surface; ones in which waves penetrate into the material of the wall; and so on. We shall have to defer discussion of the more complex structures until later; in this chapter we can discuss only the case where the plane surface separates two media, both of which can transmit pressure waves but which differ in their acoustic properties.

The situation to be analyzed is pictured in Fig. 6.4. The boundary surface is the xz plane; the region above the surface ($y > 0$) is filled with a fluid of uniform density ρ and wave velocity $c = \sqrt{1/\rho\kappa}$, which will be called the *medium*. The region below the surface ($y < 0$) is filled with material of density ρ_w and wave velocity $c_w = \sqrt{1/\rho_w\kappa_w}$, which will be called the *wall material*. As before, the incident wave is pointed at an angle ϑ with respect to the y axis; its velocity potential may be written $\Psi(T_i)$, where $T_i = t - (1/c)(x \sin\vartheta - y\cos\vartheta)$, as before. The angle of reflection is also ϑ, as before, and the amplitude of the reflected wave may differ from that of the incident wave; so we can write its velocity potential as $C_r\Psi(T_r)$, where $T_r = t - (1/c)(x \sin\vartheta + y\cos\vartheta)$, as before.

Now, however, there is also a wave moving downward into the wall material, with a dependence on time like that of the incident wave but with a

different amplitude and velocity and a direction of propagation at a different angle, ϑ_p, called the angle of refraction, or of penetration. The total velocity potential, pressure, and velocity for this case are

$$\Psi = \begin{cases} \Psi(T_i) + C_r\Psi(T_r) & \\ C_p\Psi(T_p) & \end{cases} \qquad p = \begin{cases} \rho\Psi'(T_i) + \rho C_r\Psi'(T_r) & y > 0 \\ \rho_w C_p\Psi'(T_p) & y < 0 \end{cases}$$

$$u_y = \begin{cases} \dfrac{1}{c}\cos\vartheta[-\Psi'(T_i) + C_r\Psi'(T_r)] & y > 0 \\ -\dfrac{C_p}{c_w}\cos\vartheta_p\Psi'(T_p) & y < 0 \end{cases} \tag{6.3.12}$$

where $T_p = t - (1/c_w)(x\sin\vartheta_p - y\cos\vartheta_p)$, and Ψ' is the derivative of Ψ with respect to its argument.

At the surface, $y = 0$, the pressure and the normal velocity u_y are continuous, which enables us to calculate the reflection coefficient C_r and the penetration coefficient C_p, and also to determine the angle of penetration ϑ_p. Since both incident and reflected waves, at the surface, depend on t and x via the function $\Psi'[t - (x/c)\sin\vartheta]$, for the dependence of the penetrating wave to be similar, $(1/c_w)\sin\vartheta_p$ must equal $(1/c)\sin\theta$, or

$$\sin\vartheta_p = \frac{c_w}{c}\sin\vartheta \qquad \text{and} \qquad \cos\vartheta_p = \left[1 - \left(\frac{c_w}{c}\right)^2\sin^2\vartheta\right]^{\frac{1}{2}} \tag{6.3.13}$$

which is Snell's law for acoustic waves. The continuity of pressure and normal velocity at $y = 0$ produces the following equations:

$$1 + C_r = \frac{\rho_w}{\rho}C_p \qquad \text{and} \qquad 1 - C_r = \frac{c}{c_w}\frac{\cos\vartheta_p}{\cos\vartheta}C_p$$

so that

$$C_r = \frac{\rho_w c_w\cos\vartheta - \rho c\cos\vartheta_p}{\rho_w c_w\cos\vartheta + \rho c\cos\vartheta_p} \qquad C_p = \frac{2\rho c_w\cos\vartheta}{\rho_w c_w\cos\vartheta + \rho c\cos\vartheta_p} \tag{6.3.14}$$

The formula for the reflection coefficient C_r is similar to that for a surface of local reaction, with point impedance z, as given in Eq. (6.3.5), only now the effective impedance of the surface is $z = \rho_w c_w/\cos\vartheta_p$, which depends on the angle of incidence ϑ. Thus the degree to which the surface yields to the incident wave depends on the distribution of the incident wave; the surface is aware of the waveshape; so it is one of extended reaction. However, if the velocity c_w of the sound waves in the wall material is considerably smaller than the speed of sound in the medium, then $\cos\vartheta_w = [1 - (c_w/c)^2\sin^2\vartheta]^{\frac{1}{2}}$ is practically independent of ϑ, and the wall is one of local reaction. Comparison with Eq. (6.3.5) shows that the equivalent impedance of the wall material is $\rho_w c_w\sec\vartheta$, dependent on ϑ.

On the other hand, if c_w is larger than c, then when ϑ is larger than $\sin^{-1}(c/c_w)$, the quantity called $\cos\vartheta_p$ becomes imaginary and the reflection coefficient C_r has unit amplitude; all the incident energy is reflected unless ρ_w or c_w is complex. This is the condition of total reflection, well known in optics. As long as ρ_w, c_w, and $\cos\vartheta_p$ are real, the fraction of the incident energy which enters each unit area of the surface, thus being lost from the reflected wave, is

$$(1 - C_r^2)\cos\vartheta = \frac{4\rho c\rho_w c_w \cos^2\vartheta\cos\vartheta_p}{(\rho_w c_w\cos\vartheta + \rho c\cos\vartheta_p)^2}$$

which, of course, equals the fraction $(\rho_w c/\rho c_w)C_p^2\cos\vartheta_p$ of the incident energy, per unit surface area, which is carried into the wall material by the refracted wave [the factor $\rho_w c/\rho c_w$ comes about because the intensity in the wall is $(\rho_w/c_w)|\Psi'|^2$, whereas it is $(\rho/c)|\Psi'|^2$ in the medium].

If the wall material is porous, with the acoustic properties as defined in Eqs. (6.2.23) and (6.2.26), the reflection properties of the surface depend on the frequency of the incident wave, as well as on the angle of incidence. The incident, reflected, and transmitted pressure waves are obtained by inserting the ρ_e and c_e of Eq. (6.2.25) into Eqs. (6.3.12) and (6.3.14) and by specializing the waveform to a sinusoidal shape.

$$p(x,y;t,\omega) = \begin{cases} A(\omega)(e^{-i\omega T_i} + C_r e^{-i\omega T_r}) & y > 0 \\ A(\omega)C_p e^{-i\omega T_p} & y < 0 \end{cases} \tag{6.3.15}$$

where T_i and T_r are, as before,

$$T_p = t - \frac{x}{c}\sin\vartheta + \frac{y}{c_p}\left(\cos^2\vartheta_p + i\frac{\gamma}{\omega}\right)^{\frac{1}{2}}$$

with $c^2 = 1/\rho\kappa$, $c_p^2 = 1/\rho_p\kappa_p\Omega$, $\cos^2\vartheta_p = 1 - (c_p/c)^2\sin^2\vartheta$, and $\gamma = \Phi/\rho_p$. Since c_p is usually smaller than c, $\cos\vartheta_p$ does not go to zero at grazing incidence. The reflection and penetration coefficients are

$$C_r = \frac{\rho_p c_p(\omega + i\gamma)\cos\vartheta - \rho c(\omega^2\cos^2\vartheta_p + i\gamma\omega)^{\frac{1}{2}}}{\rho_p c_p(\omega + i\gamma)\cos\vartheta + \rho c(\omega^2\cos^2\vartheta_p + i\gamma\omega)^{\frac{1}{2}}}$$

$$C_p = \frac{2\omega\rho c_p\cos\vartheta}{\rho_p c_p(\omega + i\gamma)\cos\vartheta + \rho c(\omega^2\cos^2\vartheta_p + i\gamma\omega)^{\frac{1}{2}}} \tag{6.3.16}$$

These quantities are complex, the imaginary parts being proportional to the flow resistance Φ of the porous material and inversely proportional to ω. At frequencies large compared with $\gamma = \Phi/\rho_p$, this imaginary part vanishes and

$$C_r \to \frac{\rho_p c_p\cos\vartheta - \rho c\cos\vartheta_p}{\rho_p c_p\cos\vartheta + \rho c\cos\vartheta_p} \qquad C_p \to \frac{2\rho c_p\cos\vartheta}{\rho_p c_p\cos\vartheta + \rho c\cos\vartheta_p}$$

similar to Eq. (6.3.14), only with the limiting values ρ_p, c_p used instead of

ρ_e, c_e. As indicated in Eq. (6.2.26), even at these high frequencies, the refracted wave is attenuated in accordance with a factor

$$\exp\left[y(\Phi/2\rho_p c_p \cos\vartheta_p)\right]$$

which diminishes as y increases in the negative direction, the attenuation being greatest for grazing incidence. At these higher frequencies there can

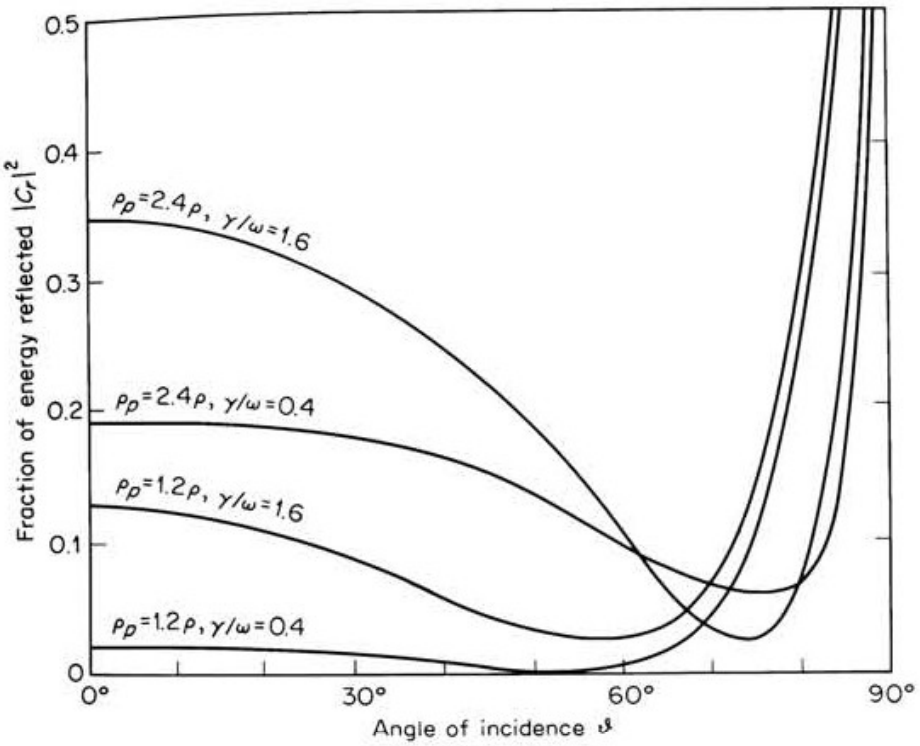

FIGURE 6.6
Fraction of energy reflected from the surface of a porous medium of effective density ρ_p, effective wave velocity $c_p = c$, and flow resistance $\Phi = \gamma\rho_p$ for frequencies $\omega/2\pi$ and angles of incidence ϑ. Compare with Fig. 6.5.

be an angle of incidence at which the reflected wave vanishes; if $c_p < c$, then $C_r = 0$ when $\cos\vartheta = (\rho c/\rho_p c_p)\cos\vartheta_p$. This is usually close to grazing incidence. At grazing incidence, of course, $|C_r|^2 \to 1$.

At very low frequencies, C_p approaches zero and nearly all the incident energy is reflected. At intermediate frequencies, the reflected wave differs from the incident wave both in phase and amplitude, depending quite strongly on frequency and angle of incidence. A few typical curves for $|C_r|^2$ as a function of angle of incidence, for different frequencies, are shown in Fig. 6.6. They are not greatly different from those of Fig. 6.5, for a surface of local reaction.

To compute the reflection of a pulse-wave from a porous surface, we parallel the derivation of Eq. (6.3.9), integrating $(e^{i\omega t_0}/2\pi)p(x,y;t,\omega)$ of Eq. (6.3.15) over ω from $-\infty$ to $+\infty$. The integrals for the reflected and

penetrant waves are complicated because of the radicals in C_r and in the wave for $y < 0$. The branch points are at $\omega = 0$ and $\omega = -i(\gamma/\cos^2 \vartheta_p)$, so that the integral for the "wake" runs between these values. The formulas become

$$p = \begin{cases} \delta(T_i - t_0) + \dfrac{\rho_p c_p \cos\vartheta - \rho c \cos\vartheta_p}{\rho_p c_p \cos\vartheta + \rho c \cos\vartheta_p}\,\delta(T_r - t_0) \\ \qquad\qquad + W_r(T_r - t_0) \qquad y > 0 \\ \dfrac{2\rho c_p \cos\vartheta}{\rho_p c_p \cos\vartheta + \rho c \cos\vartheta_p}\,\delta(T_p - t_0)\exp\dfrac{y\Phi}{\rho_p c_p \cos\vartheta_p} \\ \qquad\qquad + W_p(T_p - t_0, y) \qquad y < 0 \end{cases} \tag{6.3.17}$$

where $T_i = t - (x/c)\sin\vartheta + (y/c)\cos\vartheta$, $T_r = t - (x/c)\sin\vartheta + (y/c)\cos\vartheta$, $T_p = t - (x/c)\sin\vartheta + (y/c_p)\cos\vartheta_p$, and $\cos^2\vartheta_p = 1 - (c_p/c)^2\sin^2\vartheta$.

Both wake functions are zero when their arguments are negative; for positive arguments they have the general form

$$W_r(T) = -\frac{2}{\pi}\alpha g \cos\vartheta\cos\vartheta_p \int_0^1 \frac{(v - \beta^2\cos^2\vartheta)\sqrt{v(1-v)}\,dv\,e^{-vgT}}{\cos^2\vartheta(v - \beta^2\cos^2\vartheta_p)^2 + \alpha^2\cos^2\vartheta_p v(1-v)}$$

$$W_p(T,y) = -\frac{2g\alpha}{\pi\beta}\cos\vartheta \int_0^1 \frac{\exp\{-vg[T - (y/c_p)\cos\vartheta_p]\}v\,dv}{\cos^2\vartheta(v - \beta^2\cos^2\vartheta_p)^2 + \alpha^2\cos^2\vartheta_p v(1-v)}$$
$$\times \left\{(v - \beta^2\cos^2\vartheta_p)\cos\vartheta \sin\left[\frac{yg}{c_p}\cos\vartheta_p\sqrt{v(1-v)}\right]\right.$$
$$\left. + \alpha\cos\vartheta_p\sqrt{v(1-v)}\cos\left[\frac{yg}{c_p}\cos\vartheta_p\sqrt{v(1-v)}\right]\right\}$$

where $\alpha = \rho c/\rho_p c_p$, $\beta = c/c_p$, and $g = \Phi/\rho_p\beta^2\cos^2\vartheta_p$. Both wake functions attenuate as their arguments increase, the more rapidly, the smaller $\cos\vartheta_p$. In contrast to the reactions of the point-impedance wall of Eq. (6.3.10), the wake from a porous wall has no damped oscillations; the air in the pores yields immediately to the pulse and recovers without oscillation. The integrals for W_r and W_p cannot be expressed in closed form, but they can be integrated numerically for any given situation. Expressions for T small and T large can also be worked out, though they need not be given here.

6.4 INTERNAL ENERGY LOSS

In the foregoing analysis the effects of viscosity, heat conduction, and other dissipative processes have been neglected. As long as we deal with gases at ordinary pressures and frequencies that are not too high, this omission is not very important in most problems; calculated sound pressures agree well with experimental results over a wide range of amplitudes and frequencies.

Only when we come to calculate sound transmission over long distances in the open, or through narrow ducts, will the omission of internal dissipation become apparent. Also, in the neighborhood of sharp corners in boundaries, the dissipationless analysis will predict too large velocities.

To correct these defects we must include the effects of viscosity, heat conduction, and hysteresis in internal heat absorption, as well as the effects of nonlinearities. In this section we consider only the dissipative effects; consequently, the results, along with those of the whole chapter, will be valid only for sufficiently small amplitudes. The question of what we mean by "sufficiently small" cannot be answered until we investigate the nonlinear effects, which will be postponed until Chap. 14.

Thermal conduction and viscosity

From one point of view, the phenomena of viscosity and thermal conduction in gases are a consequence of their molecular constitution. Temperature is a manifestation of random molecular motion; the higher the temperature, the faster the average molecular speed. A region at higher temperature will have more high-speed molecules than the average, and these faster particles will diffuse out into the cooler regions, tending to equalize the temperature. As with any other diffusion, the flux $\mathbf{J}$ (in this case the flux of heat) is proportional to the rate of change of the concentration (in this case to the temperature gradient)

$$\mathbf{J}_h = -K \operatorname{grad} T \tag{6.4.1}$$

where the proportionality constant K is called the *thermal conductivity* of the gas.

Kinetic theory of gases indicates that this conductivity is simply related to other properties of the gas, such as its density ρ, its heat capacity C_v per unit mass at constant volume, the mean-free-path l of its molecules, and the speed of sound c, by the equation

$$K \simeq 1.6 \frac{l\rho c C_v}{\sqrt{\gamma}} \tag{6.4.2}$$

where γ is the constant used in Eq. (6.1.3), and discussed later, in connection with Eqs. (6.4.19). Heat conductivity of liquids and solids is, of course, a much more complicated function of the material's properties.

Since the net gain in heat in a unit volume, because of the flux $\mathbf{J}_h$, is $-\operatorname{div} \mathbf{J}_h$, and since this influx, divided by the heat capacity at constant pressure $\rho C_p = \gamma \rho C_v$ of a unit volume, is equal to the rise in temperature of the volume of gas, we have

$$\frac{\partial T}{\partial t} = \frac{K}{\rho C_p} \operatorname{div} \operatorname{grad} T \simeq \frac{lc}{\gamma^{\frac{3}{2}}} \nabla^2 T \qquad \text{(for a gas)} \tag{6.4.3}$$

which will be used to determine the loss of energy in the sound wave, caused by thermal diffusion.

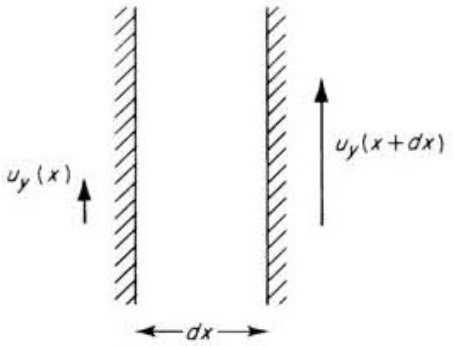

FIGURE 6.7
Rate of change of shear in a gas, producing viscous shear stress.

Viscosity, on the other hand, is produced by momentum diffusion. If one portion of the gas slides past another so that, for example, the velocity in the y direction is a function of x, as illustrated in Fig. 6.7, the slower molecules at x will diffuse over to reduce the fluid momentum at $x + dx$, and the faster ones at $x + dx$ will tend to speed up those at x. As a result, the region between x and $x + dx$ is subject to shear stress D_{xy}, proportional to $\partial u_y/\partial x$, the proportionality factor being called the *coefficient of viscosity* μ,

$$D_{xy} = -\mu \frac{\partial u_y}{\partial x} \tag{6.4.4}$$

As might be expected, since both depend on the diffusion of the gas molecules, there is a close relationship between the coefficient of viscosity and the thermal conductivity of a gas. Kinetic theory shows that, approximately,

$$K \simeq \tfrac{5}{3}\mu C_v \qquad \text{or} \qquad \mu \simeq \frac{l\rho c}{\sqrt{\gamma}} \qquad \text{(for a gas)} \tag{6.4.5}$$

where c = velocity of sound
ρ = density of gas
l = mean-free-path
γ = ratio of specific heats = C_p/C_v

To obtain the complete expression for the stress tensor $\mathfrak{P}$, introduced in Eqs. (6.2.10), we might expect its nondiagonal components P_{ij} to be proportional to the differential $\partial u_i/\partial x_j$ ($u_1 = u_x$, $x_1 = x$, $x_2 = y$, etc.), as suggested by Eq. (6.4.4). But two basic considerations prevent this. In the first place, a part of the tensor $\partial u_i/\partial x_j$ represents rotation without shear. For example, the quantity $\frac{1}{2}$ curl $\mathbf{u}$, where

$$\begin{aligned}
(\text{curl } \mathbf{u})_x &= \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \\
(\text{curl } \mathbf{u})_y &= \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \\
(\text{curl } \mathbf{u})_z &= \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}
\end{aligned} \tag{6.4.6}$$

is the vector representing the angular velocity of pure rotation of the fluid at (x,y,z). Thus we must subtract off the rotational part of tensor $\partial u_i/\partial x_j$ to obtain the true rate of strain.

$$U_{ij} = \frac{1}{2}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\frac{\partial u_j}{\partial x_i} = U_{ji} \qquad i \neq j$$

The second consideration lies in the fact that, although the nondiagonal components U_{ij} $(i \neq j)$ represent pure shear strains with no change in fluid content, the elements U_{ii} in part measure a rate of change of amount of fluid inside the element of volume around point (x,y,z). For example, the combination

$$\text{div } \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

measures the rate of outflow of fluid from unit volume around (x,y,z) [Eq. (6.1.15)]. To ensure that the shear-strain tensor does not include this outflow part, we subtract div $\mathbf{u}$ in a symmetric way from the diagonal parts of the tensor, arriving finally at

$$\mathfrak{U} = \begin{pmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{yx} & U_{yy} & U_{yz} \\ U_{zx} & U_{zy} & U_{zz} \end{pmatrix} \qquad \begin{aligned} U_{ii} &= \frac{\partial u_i}{\partial x_i} - \tfrac{1}{3}\,\text{div } \mathbf{u} \\ U_{ij} &= \frac{1}{2}\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\frac{\partial u_j}{\partial x_i} \end{aligned} \tag{6.4.7}$$

which is a measure of the rate at which one part of the fluid slides past another at point (x,y,z), excluding the effects of rotation and of efflux.

The corresponding *viscous-shear tensor* $\mathfrak{D}$ should be proportional to this; to have it correspond to the simple case of Eq. (6.4.4), we might expect $\mathfrak{D}$ to be equal to -2μ times $\mathfrak{U}$. But still another term should be added to this to obtain the complete viscous-stress tensor. As was mentioned in the discussion of Eq. (6.2.10), the total stress tensor for the fluid must also include the effects of the usual hydrostatic pressure P, which acts perpendicularly on each face of the element of fluid volume, as shown in Fig. 6.8. The stress on the x faces thus has components $P + D_{xx}$, D_{xy}, D_{xz}, and so on, for the other pairs of sides. As indicated in Eqs. (6.2.7), the acoustic pressure p is proportional to the change in density of the gas; its rate of change $\partial p/\partial t$ is proportional to the efflux div $\mathbf{u}$ of fluid. Thus the compressibility κ is analogous to the reciprocal $1/K$ of the stiffness constant of a spring. Related to the stiffness of the gas to compression, there may be a stiffness force proportional to the rate of change of compression, just as the frictional force on the simple oscillator is proportional to the oscillator's velocity.

This compressional resistance will be labeled η, and will be called the *coefficient of bulk viscosity;* the term to be included in the stress tensor is $-\eta$ div $\mathbf{u}$, representing a stress opposing the rate of change of compression.

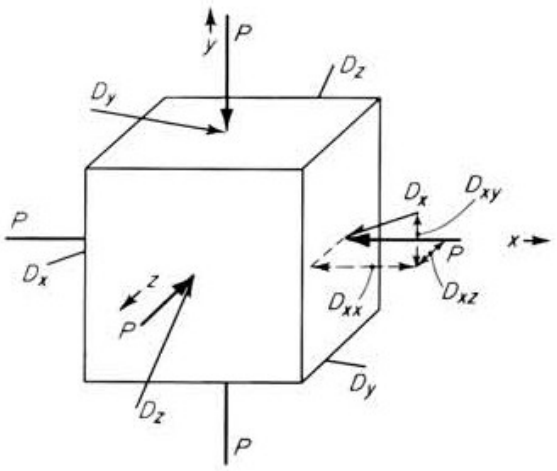

FIGURE 6.8
Total stress on an element of volume is the compressional stress P, normal to the surface, plus the various components D_{ij} of the viscous stress.

This term is too small to be measured in monatomic gases such as helium; it may be quite large in polyatomic gases at certain frequencies. The physical mechanism for most, if not all, of its effect is related to the time required to establish thermodynamic equilibrium between the translational and the rotation-vibration motions of the gas molecules. At present we simply note its presence and write down the complete expression for the components of the viscous-stress tensor $\mathfrak{D}$.

$$D_{ii} = -(\eta - \tfrac{2}{3}\mu) \operatorname{div} \mathbf{u} - 2\mu \frac{\partial u_i}{\partial x_i}$$
$$D_{ij} = -\mu \frac{\partial u_i}{\partial x_j} - \mu \frac{\partial u_j}{\partial u_i} \qquad i \neq j \tag{6.4.8}$$

Thus the stress across the faces of an element of gas, perpendicular to the x axis, as shown in Fig. 6.8, is the vector sum of the normal pressure force and the viscous force $\mathbf{D}_x$, including the bulk-viscosity term. The components of this stress tensor $\mathfrak{P}$, introduced in Eqs. (6.2.10), are

$$P_{ii} = P + D_{ii} = P - (\eta - \tfrac{2}{3}\mu) \operatorname{div} \mathbf{u} - 2\mu \frac{\partial u_i}{\partial x_i}$$
$$P_{ij} = D_{ij} = -\mu \frac{\partial u_i}{\partial x_j} - \mu \frac{\partial u_j}{\partial x_i} = P_{ji} \qquad i \neq j \tag{6.4.9}$$

which introduces viscous friction into the equations of motion.

Energy loss

The two processes thermal conduction and viscosity tend to transform the organized motion of the sound wave into the disorganized motion of heat. The energy content of the wave decreases, and the gas warms up. Since both the viscosity and the thermal conductivity of a gas are small, the energy transfer takes place slowly, and to first approximation, the shape of the wave is the same as that calculated neglecting the losses, except that the wave

amplitude diminishes exponentially with time or space or both. Thus it is useful to work out a simple first-order expression for energy loss before turning to more accurate, though more complex, formulas.

The rate of energy loss from viscosity in a volume element $dx\,dy\,dz$ can be calculated by summing the scalar product of the stress vector $D_x\,dy\,dz$ by the difference in velocities $\mathbf{u}(x) - \mathbf{u}(x + dx)$ plus similar products for the other pairs of faces; the loss rate per unit volume of gas is thus

$$D = \sum_{ij} \frac{\partial u_i}{\partial x_j} D_{ij} = (\eta + \tfrac{4}{3}\mu)(\text{div } \mathbf{u})^2 + \mu\,|\text{curl } \mathbf{u}|^2 - 4\mu\left(\frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_x}{\partial x}\frac{\partial u_z}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_z}{\partial z} - \frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial z}\frac{\partial u_z}{\partial x} - \frac{\partial u_y}{\partial z}\frac{\partial u_z}{\partial y}\right) \quad (6.4.10)$$

which turns out to be everywhere positive (as it must be, since energy cannot be gained by frictional processes).

In a plane wave, where $\mathbf{u}$ is pointed along, and is a function of, the distance along the direction of propagation (call it x), this loss rate becomes, simply,

$$D = (\eta + \tfrac{4}{3}\mu)\left|\frac{du_x}{dx}\right|^2$$

If the wave is sinusoidal, D is the time average of the loss rate per volume when rms amplitude of $\mathbf{u}$ is used, in accord with the convention of Eq. (6.2.18).

The loss of energy caused by heat conduction comes about because the fluctuations of pressure in a sound wave do not represent a condition of thermodynamic equilibrium. In the regions where p is positive, the compression has not been isothermal; there has not been time for the temperature to equalize. In fact, as we have indicated in Eqs. (6.1.3) and (6.1.5), it is more nearly correct to consider the compression to be adiabatic, to assume that no heat energy moves from the region of positive p to one of negative p. If this were strictly true, of course, no energy would be lost; the compression would be reversible. But according to Eq. (6.4.1), as soon as there is a temperature differential in the gas, heat energy flows irreversibly from the hotter to the cooler regions, and the sound wave loses energy.

This would make it seem that the loss would be greater at low frequencies, when there is a longer time per cycle for the heat to leak away. But at the higher frequencies, the shorter wavelength means that there is a shorter distance between hotter and colder regions in the wave, and thus a greater temperature gradient, which more than counterbalances the shortness of the period, as we shall see.

To compute the energy lost we refer to thermodynamic equations. The sound wave produces a threefold change of internal energy U in a portion of the gas which occupies unit volume at equilibrium. In the first place, the heat flow $\mathbf{J}_h$ may concentrate heat there of an amount $-\text{div } \mathbf{J}_h$; next, heat is generated by viscous friction of an amount D, as given in Eq. (6.4.10). And finally, U is decreased by an increase in volume of the amount $-P\,dV$, and the equation of continuity indicates that this change in volume, because of expansion, is div $\mathbf{u}$. Thus, if E is the internal energy *per unit mass* of gas, we have [using Eq. (6.4.1)]

$$\frac{dE}{dt} = \frac{1}{\rho}\frac{dU}{dt} = \frac{1}{\rho}\left[\text{div}\,(K \text{ grad } T) + D - P \text{ div } \mathbf{u}\right] \tag{6.4.11}$$

This is a statement of the first law of thermodynamics, which holds whether the process is reversible or not.

We know, of course, that the process is not exactly reversible, so that we cannot calculate change in entropy by its means. However, we can imagine suddenly isolating, at time t, a small portion of the gas near point (x,y,z) and letting it come to equilibrium. The entropy content of this portion can be computed by the second law; on a per-unit-volume basis it is given by the usual equation $dU = \rho T\,dS - P\,dV$ (with S the entropy content per unit mass). On a per-unit-mass basis (inserting $dV = \text{div } \mathbf{u}$ again), it is

$$\frac{dE}{dt} = T\frac{dS}{dt} - \frac{1}{\rho}P \text{ div } \mathbf{u}$$

We thus define an entropy content S of a unit mass of the gas, in spite of the fact that it is not quite in equilibrium; from it we can compute, with sufficient accuracy, heat production and other properties. For example, comparison with Eq. (6.4.11) shows that the change in entropy during the passage of the sound wave is

$$\begin{aligned}\rho\frac{dS}{dt} &= \frac{D}{T} + \frac{1}{T}\text{div}\,(K \text{ grad } T) \\ &= \frac{D}{T} + \frac{K}{T^2}|\text{grad } T|^2 + \text{div}\left(\frac{K}{T}\text{grad } T\right) \qquad \text{per unit volume}\end{aligned} \tag{6.4.12}$$

If we integrate the last form over a region enclosing the whole sound wave, so that grad T is zero along the region's surface, the last term in the second line integrates to zero and the total change in S over the whole disturbed region is the integral of the first two terms, both of which are positive. Therefore S continuously increases on the whole, as it must with an irreversible process (though S may decrease at some points).

An entropy increase corresponds to a change in amount of heat energy per unit volume.

$$\begin{aligned} dQ = \rho T \frac{dS}{dt} &= D + \operatorname{div}(K \operatorname{grad} T) \\ &= D + \frac{K}{T}|\operatorname{grad} T|^2 + T \operatorname{div}\left(\frac{K}{T} \operatorname{grad} T\right) \end{aligned} \tag{6.4.13}$$

which must be equal to the loss of energy per unit volume per second from the sound wave.

For a simple-harmonic plane wave, with acoustic pressure $p = \rho c u_0 e^{ik(x-ct)}$ and fluid velocity $u_x = u_0 e^{ik(x-ct)}$, where u_0 is the rms velocity amplitude and $k = \omega/c$, the viscous loss rate is, as we have shown, $D = (\eta + \frac{4}{3}\mu)\,|k u_0|^2$. If the thermal conductivity is small, we can, to the first approximation, use Eq. (6.1.3) to compute the temperature, as though the compression were purely adiabatic, and then use Eq. (6.4.13) to calculate the power lost by heat conduction. We have

$$\tau \simeq (\gamma - 1)\frac{\rho c \kappa_s}{\beta} u_x = (\gamma - 1)\frac{1}{\beta c} u_x = \frac{c}{C_p} u_x$$

$$\frac{K}{T}|\operatorname{grad} T|^2 \simeq \frac{K}{T}(\gamma - 1)^2 \left(\frac{k}{\beta c}\right)^2 |u_0|^2 = (\gamma - 1)\frac{K}{C_p}|k u_0|^2$$

where the last expressions in each line are appropriate for perfect gases. The third term of Eq. (6.4.13) is zero, on the average.

Thus both loss rates increase with the square of the frequency. For a plane wave propagating through the perfect gas, with intensity $I = \rho c\,|u_0|^2$, a fraction

$$2a \equiv \frac{dQ}{I} \simeq (\eta + \tfrac{4}{3}\mu)\frac{k^2}{\rho c}\left[1 + \frac{K}{C_p}\frac{\gamma - 1}{\eta + \frac{4}{3}\mu}\right] \tag{6.4.14}$$

of its energy is withdrawn each unit distance it traverses. Hence a is called the attenuation factor for acoustic propagation. The dimensionless ratio $(C_p\mu/K)$, called the Prandtl number, is about 0.7 for air at atmospheric pressure [note Eq. (6.4.5)]. When η is much smaller than μ, we see that the second term in brackets is approximately 0.4 for air at standard conditions. Thus viscosity and heat conduction contribute roughly equally to acoustic absorption in the gas, at all frequencies. The small quantity $(4\mu/3\rho c)$ is approximately equal to 10^{-5} cm for air at standard conditions; for sound of 1,000 cps, a is approximately $3 \times 10^{-7}\ \mathrm{cm}^{-1}$. Such sound must travel roughly 10 km in the open for its intensity to be reduced by a factor of 3 because of viscosity and thermal diffusion. As we shall see, viscosity and thermal diffusion produce much greater effects near a boundary. Also bulk absorption caused by relaxation effects (to be discussed shortly) in the presence of certain polyatomic impurities, such as water vapor, induces much

greater absorption. Consequently, we should work out the general equations of motion for sound, including viscosity and heat conduction.

The Stokes-Navier equation

The effects of viscosity in the equation of motion may be included by using the full expression for the stress tensor $\mathfrak{P}$ of Eq. (6.4.9). As was done in Sec. 6.2, we write the fluid density as $\rho + \delta$, the sum of the equilibrium density ρ and the fluctuation δ caused by the sound wave; we also write the pressure as $P + p$. If, in addition, we include a potential energy Φ caused by a possible force per unit mass of fluid, such as the force of gravity, the corrected form of the equation of motion (6.2.5) is

$$(\rho + \delta)\frac{d\mathbf{u}}{dt} \equiv (\rho + \delta)\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad})\,\mathbf{u}\right] = -\text{grad}\,(\rho\Phi) - \boldsymbol{\nabla} \cdot \mathfrak{P}$$
$$= -\text{grad}\,[p + \rho\Phi - (\eta + \tfrac{4}{3}\mu)\,\text{div}\,\mathbf{u}] - \mu\,\text{curl}\,(\text{curl}\,\mathbf{u}) \quad (6.4.15)$$

where we have used the relationships and formulas

$$\mathfrak{P} = [P - (\eta - \tfrac{2}{3}\mu)\,\text{div}\,\mathbf{u}]\mathscr{I} + \mu(\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\boldsymbol{\nabla})$$

$$\mathscr{I}_{ij} = \delta_{ij} \qquad (\boldsymbol{\nabla}\mathbf{u})_{ij} = \frac{\partial u_j}{\partial x_i} \qquad (\mathbf{u}\boldsymbol{\nabla})_{ij} = \frac{\partial u_i}{\partial x_j}$$

$$[\boldsymbol{\nabla} \cdot (\mathbf{u}\boldsymbol{\nabla})]_j = \sum_i \frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}\sum_i \frac{\partial u_i}{\partial x_i} = [\text{grad}\,(\text{div}\,\mathbf{u})]_j$$

$$[\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathbf{u})]_j = \nabla^2 u_j = [\text{grad}\,(\text{div}\,\mathbf{u})]_j - [\text{curl}\,(\text{curl}\,\mathbf{u})]_j$$

This is the Stokes-Navier equation of motion of a viscous, compressible fluid. As was done with Eq. (6.2.11), we can show that this is equivalent to the equation of continuity for the fluid momentum density,

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \boldsymbol{\nabla} \cdot (\mathfrak{P} + \mathfrak{J}) = -\text{grad}\,(\rho\Phi)$$

where the elements of tensor $\mathfrak{P}$ are given in Eq. (6.4.9), and of tensor $\mathfrak{J}$ in Eq. (6.2.10), their sum being the stress-momentum-flux tensor $\mathfrak{T}$ for the fluid, and $-\text{grad}\,(\rho\Phi)$ the applied force per unit volume of the fluid.

To include the effects of thermal diffusion, we must alter the relationship between p and δ to take into account the fact that the compression is not exactly adiabatic; that entropy now is produced locally because of the flow of heat. Thus we must recognize that density, entropy, and other properties of the gas are functions of two variables, pressure and temperature, rather than just pressure. We consider seven variables, each separated into its equilibrium value plus its small acoustic part:

Pressure:	$P + p$	Temperature:	$T + \tau$
Mass density:	$\rho + \delta$	Entropy:	$S + \sigma$ per unit mass

and the fluid velocity $\mathbf{u}$ (which is a vector and which therefore is really three variables). We are, in this chapter, assuming that the equilibrium velocity is zero, so that u is small compared with c. To determine all these, we need five equations relating them. For the present we shall write out only the first-order relationships, leaving for Chap. 14 the nonlinear effects.

The first four relationships are the equation of continuity for mass flow, which, to first order, is

$$\frac{\partial \delta}{\partial t} + \rho \operatorname{div} \mathbf{u} = 0 \tag{6.4.16}$$

and the first-order Stokes-Navier equation, really three equations,

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\operatorname{grad} p + (\eta + \tfrac{4}{3}\mu) \operatorname{grad} (\operatorname{div} \mathbf{u}) - \mu \operatorname{curl} (\operatorname{curl} \mathbf{u})$$

It can be shown that any vector function of position, such as $\mathbf{u}$, can always be uniquely separated into a longitudinal (or lamellar) part $\mathbf{u}_l$, for which curl $\mathbf{u}_l = 0$, and a transverse (or rotational) part $\mathbf{u}_t$, for which div $\mathbf{u}_t = 0$. Since the gradient of a scalar function is entirely longitudinal, i.e., since curl (grad f) $= 0$ for any f, the equation of motion can be split into two separate equations, one relating p to the longitudinal part of $\mathbf{u}$, the other giving the behavior of the transverse part of $\mathbf{u}$, unrelated to pressure waves,

$$\begin{aligned} \rho \frac{\partial \mathbf{u}_l}{\partial t} &= -\operatorname{grad} p + (\eta + \tfrac{4}{3}\mu)\nabla^2 \mathbf{u}_l \\ \rho \frac{\partial \mathbf{u}_t}{\partial t} &= -\mu \operatorname{curl} (\operatorname{curl} \mathbf{u}_t) \end{aligned} \tag{6.4.17}$$

where

$$(\nabla^2 \mathbf{F})_x = \nabla^2 F_x \qquad (\nabla^2 \mathbf{F})_y = \nabla^2 F_y \qquad \cdots$$

$$\begin{aligned} [\operatorname{grad} (\operatorname{div} \mathbf{F})]_x &= \frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) \\ &= \nabla^2 F_x + \frac{\partial}{\partial z}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x}\right) \\ &= \nabla^2 F_x + [\operatorname{curl} (\operatorname{curl} \mathbf{F})]_x \end{aligned}$$

[Eq. (6.4.6)]. Thus the two parts of the velocity solution, $\mathbf{u}_l$ and $\mathbf{u}_t$, can be solved for separately and need not be combined until we come to satisfying the boundary conditions. When we come to it we shall see that the transverse part attenuates rapidly, so that it is important only near the boundaries. The thermal-conductivity corrections and Eq. (6.4.16) have only to do with the longitudinal part of $\mathbf{u}$.

Thermodynamic relationships

The next equation, the first-order form of Eq. (6.4.12),

$$T\frac{\partial\sigma}{\partial t} \simeq K\nabla^2\tau \tag{6.4.18}$$

is the equation of continuity for heat flow. We have omitted the viscosity-loss term D because it is second order in u; the first-order effects of viscosity enter in Eqs. (6.4.17). The last two relationships between the seven variables are the equation of state, relating pressure, volume, and temperature in the gas, and the second law of thermodynamics, relating the entropy content of the gas, defined in Eq. (6.4.12), to the other variables. These equations are most conveniently written in terms of the measurable properties of the gas, such as its heat capacity C_p per unit mass at constant pressure, its isothermal compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

and coefficient of thermal expansion

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

as defined in Eqs. (6.1.1). There is also the rate of increase of pressure with temperature at constant volume, $(\partial P/\partial T)_V = \beta/\kappa_T$, which we shall call α.

In addition, there is the ratio of the specific heat at constant pressure to that at constant volume (C_P/C_V), which has been written γ. For any material, $C_V\kappa_T = C_P\kappa_S$, or $\kappa_T = \gamma\kappa_S$; so we have a means of relating properties for an adiabatic expansion (i.e., isentropic expansion) to those for an isothermal expansion. Since acoustic expansions are more nearly adiabatic than isothermal, we shall define the constant c appearing in the ensuing equations as the speed of sound in the limit of zero thermal conduction, when the compression is purely adiabatic. From Eq. (6.1.8) we have $c^2 = 1/\rho\kappa_s = 1/\gamma\rho\kappa_T$. An additional relationship between C_P and C_V, for any material, is

$$\gamma - 1 = \frac{T\beta^2}{\kappa_S C_P \rho}$$

All the quantities α, β, γ, C_P, and κ_S are functions of pressure and temperature but, if the pressure and temperature fluctuations in the sound wave are small enough for the first-order equations to be valid, we can consider these quantities to be constants, evaluated for the equilibrium values of P and T.

The pertinent quantities and relationships are then

$$\begin{aligned}
\kappa_s &= \frac{1}{\gamma\rho}\left(\frac{\partial\rho}{\partial P}\right)_T = \frac{1}{\rho c^2} \to \frac{1}{\gamma P} && \text{(for a perfect gas)} \\
\beta &= -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P \to -\frac{1}{T} && \text{(for a perfect gas)} \\
\frac{\beta V T}{C_P} &= (\gamma - 1)\frac{\kappa_s}{\beta} = (\gamma - 1)\frac{1}{\rho\beta c^2} \to (\gamma - 1)\frac{T}{\gamma P} \\
\alpha &= \left(\frac{\partial P}{\partial T}\right)_V = \frac{\beta}{\kappa_T} = \frac{\rho\beta c^2}{\gamma} \to \frac{P}{T} && \text{(for a perfect gas)}
\end{aligned} \tag{6.4.19}$$

We can also express conductivity, specific heat, and viscosity in terms of characteristic lengths l_h, l_v, and l_v' by the equations

$$\begin{aligned}
\frac{K}{\rho C_P c} &\equiv l_h \simeq 1.6\frac{l}{\sqrt{\gamma}} \qquad \frac{\mu}{\rho c} \equiv l_v \simeq \frac{l}{\sqrt{\gamma}} \\
l_v' &\equiv \frac{\eta + \frac{4}{3}\mu}{\rho c} = \left(\frac{4}{3} + \frac{\eta}{\mu}\right) l_v
\end{aligned} \tag{6.4.20}$$

The approximate formulas expressing these lengths in terms of the molecular mean-free-path l are obtained from Eqs. (6.4.2) and (6.4.5). They show that both l_h and l_v are of the order of the molecular mean-free-path, which is, roughly, 10^{-5} cm at atmospheric pressure and 20°C.

With these definitions out of the way, we return to the remaining two equations relating σ, p, τ, δ, and $\mathbf{u}$. The first is the equation of state, giving the change of density δ in terms of the acoustic pressure p and temperature τ, assumed small; the second is the equation for the change of entropy content with change of pressure and temperature:

$$\begin{aligned}
\delta &= \left(\frac{\partial\rho}{\partial P}\right)_T p + \left(\frac{\partial\rho}{\partial T}\right)_P \tau = \gamma\rho\kappa_s(p - \alpha\tau) \to \frac{\rho}{P}p - \frac{\rho}{T}\tau \\
\sigma &= \left(\frac{\partial S}{\partial T}\right)_P \tau + \left(\frac{\partial S}{\partial P}\right)_T p = \frac{C_p}{T}\left(\tau - \frac{\gamma - 1}{\alpha\gamma}p\right) \to C_p\left(\frac{\tau}{T} - \frac{\gamma - 1}{\gamma}\frac{p}{P}\right)
\end{aligned} \tag{6.4.21}$$

where the expressions introduced by arrows are those valid for a perfect gas. If p is equal to $\alpha\tau$, there will be no change in density of the gas; but if $p = [\alpha\gamma/(\gamma - 1)]\tau$, there will be no change in entropy; i.e., the compression will be adiabatic.

The modified wave equation

The five equations consisting of (6.4.16), the first of (6.4.17), and Eqs. (6.4.18) and (6.4.21) could serve to eliminate all but one of the five variables and determine a single equation governing the dependence of this variable

on space and time. This, plus the solution of the second of Eqs. (6.4.17), could then be adjusted to fit the appropriate boundary conditions. It will be more convenient to relax this condition and obtain two equations, involving two variables, which then can be solved simultaneously. The natural ones to choose are p and τ; they are more easily measured than the others, and we have already expressed δ and σ in their terms.

We first eliminate $\mathbf{u}_l$ between Eqs. (6.4.16) and (6.4.17) and then substitute for δ in terms of Eqs. (6.4.21). The final results, in terms of the constants defined in Eqs. (6.4.19) and (6.4.20), are

$$\begin{aligned} \nabla^2 p &= \frac{\gamma}{c^2}\left(\frac{\partial^2}{\partial t^2} - l_v' c\,\frac{\partial}{\partial t}\nabla^2\right)(p - \alpha\tau) \\ l_h c \nabla^2 \tau &= \frac{\partial}{\partial t}\left(\tau - \frac{\gamma - 1}{\gamma\alpha}\,p\right) \\ \rho\,\frac{\partial \mathbf{u}_l}{\partial t} &= -\text{grad}\left[p + \frac{\gamma l_v'}{c}\,\frac{\partial}{\partial t}(p - \alpha\tau)\right] \end{aligned} \tag{6.4.22}$$

After the first two equations are solved to determine p and τ, the last equation and Eqs. (6.4.21) are used to compute $\mathbf{u}_l$, δ, and σ if they are needed.

The first of Eqs. (6.4.22) is the wave equation, modified for the effects of viscosity and thermal conduction. If there were no conduction, l_h would be zero, τ would have to be equal to $(\gamma - 1)p/\gamma\alpha$, and $\gamma(p - \alpha\tau)$ would equal p. If there were no viscosity, l_v' would be zero, and the first equation would become the simple wave equation for wave velocity c. If the conductivity were infinite, τ would have to be zero, and the wave velocity in the first equation would be $c/\sqrt{\gamma} = \sqrt{1/\rho\kappa_T}$, the isothermal velocity. For intermediate cases the pressure and temperature are coupled together, the fluid tending to propagate pressure waves and the heat tending to diffuse; the coupling lies in the l_h and l_v' terms. There are thus two kinds of "waves," depending on which flow predominates. One of them corresponds to nearly adiabatic wave motion; this velocity has a real part equal to c (the adiabatic velocity) and a small imaginary part corresponding to the energy-loss term in l_h and l_v'. The other "wave" corresponds chiefly to heat diffusion; p is small compared with $\alpha\tau$, and the "wave velocity" is proportional to $\sqrt{i}$, indicating rapid attenuation.

To put numbers to these statements, we consider simple-harmonic separable motion, where all five acoustical quantities are proportional to p, which satisfies the equations

$$\nabla^2 p = -k^2 p \qquad \frac{\partial p}{\partial t} = -i\omega p$$

where the relationship between k and ω is obtained by substitution in Eqs. (6.4.22). Solutions of this sort can be combined by use of the Fourier

transform, to obtain transient and other solutions, as will be demonstrated later. When l_h and l'_v are small, the relation between k and ω can be obtained by using the statements of the preceding paragraph.

For example, the nearly adiabatic wave will have k^2 nearly equal to $(\omega/c)^2$. Inserting this into the second of Eqs. (6.4.22) shows that, to first order in $\omega l_h/c$,

$$-\frac{\omega^2 l_h}{c}\tau = -i\omega\tau + i\omega\frac{\gamma-1}{\gamma\alpha}p \qquad \text{or} \qquad \tau \simeq \frac{\gamma-1}{\gamma\alpha}\left(1 - i\frac{\omega}{c}l_h\right)p$$

Inserting this into the first of Eqs. (6.4.22) yields a first-order correction to k^2 and corresponding relationships between p and the other variables.

$$\begin{gathered}
k^2 \simeq \left(\frac{\omega}{c}\right)^2\left[1 + \frac{\omega l'_v}{c} + i(\gamma-1)\frac{\omega l_h}{c}\right] \\
\nabla^2 p = -k^2 p \qquad \frac{\partial p}{\partial t} = -i\omega p \qquad c^2 = \frac{1}{\rho\kappa_s} \\
\tau \simeq \frac{\gamma-1}{\gamma\alpha}\left(1 - i\frac{\omega}{c}l_h\right)p \qquad \delta \simeq \frac{1}{c^2}\left[1 + i(\gamma-1)\frac{\omega}{c}l_h\right]p \\
\sigma \simeq -i\frac{C_p}{T}\frac{\gamma-1}{\gamma\alpha}\frac{\omega}{c}l_h p \qquad \mathbf{u}_l \simeq \left(\frac{1}{i\omega\rho} - \frac{l'_v}{\rho c}\right)\operatorname{grad} p
\end{gathered} \tag{6.4.23}$$

where $\alpha = \beta\rho c^2/\gamma$, as indicated in Eqs. (6.4.19). This kind of wave will be called the *propagational* mode. If the motion is driven, so that ω is real, then k is complex, corresponding to a wave which attenuates in space; since l_h and l'_v are so small, the attenuation is slight. If the wave is a standing wave in space, so k is real, then ω is complex, corresponding to the damping out of the standing wave in time. In either case, τ, δ, and $\mathbf{u}$ are slightly out of phase with p; the entropy fluctuation σ is small and 90° out of phase with p. As long as ω is less than either c/l_h or c/l'_v (i.e., less than about 10^9 for air at standard conditions), the speed of propagation is closely equal to $c = \sqrt{1/\rho\kappa_s}$, corresponding to adiabatic compressibility, as assumed for Eq. (6.1.3). For $\omega l_h/c$ small but not negligible, the effective compressibility is

$$\kappa_p = \frac{\delta}{\rho p} \simeq \kappa_s\left[1 + i(\gamma-1)\frac{\omega l_h}{c}\right] \tag{6.4.24}$$

For the second mode of wave motion predicted by Eqs. (6.4.22), p is small compared to $\alpha\tau$, so that the second of Eqs. (6.4.22) is the primary one, the "zeroth-order" equation being $-l_h c k^2\tau \simeq -i\omega\tau$, or $k^2 \simeq i\omega/l_h c$. The first equation is then used to determine the magnitude of p, after which the

first-order expressions for the other quantities can be obtained for this mode:

$$k^2 \simeq \frac{i\omega}{l_h c} \qquad \nabla^2\tau = -k^2\tau \qquad \frac{\partial\tau}{\partial t} = -i\omega\tau$$

$$p \simeq \frac{i\gamma\alpha\omega}{c}(l_h - l_v')\tau \qquad \delta \simeq \frac{-\alpha\gamma}{c^2}\left[1 - \frac{i\gamma\omega}{c}(l_h - l_v')\right]\tau \tag{6.4.25}$$

$$\sigma \simeq \frac{C_p}{T}\left[1 - i(\gamma - 1)\frac{\omega}{c}(l_h - l_v')\right]\tau \qquad \mathbf{u}_l \simeq \frac{\alpha\gamma}{\rho c} l_h \operatorname{grad}\tau$$

which will be called the *thermal* mode. Constant k is large and complex, corresponding to the rapid attenuation typical of diffusion.

For liquids, or for very high frequencies in gases, we may have to solve Eqs. (6.4.22) exactly, in which case we insert $-k^2$ for ∇^2 and $-i\omega$ for $\partial/\partial t$ and solve the resulting quadratic equation for k^2. The resulting formulas are

$$k^2 = \frac{i\omega^2}{2\Omega c^2}\frac{1 - i\Upsilon - i\gamma\Omega \mp Q}{1 - i\gamma\Upsilon} \qquad \Upsilon = \frac{\omega}{c} l_v' = (\eta + \tfrac{4}{3}\mu)\frac{\omega}{\rho c^2}$$

$$\Omega = \frac{\omega}{c} l_h = \frac{K\omega}{\rho c^2 C_p} \qquad Q^2 = (1 - i\Upsilon + i\gamma\Omega)^2 - 4i(\gamma - 1)\Omega$$

$$p = \frac{\gamma\alpha}{\gamma - 1}\frac{1 - i(2\gamma - 1)\Upsilon + i\gamma\Omega \pm Q}{2(1 - i\gamma\Upsilon)}\tau \tag{6.4.26}$$

$$\tau - \frac{\gamma - 1}{\gamma\alpha} p = \frac{1 - i\Upsilon - i\gamma\Omega \mp Q}{2(1 - i\gamma\Upsilon)}\tau$$

$$p - \alpha\tau = \frac{\alpha}{\gamma - 1}\frac{2 - \gamma - i\gamma\Upsilon + i\gamma^2\Omega \pm Q}{2(1 - i\gamma\Upsilon)}\tau$$

The propagational mode corresponds to the upper signs in the formulas; for Ω and Υ small, the formulas reduce to those of Eqs. (6.4.23). The thermal mode corresponds to the lower signs; this case reduces to Eqs. (6.4.24) when Ω and Υ are small.

Plane-wave solutions

The first example of the use of these formulas will be for plane waves in free space, though the effects of viscosity and heat diffusion are usually quite small for this case. In gases, for frequencies less than the molecular-collision frequency, we can use the first-order solutions of Eqs. (6.4.23) and (6.4.25). We set $p = f(x)e^{-i\omega t}$, where $f(x)$ is a solution of $(d^2f/dx^2) + k^2 f = 0$. For a traveling wave, $f = Ae^{ikx}$, so that

$$p \simeq Ae^{i(\omega/c)(x-ct)} \exp\left\{-\frac{1}{2}\left(\frac{\omega}{c}\right)^2 [l_v' + (\gamma - 1)l_h]x\right\} \tag{6.4.27}$$

and

$$u_{lx} \simeq \left[1 - \tfrac{1}{2}i\frac{\omega}{c}l'_v + \tfrac{1}{2}i(\gamma - 1)\frac{\omega}{c}l_h\right]\frac{p}{\rho c} \qquad \text{(for the propagational mode)}$$

The wave attenuates as it proceeds, the attenuation factor being

$$a \simeq \frac{1}{2}\left(\frac{\omega}{c}\right)^2[l'_v + (\gamma - 1)l_h]$$

which is identical with Eq. (6.4.14). As noted before, for gases at normal pressures and temperatures and for frequencies less than 10^4 cps, this attenuation is very small [see, however, Eq. (6.4.50)].

In contrast, the thermal mode has a rapidly varying exponential factor,

$$\tau \simeq B\exp\left[\sqrt{\frac{\omega}{2l_h c}}(i-1)x - i\omega t\right] \qquad p \simeq \frac{i\gamma\alpha\omega}{c}(l_h - l'_v)\tau \quad (6.4.28)$$

which indicates that this mode is important only near boundaries; its effects are negligible except within a layer, next to the boundary, of thickness roughly equal to the geometric mean $\sqrt{l_h\lambda}$ of the characteristic length $l_h = K/\rho c C_p$ and the wavelength $\lambda = 2\pi c/\omega$ of the sound.

The fluid motion corresponding to the rotational or transverse solution $\mathbf{u}_t$ of Eq. (6.4.17) also is important only near a boundary. Transverse flow of this sort involves no change of density or pressure (to the first approximation, at least). A possible solution would be a shear wave,

$$u_{ty} = Ae^{ikx - i\omega t} \qquad \text{div } \mathbf{u} = \frac{\partial u_{ty}}{\partial y} = 0$$

Insertion in the second of Eqs. (6.4.17) produces

$$-\rho\omega u_{ty} = -\mu \text{ curl curl } \mathbf{u} = -\mu k^2 u_{ty} \qquad \text{or} \qquad k^2 = \frac{i\omega}{l_v c} \quad (6.4.29)$$

by use of Eqs. (6.4.6) and (6.4.20). (Note that the characteristic length l_v does not involve the bulk modulus η.) Thus one possible solution is the shear wave

$$u_{ty} = A\exp\left[\sqrt{\frac{\omega}{2l_v c}}(i-1)x - i\omega t\right] \qquad (6.4.30)$$

which dies out as rapidly as the thermal mode, since l_v is the same order of magnitude as l_h in gases. As noted earlier, $\mathbf{u}_t$ obeys a diffusion, rather than a wave equation; no wave propagation is possible.

Boundary conditions

The thermal and shear-wave modes are important near the boundaries, where they are generated because the propagational wave cannot, by itself,

fit all the boundary conditions. Now that we have improved our mathematical model of wave motion to include heat diffusion and shear viscosity, we can no longer neglect to specify the values of temperature and of tangential velocity at the boundary surface. And if the propagational mode of Eqs. (6.4.23) is adjusted to fit the ratio between pressure and normal velocity specified at the surface, it cannot at the same time satisfy the boundary conditions on the acoustic temperature and tangential velocity.

The thermal and shear modes, generated by this discrepancy, are negligible outside a thin *boundary layer* just outside the boundary surface. As noted under the preceding heading, the thicknesses of these two layers are roughly equal, at least for air at standard conditions. Equations (6.4.28) and (6.4.30) show that the thickness d_v of the viscous boundary layer and d_h of the thermal layer are

$$\begin{aligned} d_v &= \sqrt{\frac{2l_v c}{\omega}} = \sqrt{\frac{2\mu}{\rho\omega}} \simeq 0.21 \frac{1}{\sqrt{\nu}} \quad \text{cm} \\ d_h &= \sqrt{\frac{2l_h c}{\omega}} = \sqrt{\frac{2K}{\rho\omega C_p}} \simeq 0.25 \frac{1}{\sqrt{\nu}} \quad \text{cm} \end{aligned} \tag{6.4.31}$$

where ν is the frequency of the sound wave, and the last expressions on each line are for air at atmospheric pressure and room temperature.

Thus, except for fluids with high viscosity or conductivity or for capillary tubes of small diameter, the propagational mode of Eqs. (6.4.23) will adequately describe the acoustic behavior of the medium everywhere outside a boundary layer, usually a fraction of a millimeter thick, next to the boundaries. Indeed, in many cases of practical interest [for exceptions see Eq. (6.4.50)], the attenuation of the propagational mode can be neglected and the simple wave equation (6.2.8), with the relations of Eqs. (6.1.3) and (6.2.7), for τ, δ, and $\mathbf{u}$, can be used for the region outside the boundary layers. Since the majority of the power lost to viscosity and thermal conduction is lost within the boundary layers, and since these layers are so thin, we can consider the power to be lost *at* the boundary and can incorporate the effects of the other two modes into the boundary conditions on the propagational mode. Once this is done, we can usually forget the presence of the other two modes.

To find the effect of the thermal and shear modes on the boundary conditions for the propagational mode, we consider the situation shown in Fig. 6.4, with the boundary in the xz plane, confining the wave to the region $y \geqslant 0$. The propagational mode, the solution of Eqs. (6.4.23), is assumed to be a separable simple-harmonic wave, with the tangential motion of the fluid in the x direction. In other words, the wave for $y \geqslant 0$ is assumed to

have the general form defined by the following equations:

$$p_p = \psi(x)f(y)e^{-i\omega t} \qquad \frac{d^2\psi}{dx^2} = -k_t{}^2\psi$$

$$\frac{d^2f}{dy^2} = (k_t{}^2 - k^2)f \qquad \tau_p \simeq \frac{\gamma - 1}{\alpha\gamma}\psi(x)f(y)e^{-i\omega t} \tag{6.4.32}$$

$$u_{py} \simeq \frac{1}{i\omega\rho}\frac{df}{dy}\psi e^{-i\omega t} \qquad u_{px} \simeq \frac{1}{i\omega\rho}\frac{d\psi}{dx}fe^{-i\omega t}$$

where $k^2 = (\omega/c)^2$, or is given by Eqs. (6.4.23) if we need to include the viscous and thermal losses in the region outside the boundary layers. We also have assumed that we need not include the l_h and l_v' corrections in the formulas for τ and for $\mathbf{u}$, for this mode.

The boundary surface is usually fairly rigid; so we should expect the normal and tangential components of $\mathbf{u}$ to be quite small at $y = 0$. Also, the thermal conductivity of the boundary material is usually much greater than that of the medium; so we should expect the temperature fluctuation τ to be much smaller at $y = 0$ than it is outside the boundary layers. If the boundary surface is one of local reaction [see discussion preceding Eq. (6.3.4)] with an acoustic impedance $z = p/-u_y$, a reasonable approximation to the true boundary conditions at $y = 0$ would be

$$u_y = -\frac{p}{z} \qquad u_x = 0 \qquad \tau = 0 \qquad \text{at } y = 0 \tag{6.4.33}$$

But the wave of Eqs. (6.4.32) does not meet these conditions. We could adjust the value of f or of df/dy (but not both, since f must satisfy its own equation) to satisfy one of these, but not all three. Thus we must add enough of the other two modes to the wave so that the combination satisfies all three conditions. In this way the two solutions, which are negligible outside the boundary layer, affect the boundary conditions on the propagational mode.

The pure plane-wave solutions of Eqs. (6.4.28) and (6.4.30) cannot be used because at $y = 0$ there must be a dependence on x in order to match the propagational solution at all points of the surface. Thus the thermal mode must have the form

$$\tau_h = \psi(x)F_h(y)e^{-i\omega t} \qquad F_h = B\exp(ik_hx)$$

with function ψ the same as in Eqs. (6.4.32). Since, according to Eqs. (6.4.25),

$$\nabla^2\tau_h \equiv \left(\frac{d^2\psi}{dx^2}F + \psi\frac{d^2F}{dy^2}\right)e^{-i\omega t} = -\frac{2i}{d_h{}^2}\tau_h$$

or since

$$\frac{d^2\psi}{dx^2} = -k_t{}^2\psi$$

the equation for F_h must be

$$\frac{d^2F_h}{dy^2} = \left(-\frac{2i}{{d_h}^2} + {k_t}^2\right)F_h \qquad \frac{2}{{d_h}^2} = \frac{\omega}{l_h c}$$

However, the tangential part of the wave number k_t cannot be larger than $k = \omega/c = 2\pi/\lambda$ for the propagational wave to propagate. Usual acoustic wavelengths λ are of the order of tens of centimeters, whereas d_h, as we have seen, is usually of the order of tenths of millimeters or smaller. Therefore the quantity in parentheses multiplying F_h in the equation for F_h is nearly equal to $-(2i/{d_h}^2)$, and the solution which is largest at $y = 0$ and vanishes as $y \to \infty$ is $F_h = B \exp[(i-1)(y/d_h)]$, to a good approximation.

We then adjust the value of B so that this wave will just cancel out the temperature fluctuations τ_p of the propagational wave at $y = 0$, which means that $B \simeq -[(\gamma - 1)f(0)/\alpha\gamma]$, and

$$\tau_h \simeq -\frac{\gamma - 1}{\alpha\gamma} f(0)\psi(x)e^{(i-1)(y/d_h)-i\omega t}$$

$$p_h \simeq -i\frac{\omega}{c}(\gamma - 1)(l_h - l_v')\psi(x)f(0)\exp\left(iy\sqrt{\frac{i\omega}{l_h c}} - i\omega t\right)$$

$$u_{hy} \simeq -i\frac{\gamma - 1}{\rho c} l_h \sqrt{\frac{i\omega}{l_h c}}\,\tau = (1 - i)(\gamma - 1)\frac{\omega d_h}{2\rho c^2}\psi(x)f(0)e^{(i-1)(y/d_h)-i\omega t}$$

When d_h is small compared with λ, $l_h = -(\omega/c){d_h}^2$ is usually several orders of magnitude smaller than d_h; so, to first order in d_h, we can neglect the pressure and the tangential velocity components of this thermal mode. Not so with the normal component u_{hy}; since the normal gradient of F_h is so large, $u_{hy} \gg p_h/\rho c$. Therefore this mode can be adjusted to cancel out the temperature fluctuations at $y = 0$ at the expense of a nonnegligible contribution to the normal velocity (which will be taken care of later). But the tangential component u_{px} of the propagational wave is not yet canceled out.

To do this we must introduce a certain amount of shear wave, a solution of the second of Eqs. (6.4.17). We wish the tangential velocity component u_{tx} of this solution to equal $-u_{px}$ at $y = 0$; also, the divergence of $\mathbf{u}_t$ must be zero. The appropriate solution is

$$u_{tx} = -\frac{1}{i\omega\rho}\frac{\partial\psi}{\partial x} f(0)\, e^{ik_v y - i\omega t}$$

$$u_{ty} = \frac{{k_t}^2}{k_v\rho\omega}\psi(x)f(0)e^{ik_v y - i\omega t}$$

so that

$$\text{div } \mathbf{u}_t = \frac{\partial u_{tx}}{\partial x} + \frac{\partial u_{ty}}{\partial y} = 0 \qquad \text{and} \qquad \text{curl curl } \mathbf{u}_t = ({k_v}^2 + {k_t}^2)\mathbf{u}_t$$

or

$$k_v^2 = \frac{2i}{d_v^2} - k_t^2 \simeq \frac{2i}{d_v^2} = \frac{i\omega}{l_v c} = \frac{i\rho\omega}{\mu} \qquad \text{when } \lambda \gg l_v$$

since curl curl $\mathbf{u}_t$ must equal $i\rho\omega/\mu$ times $\mathbf{u}_t$. Therefore the shear mode, to be added to the others to fit the boundary conditions, is

$$u_{tx} \simeq -\frac{1}{i\omega\rho}\frac{\partial\psi}{\partial x} f(0) e^{(i-1)(y/d_v)-i\omega t}$$

$$u_{ty} \simeq (1-i)\frac{k_t^2 c^2}{\omega^2}\frac{\omega}{2\rho c^2} d_v \psi(x) f(0) e^{(i-1)(y/d_v)-i\omega t}$$

to the first order in l_h; no pressure or temperature changes accompany this mode.

Adding all three together, the combination of thermal, shear, and propagational waves which satisfy boundary conditions (6.4.33) is, to first order in the small quantities d_h and d_v,

$$p \simeq \psi(x) f(y) e^{-i\omega t} \qquad \frac{d^2\psi}{dx^2} = -k_t^2\psi \qquad \frac{d^2 f}{dy^2} = \left[k_t^2 - \left(\frac{\omega}{c}\right)^2\right] f$$

$$\tau \simeq \frac{\gamma-1}{\alpha\gamma}\psi(x)[f(y) - f(0)e^{(i-1)(y/d_h)}]e^{-i\omega t}$$

$$\delta \simeq \frac{1}{c^2}\psi(x)[f(y) + (\gamma-1)f(0)e^{(i-1)(y/d_h)}]e^{-i\omega t} \tag{6.4.34}$$

$$u_x \simeq \frac{1}{i\rho\omega}\frac{\partial\psi}{\partial x}[f(y) - f(0)e^{(i-1)(y/d_v)}]e^{-i\omega t}$$

$$u_y \simeq \frac{1}{i\rho\omega}\psi(x)\left\{\frac{\partial f}{\partial y} + i(1-i)\frac{\omega^2}{2c^2}\left[\frac{k_t^2 c^2}{\omega^2} d_v e^{(i-1)(y/d_v)} + (\gamma-1) d_h e^{(i-1)(y/d_h)}\right] f(0)\right\} e^{-i\omega t}$$

Inside the boundary layer the thermal and shear waves are important, rising rapidly from nearly zero at the outer edge to a sufficient size, at $y = 0$, to cancel the temperature and tangential-velocity oscillations of the primary propagational wave. Indeed, unless the region enclosing the sound waves is so large that the minute attenuation factor of Eq. (6.4.27) becomes noticeable, we can neglect thermal and viscous effects in the main portion of the region [except perhaps for relaxation effects; see Eq. (6.4.50)]. The only place these effects show up is inside the boundary layer, and the one effect they have on the propagational wave is on its boundary conditions.

Inside the boundary layer we find an effective compressibility, given by

$$\kappa_h = \frac{\delta}{\rho p} \simeq \frac{1 + (\gamma-1)e^{(i-1)(y/d_h)}}{\rho c^2} \tag{6.4.35}$$

if we assume that d_h is so small compared with λ that $f(d_h) \simeq f(0)$. This compressibility varies from $\gamma/\rho c^2 = \kappa_T$ at $y = 0$ to $1/\rho c^2 = \kappa_s$ for $y \gg d_h$, as expected; the compression is isothermal at $y = 0$ and is nearly adiabatic outside the layer.

If the boundary surface is one of local reaction, with acoustic impedance $p/-u_y = z$, Eqs. (6.4.34) and (6.4.33) show that the ratio between p and its normal derivative at the surface must be

$$\begin{aligned} p_p &\equiv \psi(x) f(0) \\ &\simeq \frac{-z}{i\rho\omega}\,\psi(x)\left\{\left(\frac{df}{dy}\right)_0 + \frac{i\omega^2}{2c^2}(1-i)\left[\left(\frac{k_t c}{\omega}\right)^2 d_v + (\gamma-1)d_h\right] f(0)\right\} \end{aligned}$$

or

$$\begin{aligned} \left(\frac{\partial p_p}{\partial n}\right)_0 &\equiv \psi(x)\left(\frac{df}{dy}\right)_0 \\ &\simeq -i\omega\rho\left\{\frac{1}{z} + \frac{1-i}{\rho c}\left[(\gamma-1)\sqrt{\frac{\omega l_h}{2c}} + \left(\frac{k_t c}{\omega}\right)^2 \sqrt{\frac{\omega l_v}{2c}}\right]\right\} p_p \qquad y = 0 \end{aligned} \tag{6.4.36}$$

with the dimensionless quantities

$$\sqrt{\frac{\omega l_h}{2c}} = \frac{\omega}{2c} d_h = \sqrt{\frac{K\omega}{2\rho c C_p}} = \sqrt{\frac{\nu}{\nu_h}}$$

$$\sqrt{\frac{\omega l_v}{2c}} = \frac{\omega}{2c} d_v = \sqrt{\frac{\mu\omega}{2\rho c}} = \sqrt{\frac{\nu}{\nu_v}}$$

where, for air at standard conditions, $\nu_h \simeq 1.9 \times 10^9$ cps and $\nu_v \simeq 2.7 \times 10^9$ cps. These are the sole vestiges of the effects of viscosity and thermal conduction on the acoustic wave outside the boundary layer, at least to the first order in d_h and d_v. The sole boundary condition to be satisfied is this relation between the value and the normal derivative of the propagational wave pressure at the boundary.

Equation (6.4.36) establishes several conclusions of interest, perhaps the only conclusions we need to carry forward to later chapters. In the first place, if the boundary surface is moderately "soft," if the magnitude of its specific acoustic admittance $|\rho c/z|$ [Eqs. (6.3.4)] is large compared with the quantity in square brackets in (6.4.36), we need not consider the effects of thermal conduction and viscosity at all; even the surface effects are negligible compared with those of the wall's admittance. Second, we see that the boundary effect of thermal diffusion is, to the first order in d_h, completely equivalent to that of an additional specific acoustic conductance ξ_h and susceptance σ_h of the boundary surface, where $\xi_h = \sigma_h = (\gamma - 1)\sqrt{\omega l_h/2c}$. The addition of this to the admittance of the boundary surface itself will account for the effect on the propagational mode of thermal diffusion in the boundary layer.

The equivalence for the viscous losses is not so simple, because of the factor $k_t^2c^2/\omega^2$. Thermal losses are caused by compression, and thus are related to the normal velocity of the medium, the same pair of variables which enter into the specification of acoustic impedance. Viscous losses, on the other hand, are related to the tangential velocity of the medium just outside the boundary layer; the relationship between tangential velocity and pressure is proportional to $k_t c/\omega$. This point will be discussed in more detail under the next heading.

Power lost at a surface

The power absorbed in the thermal boundary layer can most easily be calculated by using the equivalent conductance $\xi_h/\rho c$ mentioned under the previous heading. As with any surface of local reaction, the power lost per unit area of surface is the conductance times the square of the rms value of the pressure at the surface. The thermal loss per unit area at the boundary is thus

$$L_{bh} = \frac{\xi_h}{\rho c}\,|p(y=0)|^2 \simeq \left(\frac{\gamma-1}{2\rho c^2}\right)\omega d_h\,|p_p(y=0)|^2 \qquad (6.4.37)$$

This same expression can also be obtained from the formula for compressibility of Eq. (6.4.35). The acoustic energy of compression per unit volume is $-(P\,dV/V) = (P+p)(\delta/\rho)$; the rate of loss of energy per unit volume is then the real part of $-\dfrac{P+p}{\rho}\dfrac{\partial\delta^*}{\partial t}$, which equals $-\mathrm{Re}\,[(P+p)\kappa(\partial p^*/\partial t)]$. The term $-P\kappa(\partial p^*/\partial t)$ averages out to zero over a cycle, but $-p\kappa(\partial p^*/\partial t)$ has a real part if κ has an imaginary part. From Eq. (6.4.35) we see that κ does have an imaginary part in the boundary layer; thus the power loss in the thermal layer, per unit area of surface, is

$$-\mathrm{Re}\int_0^\infty p\kappa\frac{\partial p^*}{\partial t}\,dy \simeq -\mathrm{Re}\int_0^\infty \frac{i\omega}{\rho c^2}(\gamma-1)e^{(i-1)(y/d_h)}\,|p_p|^2\,dy$$

$$= -\mathrm{Re}\left[\frac{(\gamma-1)\omega}{2\rho c^2}\,d_h(i-1)\right]|p_p|^2 = L_{bh} \quad \text{of Eq. (6.4.37)}$$

The integrand, the power lost per volume in the boundary layer, is zero at $y=0$, rises to a maximum at $y=\pi d_h/4$, and drops exponentially to zero as $y\to\infty$. In the integration we have assumed that d_h is so much smaller than λ that p_p is practically independent of y within the boundary layer; so $|p|^2$ can be taken outside the integral.

The power lost by viscosity, per unit volume, is of course the function D defined in Eq. (6.4.10). In the present case, to first order, the velocity tangential to the surface, producing the shear, is

$$\frac{i}{\rho\omega}\frac{\partial\psi}{\partial x}\,f(0)e^{(i-1)(y/d_h)-i\omega t}$$

and the power lost in a unit area of boundary layer is

$$\int_0^\infty D\,dy \simeq \mu\int_0^\infty \left|\frac{\partial u_x}{\partial y}\right|^2 dy = \tfrac{1}{2}\rho\omega d_v{}^2 \frac{2}{\rho^2\omega^2 d_v{}^2}\left|\frac{\partial\psi}{\partial x}\right|^2 |f(0)|^2 \int_0^\infty e^{-2y/d_v}\,dy$$

where we have used Eq. (6.4.31) to express μ in terms of d_v. In this case the loss is greatest next to the surface, and drops off exponentially as y increases. Averaging the loss over x, we use the relation

$$\int_0^X \left|\frac{d\psi}{dx}\right|^2 dx = \int_0^X k_t{}^2 |\psi|^2\,dx + \int_0^X \frac{d}{dx}\left(\psi\frac{d\psi^*}{dx}\right) dx \to X k_t{}^2 |\psi_{\text{avg}}|^2 \qquad X\to\infty$$

so that the mean power loss per unit area, averaged over the surface area, from viscous forces, is

$$L_{bv} \simeq \frac{k_t{}^2}{\rho\omega}|\psi f|^2 \int_0^\infty e^{-2y/d_v}\,dy = \left(\frac{k_t c}{\omega}\right)^2\left(\frac{\omega}{2\rho c^2}\right) d_v\,|p(y=0)|^2 \qquad (6.4.38)$$

which also can be obtained by considering the viscous "surface conductance" to be the term

$$\frac{1}{2}\frac{\omega}{\rho c^2}\left(\frac{k_t c}{\omega}\right)^2 d_v = \frac{1}{\rho c}\left(\frac{k_t c}{\omega}\right)^2\sqrt{\frac{\omega l_v}{2c}}$$

as implied by Eq. (6.4.38). Multiplying this by $|p(y=0)|^2$ results in the energy loss.

If $u_{tx} = -(1/i\omega\rho)(d\psi/dx)f(0)e^{-i\omega t}$ is the component of the shear wave parallel to the surface at $y=0$, $-u_{tx}$ is the corresponding component of the propagational wave there, since tangential $\mathbf{u}$ must be zero at the surface. When d_v is much smaller than a wavelength, the tangential component of the propagational wave, just outside the boundary layer, is thus $-u_{tx}$. Consequently, an alternative formula to Eq. (6.4.38) for the viscous surface power loss per unit area is

$$L_{bv} \simeq \tfrac{1}{2}\rho\omega d_v\,|\mathbf{u}_{\tan}|^2 \qquad (6.4.39)$$

where $\mathbf{u}_{\tan}(= -u_{tx}$ in the example) is the rms amplitude of the tangential component of the propagational wave just outside the boundary layer. This formula also allows us to calculate the viscous surface losses without having to deal with the details of the shear wave. It is generally applicable for wave frequencies such that $d_v \ll \lambda$.

The power lost per unit volume in the region outside the boundary layer is another order of magnitude smaller. As already indicated in Eq. (6.4.14), for a plane wave of intensity $I = |p|^2/\rho c = \rho c\,|\mathbf{u}|^2$, this loss rate is

$$L_m = \left(\frac{\omega}{c}\right)^2[l_v' + (\gamma-1)l_h]I = \frac{1}{2}\left(\frac{\omega}{c}\right)^3[(d_v')^2 + (\gamma-1)(d_h)^2]I$$

where d_v and d_h are given in Eqs. (6.4.31) and $(d_v')^2 = [\frac{4}{3} + (\eta/\mu)](d_v)^2$.

Boundary losses in plane-wave reflection

As an example of the utility of Eq. (6.4.36), which substitutes a simple ratio between propagational pressure and normal gradient of propagational pressure at the surface for the details of viscous and thermal forces inside the boundary layer, we shall work out the reflection of a plane wave from a plane surface. The procedure is similar to that for Eq. (6.3.5); only we use Eq. (6.4.36) for the boundary conditions, instead of the formula $(\partial p/\partial n)_0 = -(i\omega\rho/z)p(\text{surface})$, which was used for the surface of local reaction when viscous and thermal losses are neglected.

For a plane wave of frequency $\omega/2\pi$ incident on the yx plane at angle of incidence ϑ, as shown in Fig. 6.4, the pressure outside the boundary layer is

$$p_p(x,y,t) = A\exp\left(\frac{i\omega x}{c}\,x\sin\vartheta - \frac{i\omega y}{c}\cos\vartheta - i\omega t\right) + AC_r\exp\left(\frac{i\omega x}{c}\sin\vartheta + \frac{i\omega y}{c}\cos\vartheta - i\omega t\right) \quad (6.4.40)$$

where C_r is, as before, the reflection coefficient for the pressure wave. If C_r is unity, the reflected pressure equals the incident pressure at the surface, the two adding to produce a pressure amplitude $2A$ there. If $C_r = -1$, the reflected wave just cancels the incident wave at the surface, and the pressure there is only the small amplitude caused by the thermal wave inside the boundary layer.

We now compare this formula with those of Eqs. (6.4.32) for the assumed form of the propagational wave and find that

$$\psi(x) = Ae^{i(\omega x/c)\sin\vartheta} \qquad f(y) = e^{-i(\omega y/c)\cos\vartheta} + C_r e^{i(\omega y/c)\cos\vartheta}$$

$$k_t^2 = \left(\frac{\omega}{c}\right)^2\sin^2\vartheta \qquad \left(\frac{\omega}{c}\right)^2\cos^2\vartheta = \left(\frac{\omega}{c}\right)^2 - k_t^2$$

if we neglect the attenuation of the propagational wave in free space (i.e., if we neglect small quantities of the order of $l_v = \omega d_v^2/2c$ in the formula for k^2). Thus, for the reflection of a plane wave, the factor $(k_t c/\omega)^2$, which enters the viscosity term in the boundary conditions, is simply the square of the sine of the angle of incidence ϑ. Inserting these values into Eq. (6.4.36), we find, for the reflection coefficient,

$$C_r \simeq \frac{\cos\vartheta - (\rho c/z) - (\xi_b - i\sigma_b)}{\cos\vartheta + (\rho c/z) + (\xi_b - i\sigma_b)} \quad (6.4.41)$$

where

$$\xi_b = \sigma_b = \frac{\omega d_v}{2c}\sin^2\vartheta + (\gamma - 1)\frac{\omega d_h}{2c}$$

$$= \sqrt{\frac{\mu\omega}{2\rho c}}\sin^2\vartheta + (\gamma - 1)\sqrt{\frac{K\omega}{2\rho c C_p}}$$

are the effective acoustic conductance and susceptance of the boundary layer, to first order in the small quantities d_h and d_v.

This is to be compared with Eq. (6.3.5). As mentioned previously, the correction for heat conduction acts exactly the same as an additional acoustic admittance of the surface, but the viscosity correction has a different dependence on angle of incidence; to first order there is no viscous absorption for a normally incident wave $\vartheta = 0$. Reference to Eq. (6.3.6) shows that the energy absorbed per unit area of surface, from a plane wave of intensity $I = |A|^2/\rho c$, incident at angle ϑ, during its reflection, is

$$I(1 - |C_r|^2)\cos\theta \simeq \frac{4I(\xi + \xi_b)\cos^2\vartheta}{(\cos\vartheta + \xi + \xi_b)^2 + (\sigma + \sigma_b)^2} \qquad (6.4.42)$$

where $\rho c/z = \xi - i\sigma$. This also is valid only to first order in d_v and d_h.

If the surface is absolutely rigid, so that $\rho c/z = 0$, the first-order approximation to the energy absorbed per unit area is

$$4I\xi_b = \frac{4\,|A|^2}{\rho c}\,\xi_b \simeq \frac{|p(x,0,t)|^2}{\rho c}\,\xi_b = L_{bv} + L_{bh}$$

This formula checks with those of Eqs. (6.4.37) and (6.4.38).

Molecular energy equipartition

The foregoing discussion has covered, sufficiently for the needs of this book, the energy losses caused by shear friction and thermal diffusion. The magnitude, the dependence on frequency and temperature of these losses, and their relation with the kinetics of molecular motion have been set forth. A similar discussion cannot be given for liquids; molecular detail regarding the heat and momentum diffusion in liquids is not well understood. All we can do is to use experimentally determined constants in the formulas we have derived, or, in many cases, to use the formulas backward to determine empirical values for μ, γ, l_h, and l_v from measurements of energy loss. However, in the case of the energy loss related to the bulk modulus η of gases, present in Eqs. (6.4.8) and (6.4.20), additional discussion is desirable.

As we have seen in Eq. (6.4.24), the effect of heat conduction on sound propagation can be expressed in terms of a complex compressibility. Its imaginary part is proportional to the ratio between mean-free-path and wavelength, in other words, to the ratio between the period of the sound oscillation and the mean-free-time between molecular collisions. The pressure change resulting from a reduction of volume occupied is not quite in phase with the change in density, and energy is thereby changed to heat. Through interaction with the surrounding gas the compressed portion loses heat by diffusion, and a corresponding "relaxation" of the sound pressure results.

The effect of viscosity involves a similar diffusion of momentum from the moving part to the rest of the gas. These effects are irreversible transfers

of potential and kinetic energy from the organized motion of the sound wave to the disorganized translational motion of the molecules known as heat. The attenuation caused by viscosity and heat conduction is thus often referred to as translational relaxation effects. Measurements of sound propagation in monatomic gases have shown that the attenuation is in agreement with that predicted by translational relaxation, as presented in the foregoing portion of this section.

In diatomic and polyatomic gases, however, viscosity and heat conduction are only part of the sound-attenuating mechanism. In addition, there is an irreversible energy transfer from the collective motion of the sound wave to other degrees of freedom of the molecules, other than the translational ones. It is easy to see qualitatively how this attenuation will come about. Imagine a volume element of gas to be compressed quickly. The translational motion of the molecules will adjust nearly instantaneously, and since the pressure of the gas is determined solely by this translational motion, the pressure adjusts immediately to the density change. After the compression is completed, some of the translational energy of the molecules goes more slowly into rotational and vibrational energy, and the pressure decreases accordingly. If the gas is allowed to expand to its initial volume, the work delivered by the gas during the expansion is less than that absorbed during the compression. Therefore, in each such cycle, there will be a net energy transfer to the internal degrees of freedom of the molecules, which will heat the gas and attenuate the sound.

The magnitude of this energy loss will depend on the ratio between the length of the cycle and the "relaxation time" of the transfer from translational to internal energy of the molecules. In the extreme case, when the gas is expanded immediately after compression, before such transfer can take place, the gas will behave as a monatomic gas, and only viscosity and heat conduction will cause attenuation. On the other hand, if both compression and expansion take place very slowly, so that thermal equilibrium between all internal modes of motion is continuously established, relaxational attenuation is again absent. Thus the relaxational attenuation will be important in some intermediate range of frequencies.

Frequency dependence of the specific heat

The effect of the delay in distribution of energy among the various degrees of freedom of the molecules is most clearly displayed in the frequency dependence of the heat capacity of the gas. Thence, via the ratio γ of Eqs. (6.4.19), it can be related to the compressibility of a gas subjected to simple-harmonic pressure variations. The specific heat at constant volume (strictly speaking, specific heat is the heat capacity of a mole of the gas; the quantity C_v used here is the heat capacity of a *unit mass* of the gas) is the rate of change of the internal energy of the gas with temperature, $(\partial E/\partial T)_v$, at constant volume (i.e., constant density).

In thermodynamics we assume that the temperature is changed slowly enough so that all the degrees of freedom of the molecules can take up their share of the increased energy. In this case the energy absorbed during a unit rise in T by the nth degree of freedom of a unit mass of gas molecules can be written $(R/2M)F_n(T)$, where R is the gas constant and M is the molecular weight. Function F_n is zero at $T = 0$, rises rapidly at temperatures near T_n, the characteristic temperature for the nth degree of freedom, and for $T \gg T_n$ reaches an asymptotic value of unity; in the limit each mode absorbs $\frac{1}{2}R$ per mole per degree rise.

In the range of temperatures of interest to us, the three F_n's corresponding to the translational degrees of freedom of the molecules (those contributing to the pressure) have long since reached their asymptotic values, whereas those for the electronic motions have not yet begun to rise from near zero. Therefore we can write the heat capacity at constant volume of the gas as

$$C_V = \left(\frac{\partial E}{\partial T}\right)_v = \frac{R}{M}\left[\tfrac{3}{2} + \tfrac{1}{2}\sum_n F_n(T)\right]$$

where the sum over n includes all the internal degrees of freedom of the molecule having to do with the nuclear motions, the rotational ones (zero, two, or three of them, depending on whether the molecules are mono-, di-, or polyatomic), and the vibrational ones, corresponding to the various normal modes of vibration of the nuclei, as discussed in Chap. 3 (the number of such degrees of freedom being zero for a monatomic, one for a diatomic, and $3N - 6$ for an N-atomic molecule). For all molecules except hydrogen, the F_n's for the rotational degrees of freedom (if any) have also reached their unit asymptotic values by 0°C. Usually, the F_n's for the vibrational modes are less than unity at room temperatures.

Any book on thermodynamics will demonstrate that the difference between the specific heats at constant pressure and at constant volume, for a perfect gas, is R, the gas constant, the difference being caused by the expansion of the gas when heated at constant pressure, the expansion requiring extra energy. Therefore, for a unit mass of gas,

$$C_P = C_V + \frac{R}{M} = \frac{R}{M}\left[\tfrac{5}{2} + \tfrac{1}{2}\sum_n F_n(T)\right]$$

and the ratio between these heat capacities, the quantity γ entering the equation for the velocity of sound, is

$$\gamma = \frac{C_P}{C_V} = \frac{\frac{5}{2} + \frac{1}{2}\sum F_n(T)}{\frac{3}{2} + \frac{1}{2}\sum F_n(T)} \tag{6.4.43}$$

Once one works out, from statistical mechanics, the expressions for the F_n's as functions of T, then the dependence of γ on T, for reversible processes, can be calculated.

As pointed out earlier, however, the gas during a cycle of a sound wave does not have time to come to thermodynamic equilibrium. The increase in density immediately increases the energy of translational motion, but the internal degrees of freedom, represented by the F_n's, take time to absorb their component shares of energy. The nth degree of freedom has its characteristic time of response, called its *relaxation time* t_n, to reach equilibrium. The rotational modes, being more affected by collisions, reach equilibrium fairly quickly, but the values of t_n for the vibrational modes are considerably longer, some of them being of the order of magnitude of a few milliseconds for standard pressures and temperatures.

Statistically, the rate of approach of the energy E_n of the nth degree of freedom to its equilibrium value E_n^e is proportional to the difference $E_n - E_n^e$, the proportionality factor being the reciprocal of the relaxation time.

$$\frac{dE_n}{dt} = -\frac{1}{t_n}(E_n - E_n^e) \tag{6.4.44}$$

Thus, when left alone, the energy approaches E_n^e exponentially:

$$E_n = E_n^e + (E_n^i - E_n^e)e^{-t/t_n}$$

relaxing from its initial, nonequilibrium value E_n^i to E_n^e in a time roughly equal to t_n. In the presence of a simple-harmonic sound wave, however, the gas is not left alone; energy is alternately given and withdrawn at a rate determined by the factor $e^{-i\omega t}$. As long as the acoustic pressure p is small compared with the equilibrium pressure P, the situation is not far from equilibrium and it is not a bad approximation to assume that $E_n^e = E_n^0 + \epsilon_n e^{-i\omega t}$, where ϵ_n is the amplitude of the energy change in the nth degree of freedom which would be produced by the sound wave *if* equilibrium were attained at each instant of the cycle. Since the internal modes cannot follow so rapidly, the actual energy content E_n of the nth mode during the cycle is obtained from Eq. (6.4.44).

$$E_n = E_n^0 + \frac{\epsilon_n}{1 - i\omega t_n} e^{-i\omega t}$$

If the "driving force" is a periodic temperature change $\tau = \tau_0 e^{-i\omega t}$, the equilibrium amplitude ϵ_n of the internal energy change would be C_v or C_p times τ, and the amplitude of the actual change would be this product times $(1 - i\omega t_n)^{-1}$. Thus we can talk about a *dynamic* heat capacity, the ratio of internal energy to temperature change for a simple-harmonic temperature fluctuation in a gas,

$$C_V = \frac{R}{M}\left(\tfrac{3}{2} + \tfrac{1}{2}\sum_n \frac{F_n}{1 - i\omega t_n}\right)$$

and

$$C_P = \frac{R}{M}\left(\tfrac{5}{2} + \tfrac{1}{2}\sum_n \frac{F_n}{1 - i\omega t_n}\right)$$

and finally, a ratio of specific heats appropriate for such fluctuations,

$$\gamma(T,\omega) = \frac{5 + \sum [F_n(T)/(1 - i\omega t_n)]}{3 + \sum [F_n(T)/(1 - i\omega t_n)]} = \frac{\gamma + \frac{1}{2}(\gamma - 1) \sum \dfrac{i\omega t_n F_n}{1 - i\omega t_n}}{1 + \frac{1}{2}(\gamma - 1) \sum \dfrac{i\omega t_n F_n}{1 - i\omega t_n}} \qquad (6.4.45)$$

where γ is the equilibrium value defined in Eq. (6.4.43).

Relaxation attenuation

This dynamic value of γ should be used in computing the compressibility $\kappa_s = \kappa_T/\gamma$ for the processes in the sound wave and in computing the speed of sound $c = \sqrt{1/\rho\kappa_s}$. The attenuation of a plane wave of sound is then the imaginary part of the wavenumber $k = \omega/c$. In the present derivation of c and k, we have neglected the effects of viscosity and of thermal conduction; therefore, for the whole expression for the propagational mode, we should combine the results of Eqs. (6.4.23) and (6.4.45), obtaining for the square of the wavenumber

$$k^2 \simeq \gamma \left(\frac{\omega}{c_0}\right)^2 \frac{1 + \frac{1}{2}(\gamma - 1) \sum \dfrac{i\omega t_n F_n}{1 - i\omega t_n}}{\gamma + \frac{1}{2}(\gamma - 1) \sum \dfrac{i\omega t_n F_n}{1 - i\omega t_n}} \left[1 + i\frac{\omega}{c} l'_v + i(\gamma - 1)\frac{\omega}{c} l_h\right] \qquad (6.4.46)$$

where

$$c_0^2 = \frac{\gamma}{\kappa_T \rho} = \frac{\gamma P}{\rho} \qquad \text{(for a perfect gas)}$$

and where

$$l'_v = \frac{\eta + \frac{4}{3}\mu}{\rho c} \qquad \text{and} \qquad l_h = \frac{K}{\rho c C_p}$$

The formula for a plane wave in the x direction would be $e^{ikx - i\omega t}$; so the imaginary part of k is the attenuation constant a of Eq. (6.4.14), the reciprocal of the distance within which the wave amplitude is attenuated by a factor e^{-1}. The ratio between ω and the real part of k is the phase velocity of the wave, to be discussed in detail in the latter part of Sec. 9.1. For low frequencies, such that $\omega t_n \ll 1$ for all n's, the fraction in Eq. (6.4.46) can be approximated; to the first order in ωt_n, l'_v, and l_h, the expression for the wavenumber is

$$k \simeq \frac{\omega}{c_0}\left\{1 + \frac{i\omega}{2\rho c_0^2}\left[\frac{(\gamma - 1)^2}{2\kappa_T} \sum_n t_n F_n(T) + \eta + \tfrac{4}{3}\mu + (\gamma - 1)\frac{K}{C_p}\right]\right\} \qquad (6.4.47)$$

the attenuation constant being proportional to the quantity in the square brackets, and the phase velocity differing from c_0 only to second order in the small quantities. At low frequencies all effects enter linearly, each degree of freedom making its contribution independently of the others.

Thus the quantity

$$\eta_r = \frac{(\gamma - 1)^2}{2\kappa_T} \sum t_n F_n(T)$$

enters the formulas for attenuation on a par with the bulk modulus η. Since neither η_r nor η enters the shear and conductivity effects near a boundary except in second order, it is hard to distinguish between η_r and η. Factor η is an assumed internal friction of compression, and η_r comes from the hysteresislike loss to internal modes during a compression cycle. Of course, there are no internal degrees of freedom for monatomic gases; so η_r should be zero in this case. But the internal-frictional term η for a monatomic gas is so small that it is doubtful whether it is really present; and with di- and polyatomic gases, one cannot distinguish η_r from η. Thus it is doubtful whether there should be a separate factor η; we might as well consider η_r to *be* the η introduced earlier in this section.

If we insert the expressions for μ and for K/C_p in terms of molecular mean-free-path l, given in Eqs. (6.4.2) and (6.4.5), into the formula for the nondimensional quantity $a\lambda = (c_0/\omega)\mathrm{Im}\, k$, the attenuation per wavelength,

$$a\lambda \simeq \pi \frac{(\gamma - 1)^2}{2\gamma} \sum_n \omega t_n F_n + \tfrac{4}{3}\pi \frac{\omega}{\sqrt{\gamma}} \frac{l}{c_0} + \pi \frac{\gamma - 1}{\gamma^{\frac{3}{2}}} \omega \frac{l}{c_0}$$

we see that the viscosity and thermal effects are quite similar to the relaxation effects. The mean time between collisions l/c_0 (since c_0 is roughly equal to the mean speed of the molecules; see page 229) is equivalent to a relaxation time; indeed, it *is* the relaxation time for the translational degrees of freedom of the molecules. Since the relaxation time for the rotational degrees of freedom is not much longer than the time between collisions, and since F_n for the rotational terms is nearly unity at room temperatures, the rotational terms in the sum over n are of the same order of magnitude as the viscosity and thermal-diffusion terms (i.e., the translational terms). At standard conditions of temperature and pressure, the time between collisions in air is roughly 10^{-9} sec; so the rotational and translational contributions to acoustic attenuation are extremely small for frequencies less than about a megacycle.

Vibration-relaxation attenuation

As was stated on page 277, we thus find that the attenuation caused by viscosity, thermal diffusion, and rotational relaxation, away from boundaries, can usually be neglected (the boundary effects, as we have seen, are several orders of magnitude larger). This is not true for the vibrational modes, however, particularly for polyatomic gases. Although the factors F_n for these modes are small, between 0.1 and 0.01 for room temperatures, the relaxation times are much longer than the rotational times; it takes many collisions to excite some vibrations. For the oxygen molecule, for example,

t_n is about 10^{-3} sec. However, the presence of an impurity, such as water vapor, is known to decrease t_n so its inverse is in the acoustic frequency range.

Since the F_n's for the vibrational modes are small at room temperatures, we can expand the fraction of Eq. (6.4.46) for these terms also, but we cannot neglect the higher orders of ωt_n. The complete expression for kc_0/ω, using appropriate approximations, is

$$\frac{kc_0}{\omega} \simeq 1 - \frac{(\gamma-1)^2}{4\gamma} \sum_{\text{vib}} \frac{(\omega t_n)^2 F_n}{1+(\omega t_n)^2} + i\frac{(\gamma-1)^2}{4\gamma} \sum_{\text{vib}} \frac{\omega t_n F_n}{1+(\omega t_n)^2}$$

$$+ i\frac{(\gamma-1)^2}{4\gamma} \sum_{\text{rot}} \omega t_n F_n + \tfrac{4}{3} i \frac{\omega\mu}{\rho c_0^2} + i(\gamma-1)\frac{\omega K}{\rho c_0^2 C_p} \qquad (6.4.48)$$

where the sum has been split into a rotational and a vibrational part, as labeled below the summation signs.

The last three terms can be neglected, at standard conditions, if there are any vibrational terms present. In fact, for air containing moderate amounts of molecules such as H_2O or CO_2, the vibrational terms for O_2 and N_2 vibrations cannot be neglected but, in reality, provide the largest contributions in Eq. (6.4.48) (multiplied, of course, by the factor corresponding to their concentration, which can be included in the factors F_n).

The first two terms in this expression are real and represent the effect of the vibration relaxation on the speed of sound. The phase velocity is

$$\frac{\omega}{\operatorname{Re} k} \simeq c_0 \left[1 + \frac{(\gamma-1)^2}{4\gamma} \sum_{\text{vib}} \frac{(\omega t_n)^2 F_n}{1+(\omega t_n)^2}\right] \qquad (6.4.49)$$

At frequencies well below $1/t_n$ for all n's this is equal to $c_0 = \sqrt{\gamma P/\rho}$, the velocity in dry air. As the frequency is increased, this phase velocity increases, approaching a somewhat higher value asymptotically. Thus moist air is a dispersive medium for sound waves (see the latter part of Sec. 9.1).

The attenuation per wavelength

$$2\pi \operatorname{Im} \frac{kc_0}{\omega} \simeq \pi \frac{(\gamma-1)^2}{2\gamma} \sum_{\text{vib}} \frac{\omega t_n F_n}{1+(\omega t_n)^2} \qquad (6.4.50)$$

also is small at low frequencies, rising linearly with ω. But it reaches a maximum, for each vibrational mode, at $\omega = 1/t_n$; when the period of oscillation of the sound is equal to 2π times the relaxation time, a maximum amount of energy is converted into heat because of the relaxation hysteresis. For still higher frequencies the attenuation decreases again, approaching zero as $\omega \to \infty$, as mentioned on page 295.

Problems

1 (*a*) A plane wave in air has an intensity of 100 ergs per sec per sq cm. What is the magnitude of the temperature fluctuation in the air, caused by the wave? (*b*) At $t = 0$ a sound wave has the form of a step function, with $P = P_0 + p$ for $x < 0$ and $P = P_0$ for $x > 0$. Explain qualitatively how heat conduction would change the shape of this step function as it travels in the positive x direction.

2 As shown in Eq. (6.4.14), the attenuation of a plane wave depends on the size of the shear modulus μ. Explain how shear stresses can arise in a plane wave.

3 Suppose the air, which was initially at rest at the point x_0, y_0, z_0 has displacement $X - x_0 = AF(ct - x_0)$ in the x direction, where

$$F(q_0) = \begin{cases} 0 & q_0 < 0 \\ \alpha q_0 & 0 < q_0 < \dfrac{1}{\alpha} \\ 2 - \alpha q_0 & \dfrac{1}{\alpha} < q_0 < \dfrac{2}{\alpha} \\ 0 & \dfrac{2}{\alpha} < q_0 \end{cases} \qquad \text{where} \quad \begin{aligned} q &= ct - x \\ q_0 &= ct - x_0 \end{aligned}$$

Show that the displacement of the portion of the air, which happens to be at point x at time t, is

$$D(q) = A \begin{cases} 0 & 0 > q \\ \dfrac{\alpha q}{1 - \alpha A} & 0 < q < \dfrac{1}{\alpha} - A \\ \dfrac{2 - \alpha q}{1 + \alpha A} & \dfrac{1}{\alpha} - A < q < \dfrac{2}{\alpha} \\ 0 & \dfrac{2}{\alpha} < q \end{cases}$$

Show that the fluid velocity at point x at time t is $+c\alpha A$ for $0 < q < (1/\alpha) - A$ and is $-c\alpha A$ for $(1/\alpha) - A < q < (2/\alpha)$; zero otherwise. Show that the pressure there is $\rho c^2(\partial D/\partial q)$. What happens to the displacement and pressure wave when $\alpha A \to 1$? Plot D, u, and p as functions of x for $t = 0$ and $\alpha A \to 1$. What is the physical meaning of this result? What happens when $\alpha A > 1$?

4 By the usual methods of the calculus of variations, show that Eq. (6.2.14) is the Lagrange-Euler equation for the Hamilton principle

$$\delta \int dt \iiint L(\psi, \psi_x, \psi_y, \psi_z, \psi_t)\, dx\, dy\, dz = 0$$

where the subscript denotes differentiation, as in Eq. (6.2.14), and when the variation of ψ does not alter ψ at the limits of the integration in space or time. Show that adding to L the divergence of a vector $\mathbf{A}$, which goes to zero at the limits of integration, leaves Eq. (6.2.14) unchanged.

5 Show that if a Lagrange density L is a function of second derivatives ψ_{xy}, etc., as well as of ψ and its first derivatives, the Lagrange-Euler equation for the

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Hamilton principle, $\int L\,dt\,dV = 0$, is

$$\frac{\partial L}{\partial \psi} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial L}{\partial \psi_i} + \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\frac{\partial L}{\partial \psi_{ij}} = 0$$

instead of Eq. (6.2.14), where $\psi_1 = \partial\psi/\partial x$, $\psi_{24} = \partial^2\psi/\partial y\,\partial t$, etc. Show that, if L is a quadratic function of four functions p, its adjoint p^a, τ, and its adjoint τ^a [see discussion of Eq. (6.2.28)] and their first and second derivatives, there will be four Lagrange-Euler equations, one

$$\frac{\partial L}{\partial p^a} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial L}{\partial p_i{}^a} + \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\frac{\partial L}{\partial p_{ij}{}^a} = 0$$

for function p, and three more, one for p^a, one for τ, and one for its adjoint τ^a. Write out the formulas for the elements W_{ij} of the stress-energy tensor, and show how they are related by equations of continuity.

6 Use the results of Prob. 5 to show that Eqs. (6.4.22) are Lagrange-Euler equations for p and τ, corresponding to a Lagrange density function

$$\begin{aligned} L = &-\frac{\gamma}{2c^2}\frac{\partial p}{\partial t}\frac{\partial p^a}{\partial t} + \frac{\gamma^2\alpha^2 l_h}{2(\gamma-1)c^2}\frac{\partial \tau}{\partial t}\frac{\partial \tau^a}{\partial t} \\ &+ \tfrac{1}{2}\operatorname{grad} p \cdot \operatorname{grad} p^a - \frac{\gamma l_v'}{4c}\left(\frac{\partial p}{\partial t}\nabla^2 p^a - \frac{\partial p^a}{\partial t}\nabla^2 p\right) \\ &- \frac{\alpha^2\gamma^2 l_h}{4(\gamma-1)c}\left(\frac{\partial \tau}{\partial t}\nabla^2\tau^a - \frac{\partial \tau^a}{\partial t}\nabla^2\tau\right) + \frac{\alpha\gamma}{2c^2}\left(\frac{\partial p}{\partial t}\frac{\partial \tau^a}{\partial t} + \frac{\partial p^a}{\partial t}\frac{\partial \tau}{\partial t}\right) \\ &+ \frac{\alpha\gamma l_h}{4c}\left(\frac{\partial p}{\partial t}\nabla^2\tau^a - \frac{\partial p^a}{\partial t}\nabla^2\tau + \frac{\partial \tau}{\partial t}\nabla^2 p^a + \frac{\partial \tau^a}{\partial t}\nabla^2 p\right) \end{aligned}$$

which takes into account viscosity and heat conduction (note the arrangement of signs in the last parentheses). Compute the stress-energy tensor W_{ij} [Eqs. (6.2.15)]. Discuss the physical significance of the formulas for W_{tt} and the vector $\mathbf{I}$ with x component W_{tx}.

7 One method of measuring the normal impedance of a boundary (as a function of the angle of incidence of the sound wave) is the following: Imagine the (complex) sound-pressure amplitude p_∞, that is, magnitude and phase (relative to some reference phase) to be measured at the surface of a perfectly reflecting surface (infinite impedance). Keeping the sound source the same, the surface is then replaced by the surface to be measured, and the complex surface pressure is again measured. Denote this by p_ζ. The complex number $w = p_\zeta/p_\infty$ is thus determined from the experiments.

(*a*) Show that the relationship between the normal impedance ζ (to be determined) and the measured quantity w is

$$\zeta = \frac{w}{1 - w\cos\vartheta}\frac{1}{\cos\vartheta}$$

where ϑ is the angle of incidence of the wave.

(*b*) If we set $\zeta \cos \vartheta = u + iv$ and $w = |w|e^{i\gamma}$, show that the relation between $\zeta \cos \vartheta$ and w can be represented as a set of orthogonal circles in the uv plane, circles for constant $|w|$ and constant γ. These circles are, with $[|w|^2/(1 - |w|^2)] = A$,

$$(u - A)^2 + v^2 = A(1 + A) \qquad \text{(constant } |w| \text{ circles)}$$

$$(v - \tfrac{1}{2} \cot \gamma)^2 + (u + \tfrac{1}{2})^2 = \tfrac{1}{4} \sin^{-2} \gamma \qquad \text{(constant } \gamma \text{ circles)}$$

8 The region $y < 0$ is filled with porous material of porosity Ω and flow resistance Φ; the region for $y > 0$, with air of characteristic impedance ρc and wave velocity c. Show that the effective acoustic impedance $z(\vartheta) = -(p/u_y)$, which the interface $y = 0$ presents to a plane wave of sound of frequency $\omega/2\pi$, incident at angle of incidence ϑ, is

$$\rho_p c_p \left[\frac{1 + i(\Phi/\rho_p \omega)}{1 - (c_p/c)^2 \sin^2 \vartheta} \right]^2$$

where ρ_p and c_p are defined in connection with Eq. (6.2.24). Show that, when $c_p \ll c$ and $\Phi \ll \rho_p \omega$, the interface behaves like a locally reacting surface of acoustic resistance $\rho_p c_p$ and stiffness reactance $-(c_p \Phi/2\omega)$.

9 The xz plane is a rigid wall; the region between $y = 0$ and $y = l$ is occupied by a material with acoustic parameters ρ_e and c_e, which may be complex quantities. Show that the ratio of reflected to incident intensity, for a normally incident wave, is

$$|C_r|^2 = \left| \frac{\rho_e c_e + i\rho c \tan (\omega l/c_e)}{\rho_e c_e - i\rho c \tan (\omega l/c_e)} \right|^2$$

10 A panel of porous material 3 cm thick is glued on one side to a plane rigid wall; the other side is open to the air. The acoustic parameters of the material, as defined in connection with Eq. (6.2.24), are $\rho_p = 2\rho$, $c_p = \frac{1}{2}c$, $\Phi/\rho c = 100 \text{ m}^{-1}$. A plane wave of frequency $\omega/2\pi$ is normally incident on this coated wall. Plot the fraction of incident acoustic energy which is reflected, as a function of frequency, from 0 to 10,000 cps.

11 Plot against angle of incidence ϑ the fraction of energy $|C_r|^2$ reflected from the plane interface between the air (ρ,c) and a medium of density $\frac{2}{5}\rho$ and sound velocity $\frac{5}{4}c$, when the incident wave comes from the air side. Plot $|C_r|^2$ when $\rho_w = \frac{8}{5}\rho$, $c_w = \frac{5}{4}c$.

12 Suppose a unit-pulse pressure wave $\delta(T_i - t_0)$, where

$$T_i = t - (1/c)(x \sin \vartheta - y \cos \vartheta)$$

impinges on the surface ($y = 0$) of a porous material of the sort discussed in Prob. 8, where $z = \rho_p c_p + i(c_p \Phi/2\omega)$. Show that the reflected pressure wave is

$$\delta(T_r - t_0) + \frac{\rho c c_p \Phi \cos \vartheta}{(\rho c + \rho_p c_p \cos \vartheta)^2} \exp \left[\frac{-c_p \Phi \cos \vartheta}{2\rho c + 2\rho_p c_p \cos \vartheta} (T_r - t_0) \right]$$

when $t > 0$, where $T_r = t - (1/c)(x \sin \vartheta + y \cos \vartheta)$. Discuss the physical significance of the dependence on ϑ, t, and Φ/ρ_p.

13 A fluid having viscosity and thermal conductivity is confined between two rigid plates, one at $y = +\frac{1}{2}a$, the other at $y = -\frac{1}{2}a$. A sound wave is sent in the x direction in the fluid, with frequency $\omega/2\pi$. Show that the three types of

waves, which together fit the boundary conditions $\tau = 0$ and $\mathbf{u} = 0$ at $y = \pm\frac{1}{2}a$, are, to first order in $kd_v = \sqrt{\frac{1}{2}kl_v}$ and $kd_h = \sqrt{\frac{1}{2}kl_h}$ [see Eq. (6.4.31)],

$$p_p = P \exp(i\mu x - i\omega t) \cosh(\beta y) \qquad \tau_p \simeq \frac{\gamma - 1}{\gamma\alpha} p_p$$

$$u_{px} = \frac{\mu}{\rho c k} p_p \qquad \text{(propagational wave)}$$

$$u_{py} = -\frac{i\beta P}{\rho c k} \exp(i\mu x - i\omega t) \sinh(\beta y)$$

$$p_h \simeq -i\gamma\alpha k l_h \tau_h$$

$$\tau_h \simeq \frac{\gamma - 1}{\gamma\alpha} P \frac{\cosh[(1 - i)y/d_h]}{\cosh[(1 - i)a/2d_h]} \exp(i\mu x - i\omega t) \cosh\frac{\beta a}{2}$$

$$u_{hx} \simeq 0 \qquad \text{(thermal wave)}$$

$$u_{hy} \simeq -(1 - i)kd_h \frac{\gamma - 1}{2\rho c} P \frac{\sinh[(1 - i)y/d_h]}{\cosh[(1 - i)a/2d_h]} \exp(i\mu x - i\omega t) \cosh\frac{\beta a}{2}$$

$$u_{tx} \simeq \frac{-\mu P}{\rho c k} \frac{\cosh[(1 - i)y/d_v]}{\cosh[(1 - i)a/2d_v]} \exp(i\mu x - i\omega t) \cosh\frac{\beta a}{2} \qquad \text{(shear wave)}$$

$$u_{ty} \simeq -(1 - i) \frac{\mu^2 d_v}{\rho c k} P \frac{\sinh[(1 - i)y/d_v]}{\cosh[(1 - i)a/2d_v]} \exp(i\mu x - i\omega t) \cosh\frac{\beta a}{2}$$

where $k = \omega/c$, and the wavenumber $\mu = \sqrt{k^2 + \beta^2}$. Show that, in order that $u_x = 0$ at $y = \pm\frac{1}{2}a$,

$$\beta^2 a^2 \simeq k^2 a^2 \frac{(\gamma - 1)F(d_h/a) + F(d_v/a)}{1 - F(d_v/a)}$$

if βa is smaller than about $\frac{1}{2}$, where

$$F(x) = (1 + i)x \tanh\frac{1 - i}{2x} \to \begin{cases} (1 + i)x & x \ll 1 \\ 1 + \dfrac{i}{6x^2} & x \gg 1 \end{cases}$$

14 A plane sheet of cloth has flow resistance R such that the air velocity through it equals $1/R$ times the pressure difference on the two sides of the sheet. This sheet is suspended along the xz plane, the surface of a rigid wall is a distance behind it in the plane $y = -l$, with air in the interspace. Show that the reflection coefficient of Eqs. (6.3.5) and (6.3.13) is

$$C_r = \frac{R\cos\vartheta + i\rho c \cot(kl\cos\vartheta) - \rho c}{R\cos\vartheta + i\rho c \cot(kl\cos\vartheta) + \rho c}$$

Discuss the dependence of the fraction of incident energy which is absorbed on kl, R, and ϑ.

15 Discuss the similarities and differences between the C_r of Prob. 14 and the C_r for a covered wall of the sort described in Prob. 10, with arbitrary values of ρ_p, c_p, Φ, and panel thickness l. What are the advantages and disadvantages of the two sorts of acoustic covering?

16 A plane sound wave with angle of incidence $\vartheta = 60°$ is reflected from a plane wall of acoustic impedance z, and the sound pressure at the surface is measured in amplitude and phase. The experiment is then repeated, with the absorptive wall replaced by a perfectly rigid one. The mean-square pressure at the hard wall is found to be twice that for the absorbing wall, but the phase difference is zero. Use the results of Prob. 7 to compute the impedance of the absorbing wall. What fraction of the incident energy does it absorb?

17 Suppose the sheet of Prob. 14 is light enough so that its motion as a whole must be included. Show that the normal air velocity at its surface is $(1/R) + (i/\omega c)$ times the pressure drop across the sheet. Compute C_r and compare its dependence on kl, ϑ, and R with that of Prob. 14.

18 It is known that the bulk viscosity of nitrogen is about 0.8 times the ordinary shear viscosity. From this information and the discussion on page 299 determine the relaxation time of the rotational modes of motion in nitrogen. At 20°C and 1 atm the shear viscosity coefficient in nitrogen is $\mu = 1.78 \times 10^{-4}$ cgs units.

19 Carbon dioxide, CO_2, has its atoms along a line. The characteristic (Debye) temperature T_n (see page 296) for the transverse vibrational mode of motion of CO_2 is $T_n = 960°K$. (The longitudinal modes have considerably higher Debye temperatures.) Calculate and plot as functions of frequency the speed of sound and the attenuation per wavelength in CO_2 resulting from the transverse mode when the relaxation frequency $\frac{1}{2}\pi t_n$ of this mode at 20°C and 1 atm is known to be about 30,000 Hz. For a vibrational mode the specific heat function F_n of Eq. (6.4.43) is known from statistical mechanics to be

$$F_n = \left(\frac{x}{e^x - 1}\right)^2 e^x \qquad \text{vibration}$$

where $x = T_n/T = h\nu/k_B$, ν = vibrational frequency
h = Planck's constant
k_B = Boltzmann's constant
T = absolute temperature

20 With reference to the discussion leading to Eqs. (6.4.48) to (6.4.50), calculate and plot as functions of frequency the speed of sound and attenuation per wavelength in air at some different relative humidities. It is known that the average relaxation time of vibration in air depends on the relative humidity H as given by $\nu_n = (2\pi t_n)^{-1} \simeq 1 + 30H^2$, where H is measured in percent at a temperature of 20°C. (At 20°C the fraction of H_2O molecules in air is $H/4{,}320$.) In particular, consider $H = 18$ (corresponding to $\nu_n \simeq 10{,}000$ Hz), $H = 50$, and $H = 80$. On the basis of the corresponding curves discuss the role of water vapor on acoustic dispersion in the atmosphere as compared with the effects of viscosity and heat conduction.

CHAPTER

7

THE RADIATION OF SOUND

7.1 POINT SOURCES

Sound waves are generated by the vibration of any solid body in contact with the fluid medium, or by vibratory forces acting directly on the fluid, or by the violent motion of the fluid itself, as from a jet, or by oscillatory thermal effects, as would be produced by a modulated laser beam. In each case energy is transferred from the source to the fluid. The characteristics of the source determine the frequency and directional characteristics of the generated sound field; reciprocally, the directional properties of the field can be used to shed light on the nature of the source.

Of course, any motion of one portion of the fluid medium is transmitted to other parts of the medium; the motion in the frontal region of a sound wave at one instant may be regarded as the "source" of the subsequent wave motion, farther along in the medium. But in this case the energy of the wave is just being transmitted from one part of the medium to another; no new energy is being introduced. In the cases mentioned in the first paragraph, however, energy originally not acoustic is being changed to acoustic energy at the source, to be radiated outward and lost to the source. In the case of the jet, for example, the high-speed, organized motion of the jet is transformed by nonlinear processes into heat and acoustic energy, the sound being transmitted outward into the otherwise quiescent surrounding medium. From the point of view of acoustics, therefore, a *source* is a region of space, in contact with the fluid medium, where new acoustic energy is being generated, to be radiated outward as sound waves.

In this chapter we shall not go into the details of the nonlinear effects which sometimes generate the acoustic energy, but shall devote our attention to the generated sound waves as they move outward from the source. We shall assume that the fluid medium outside the source region is initially uniform and at rest and that the acoustic pressure generated outside the source is small enough so that the first-order equations of sound, derived in the preceding chapter, are valid in the region outside the source. We also

shall concentrate on wave propagation in an infinite medium; sound propagation in ducts and rooms will be taken up in a later chapter.

Curvilinear coordinates

Since the wave motion near a source is seldom a plane wave, we must extend the form of our first-order equations (6.2.7) and (6.2.8) to more appropriate coordinate systems. The most useful, aside from the usual rectangular system x, y, z, are the spherical coordinates r, ϑ, φ, where

$$x = r \sin\vartheta \cos\varphi \qquad y = r \sin\vartheta \sin\varphi \qquad z = r \cos\vartheta$$

These are appropriate when the source region is concentrated in a small region of space. The circular cylindrical coordinates w, ϕ, z, where

$$x = w \cos\phi \qquad y = w \sin\phi \qquad z = z$$

are appropriate when the source region is extended in one dimension and concentrated in the other two. (When both spherical and cylindrical coordinates are used, we shall use w for the cylindrical radius; when spherical coordinates are not introduced, we shall use r for the cylindrical radius.)

As with the two-dimensional coordinates for the membrane, discussed in connection with Eq. (5.3.3), the differential operators, expressing the divergence, gradient, and Laplacian of the properties of the fluid, have different forms in the different coordinates. For example, the gradient $\operatorname{grad} f \equiv \nabla f$ of a scalar quantity, such as the pressure or the velocity potential, which enters the basic equations

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\operatorname{grad} p \qquad \text{and} \qquad \mathbf{u} = -\operatorname{grad} \Psi$$

is a vector with components $\nabla_x f, \nabla_y f, \nabla_z f$ along the rectangular axes, components $\nabla_w f, \nabla_\phi f, \nabla_z f$ along the cylindrical coordinate axes, and components $\nabla_r f, \nabla_\vartheta f, \nabla_\varphi f$ along the radius from the origin, along a perpendicular to r in the rz plane and along a perpendicular to the rz plane, respectively. The expressions for these components are

$$\operatorname{grad}_x(\ ) = \frac{\partial}{\partial x} \qquad \operatorname{grad}_y(\ ) = \frac{\partial}{\partial y} \qquad \operatorname{grad}_z(\ ) = \frac{\partial}{\partial z}$$

(in rectangular coordinates)

$$\operatorname{grad}_w(\ ) = \frac{\partial}{\partial w} \qquad \operatorname{grad}_z(\ ) = \frac{\partial}{\partial z} \qquad \operatorname{grad}_\phi(\ ) = \frac{1}{w}\frac{\partial}{\partial \phi} \tag{7.1.1}$$

(in cylindrical coordinates)

$$\operatorname{grad}_r(\ ) = \frac{\partial}{\partial r} \qquad \operatorname{grad}_\vartheta(\ ) = \frac{1}{r}\frac{\partial}{\partial \vartheta}; \qquad \operatorname{grad}_\varphi(\ ) = \frac{1}{r \sin\vartheta}\frac{\partial}{\partial \varphi}$$

(in spherical coordinates)

The second of the basic equations (6.2.7) involves the divergence of the vector **u**. In the three coordinate systems the divergence of vector **A** is

$$\begin{aligned}
&\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} && \text{(in rectangular coordinates)}\\
&\frac{1}{w}\frac{\partial}{\partial w}(wA_w)+\frac{1}{w}\frac{\partial}{\partial \phi}(A_\phi)+\frac{\partial}{\partial z}(A_z) && \text{(in cylindrical coordinates)}\\
&\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\vartheta}\frac{\partial}{\partial \vartheta}(A_\vartheta\sin\vartheta)+\frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}(A_\vartheta) && \text{(in spherical coordinates)}
\end{aligned} \tag{7.1.2}$$

where A_r, A_ϑ, A_φ are the components of **A** along the three axial directions in spherical coordinates, etc.

The Laplace operator $\nabla^2 \equiv \text{div grad}$ thus has three forms in the three different systems:

$$\nabla^2 = \begin{cases}
\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2} & \text{(in rectangular coordinates)}\\[2ex]
\dfrac{1}{w}\dfrac{\partial}{\partial w}\left(w\dfrac{\partial}{\partial w}\right)+\dfrac{1}{w^2}\dfrac{\partial^2}{\partial \phi^2}+\dfrac{\partial^2}{\partial z^2} & \text{(in cylindrical coordinates)}\\[2ex]
\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right)+\dfrac{1}{r^2\sin\vartheta}\dfrac{\partial}{\partial \vartheta}\left(\sin\vartheta\dfrac{\partial}{\partial \vartheta}\right) & \\
\qquad +\dfrac{1}{r^2\sin^2\vartheta}\dfrac{\partial^2}{\partial \varphi^2} & \text{(in spherical coordinates)}
\end{cases} \tag{7.1.3}$$

The first-order equations of motion and the second-order equations for the acoustic energy density and intensity of Eqs. (6.2.5) are given in standard form in terms of these operators:

$$\rho\frac{\partial \mathbf{u}}{\partial t}=-\text{grad}\,p \qquad \mathbf{u}=-\text{grad}\,\Psi$$

$$\frac{\partial p}{\partial t}=-\text{div}\,\mathbf{u} \qquad p=\rho\frac{\partial \Psi}{\partial t}$$

$$\frac{\partial^2 p}{\partial t^2}=c^2\nabla^2 p \qquad \frac{\partial^2 \Psi}{\partial t^2}=c^2\nabla^2\Psi$$

$$w=\tfrac{1}{2}\rho\left(|\text{grad}\,\Psi|^2+\frac{1}{c^2}\left|\frac{\partial \Psi}{\partial t}\right|^2\right)=\tfrac{1}{2}\rho\,|\mathbf{u}|^2+\frac{1}{2}\frac{1}{\rho c^2}|p|^2$$

$$\mathbf{I}=-\rho\,\text{Re}\left(\frac{\partial \Psi^*}{\partial t}\,\text{grad}\,\Psi\right)=\text{Re}\,(\mathbf{u}^*p)$$

They can be expressed in different form for the different coordinate systems by using Eqs. (7.1.1) to (7.1.3).

In extension of the remarks on page 193 on the physical meaning of the Laplacian, we should point out that the Laplacian in three dimensions measures the negative of the concentration of a quantity. The value of $\nabla^2 p$ at a point is proportional to the difference between the average pressure near the point and the pressure right at the point. When this is negative, there is a concentration of pressure at the point; when it is positive, there is a lack of concentration there. The wave equation simply states, in a quantitative manner, that if there is a concentration of pressure at some point, the pressure there will tend to diminish.

The simple source

If the source region is compact and the generating motion has no preferred direction, it will produce a wave which spreads spherically outward. If the medium is infinite in extent, the waveform will depend on the distance r from the center of the source, but will be independent of the spherical angles ϑ and φ, as though the wave were generated by a sphere, centered at the origin, uniformly expanding and contracting. The wave equation in this case is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

A general solution of this equation which is finite everywhere except at $r = 0$ is

$$p = \frac{1}{r}F(r - ct) + \frac{1}{r}f(r + ct)$$

consisting of a wave of arbitrary form going outward from the center and another wave focusing in on the center.

This solution, except for the factor $1/r$, is similar to Eq. (4.1.2) for waves on a string and to the equation for plane waves of sound given in Eq. (6.3.3). This means that spherical waves are more like plane waves than they are like cylindrical waves. Plane waves do not change shape or size as they travel; spherical waves do not change shape as they spread out, but they do diminish in amplitude because of the factor $1/r$; whereas cylindrical waves change both shape and size as they go outward, leaving a wake behind. Figure 5.8 shows that if a cylinder sends out a pulse of sound, the wave as it spreads out has a sharp beginning but no ending; the pressure at a point r from the axis is zero until a time $t = r/c$ after the start of the pulse, but the pressure does not settle back to its equilibrium value after the crest has gone by. With both plane and spherical waves the wave for a pulse has a sharp beginning and ending, the pressure settling back to equilibrium value after the pulse has gone past.

This behavior is another example of the general law (proved in books on the mathematics of wave motion) that waves in an odd number of dimensions (one, three, five, etc.) leave no wake behind them, whereas waves in an even number of dimensions (two, four, etc.) do leave wakes.

Spherical waves do resemble circular waves on a membrane, however, in that they have infinite values at $r = 0$. As we have seen on page 196, this simply means that the size of the source must be taken into account; every actual source of sound has a finite size, so that the wave motion never extends in to $r = 0$ where it would be infinite.

Suppose that a sphere of average radius a is expanding and contracting so that the radial velocity of its surface is everywhere the same function of time $U(t)$. The rate of flow of air away from the surface of the sphere, in every direction, is $4\pi a^2 U(t) = S(t)$. To obtain an expression for the pressure wave radiated from the sphere, we write Newton's equation as $\rho(\partial u_r/\partial t) = -(\partial p/\partial r)$. If p is chosen to be an arbitrary outgoing wave $p = P(r - ct)/r$, the requirement at the surface of the sphere,

$$\frac{1}{r}\frac{\partial P}{\partial r} - \frac{P}{r^2} = -\frac{\rho}{4\pi a^2}\frac{\partial S}{\partial t} \qquad \text{at } r = a$$

serves to determine the shape of the wave P.

If the vibrating sphere is very small (more specifically, if a is small compared with the wavelength of the sound radiated), the sphere is called a *simple source* of sound. In this case P/r is much larger than dP/dr at $r = a$, and $P \simeq \dfrac{\rho}{4\pi}\dfrac{dS}{dt}$ at $r = a$. The pressure wave at a distance r from the center of the simple source is therefore

$$p \simeq \frac{\rho}{4\pi r} S'\left(t - \frac{r}{c}\right) \tag{7.1.4}$$

where $S'(z) = (d/dz)S(z)$. The function S gives the instantaneous value of the total flow of air away from the center of the source. The pressure at the distance r is proportional to the rate of change of this flow a time r/c earlier. If, for example, the flow outward started suddenly, being zero for $t < 0$ and 1 for $t > 0$, the generated pressure wave would be a pulse, $(\rho/4\pi r)\,\delta[t - (r/c)]$; a steady flow outward produces no sound; only a change in flow can be heard.

The periodic simple source

In the special case when the source is simple-harmonic, so that the flow outward from the origin is $S_\omega e^{-i\omega t}$, the resulting wave motion, its mean-square

energy and intensity [Eqs. (6.2.21)], and total power radiated Π are

$$p = p_s(\omega)e^{-i\omega t} = -\frac{ik\rho c}{4\pi r} S_\omega e^{ik(r-ct)} \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$u_r = \frac{-1}{4\pi r^2}(ikr - 1)S_\omega e^{ik(r-ct)} = \frac{p}{\rho c}\left(1 + \frac{i\lambda}{2\pi r}\right)$$

$$w = \rho\left(\frac{1}{4\pi r^2}\right)^2 |S_\omega|^2 [(kr)^2 + \tfrac{1}{2}] = \rho\left(\frac{1}{2\lambda r}\right)^2 |S_\omega|^2 \left[1 + \frac{1}{2}\left(\frac{\lambda}{2\pi r}\right)^2\right] \quad (7.1.5)$$

$$I_r = \rho c\left(\frac{1}{2\lambda r}\right)^2 |S_\omega|^2 = \frac{|p|^2}{\rho c} \qquad I_\vartheta = I_\varphi = 0$$

$$\Pi = (4\pi r^2)I_r = \rho c \frac{\pi}{\lambda^2} |S_\omega|^2 = \frac{\rho\omega^2}{4\pi c} |S_\omega|^2$$

The pressure amplitude is inversely proportional to the distance r from the center of the sphere; the fluid velocity is entirely radial, being in phase with the pressure at distances large compared with λ and out of phase in the region $r < \lambda$. The time-average intensity $\mathbf{I}$ is also radial, proportional to the square of the pressure amplitude, and thus independent of time; its divergence is zero except at $r = 0$. The total outflow of energy per second from the source is $\pi\rho c(|S_\omega|/\lambda)^2$, independent of r; thus, for a given source strength S_ω, the power radiated is proportional to the square of the frequency.

At distances r from the source which are large compared with the wavelength, the relationships between p, $\mathbf{u}$, and $\mathbf{I}$ approach the simple ones given in Eqs. (6.3.3) for a plane wave, $u_r = p/\rho c$, $w_r = |p|^2/\rho c^2$, and $I_r = |p|^2/\rho c$. The sound field in this region is called the *far field*. Here the fluid velocity is in phase with the pressure, so that all the energy density is radiant, moving outward with the speed of sound, since $I_r = cw_r$.

In contrast, the *near field*, in the region $r \ll \lambda$, has a large velocity component out of phase with the pressure, which is responsible for the presence of additional mean-square energy density $w_x = \frac{1}{2}\rho(|S|/4\pi r^2)^2$, reactive energy which does not radiate outward. The radiant energy in a shell of unit thickness around the source is $\rho k^2 |S_\omega|^2/4\pi$, independent of the radius of the shell. The reactive energy stays close to the source; the total amount outside a radius r_m is $\dfrac{\rho |S_\omega|^2}{8\pi} \dfrac{1}{r_m}$, inversely proportional to r_m but independent of ω.

Two additional properties of the sound field from a simple source will be of utility in our later discussions. The first is the obvious fact that the source need not be at the origin of coordinates. If the *source point* is at $\mathbf{r}_0 = (x_0, y_0, z_0)$, then the pressure at the *measurement point* $\mathbf{r} = (x,y,z)$ is

$$p_\omega(\mathbf{r} \mid \mathbf{r}_0)e^{-i\omega t} = -ik\rho c S_\omega g_\omega(\mathbf{r} \mid \mathbf{r}_0)e^{-i\omega t} \quad (7.1.6)$$

where

$$g_\omega(\mathbf{r} \mid \mathbf{r}_0) = \frac{1}{4\pi R} e^{ikR}$$

and

$$R^2 = |\mathbf{r} - \mathbf{r}_0|^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

The other quantities in Eq. (7.1.5) can be obtained by substituting for r the distance R between source and measurement point. The quantity $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$ is the Green's function for an infinite medium [Eq. (5.1.4)]; we shall discuss its properties and uses later.

The second point to note about the field from a simple source is that it is symmetric with respect to interchange of $\mathbf{r}$ and $\mathbf{r}_0$. In other words, the pressure at the measurement point $\mathbf{r}$, caused by a source at $\mathbf{r}_0$, is equal to the pressure which would be measured at $\mathbf{r}_0$ if the source were placed at $\mathbf{r}$. The Green's function is said to conform to the *principle of reciprocity*.

Dipole and quadrupole sources

To generate the sound field we have just described requires that we, somehow, periodically introduce and withdraw fluid from a small region of space, the source point, or else locate there a sphere which can alternately expand and contract. Although this condition is approximately fulfilled by many musical instruments and some loudspeakers (as we shall see later), in many other cases sound is produced by moving a portion of the fluid back and forth, no new fluid being introduced. The simplest arrangement of this sort can be simulated by two simple sources, close together, opposite in sign and equal in magnitude. In this way fluid is being sucked in by one as it is being ejected by the other, as though it were simply being moved from the one source point to the other. Such a source is appropriately called a *dipole source*.

The resulting radiation field can be derived from that for a simple source. For the simple-harmonic case, if the source of strength S_ω is at $\frac{1}{2}\mathbf{d}$ and the one of strength $-S_\omega$ is at $-\frac{1}{2}\mathbf{d}$, and if $\mathbf{d}$, the vector distance between the two sources, is very small compared with λ, then, from Eq. (7.1.6) by Taylor's theorem,

$$
\begin{aligned}
p = p_d(\omega)e^{-i\omega t} &= -ik\rho c S_\omega[g_\omega(\mathbf{r} \mid \tfrac{1}{2}\mathbf{d}) - g_\omega(\mathbf{r} \mid -\tfrac{1}{2}\mathbf{d})]e^{-i\omega t} \\
&= -ik\rho c D_\omega \ \{\mathrm{grad}_0\,[g_\omega(\mathbf{r} \mid \mathbf{r}_0)]\}_{r_0=0} e^{-i\omega t} \\
&= -k^2 D_\omega \frac{\rho c}{4\pi r}\cos\vartheta\left(1 + \frac{i}{kr}\right)e^{ikr-i\omega t} \\
u_r &= -\frac{k^2 D_\omega}{4\pi r}\cos\vartheta\left(1 + \frac{2i}{kr} - \frac{2}{k^2r^2}\right)e^{ikr-i\omega t} \\
u_\vartheta &= i\frac{kD_\omega}{4\pi r^2}\sin\vartheta\left(1 + \frac{i}{kr}\right)e^{ikr-i\omega t} \qquad (7.1.7) \\
w &= \rho\left(\frac{k^2\,|D_\omega|}{4\pi r}\right)^2\left[\cos^2\vartheta + \frac{1}{2}\left(\frac{1}{kr}\right)^2 + \frac{1}{2}\left(\frac{1}{kr}\right)^4(1 + 3\cos^2\vartheta)\right] \\
I_r &= \rho c\left(\frac{k^2\,|D_\omega|}{4\pi r}\right)^2\cos^2\vartheta \qquad I_\vartheta = I_\varphi = 0 \\
\Pi &= 2\pi r^2\int_0^\pi I_r \sin\vartheta\, d\vartheta = \rho c\,\frac{4\pi^2}{3\lambda^4}\,|D_\omega|^2 = \frac{\rho\omega^4}{12\pi c^3}\,|D_\omega|^2
\end{aligned}
$$

where $\mathbf{D}_\omega$, the vector *dipole strength*, equals $S_\omega \mathbf{d}$, the strength of the component simple sources times the vector distance between them, and ϑ is the angle between $\mathbf{D}_\omega$ and the radius vector $\mathbf{r}$ to the measurement point. As would be expected, this field has directionality, represented by the dependence on angle ϑ.

Even the far field from a dipole has directionality; when $kr \gg 1$, we have $p \to -\rho c(k^2 D_\omega/4\pi r) \cos \vartheta e^{ikr-i\omega t}$, the amplitude varying inversely with r, as with the far field from the simple source, but having its maximum along the dipole axis and dropping to zero at $\vartheta = \frac{1}{2}\pi$, in the plane through the dipole, normal to $\mathbf{D}_\omega$. However, the far field still has the simple relationship valid for a plane wave; to the first order in $1/kr$, $u_r \to p/\rho c$ and $u_\vartheta \to u_\varphi \to 0$; to the second order, $w \to |p|^2/\rho c^2 = \rho |u_r|^2$ and $I_r \to wc$. At these large distances the energy density is all radiant, traveling with velocity c. The total power radiated is more strongly dependent on frequency than is that from the simple source, being proportional to ω^4.

The near field is more complicated in form than that for the simple source; the fluid motion is not purely radial, and the reactive energy is relatively larger and more concentrated near the origin. The radiant energy in a shell of unit thickness is $\rho k^4 |D_\omega|^2/12\pi$, independent of r but proportional to ω^4. The total reactive energy outside radius r_m is $\rho |D_\omega|^2/12\pi r_m{}^3$ when $kr_m \ll 1$, independent of ω but inversely proportional to $r_m{}^3$.

If the dipole is centered at the point $\mathbf{r}_0 = (x_0, y_0, z_0)$, instead of at the origin, and if the axis of spherical coordinates is not pointed along the dipole axis, the dipole vector $\mathbf{D}_\omega$ has components D_x, D_y, D_z, and the pressure wave at point $\mathbf{r}$ can be written in terms of the radius vector $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ and the spherical angles ϑ_R, φ_R of vector $\mathbf{R}$.

$$p = -ik\rho c \mathbf{D}_\omega \cdot \operatorname{grad}_0 [g_\omega(\mathbf{r} \mid \mathbf{r}_0)] e^{-i\omega t} = -ik\rho c \mathbf{D}_\omega \cdot \mathbf{g}'_\omega(\mathbf{r} \mid \mathbf{r}_0) e^{-i\omega t} \quad (7.1.8)$$

where

$$g'_x = \sin \vartheta_R \cos \varphi_R |g'_\omega| \qquad g'_y = \sin \vartheta_R \sin \varphi_R |g'_\omega| \qquad g'_z = \cos \vartheta_R |g'_\omega|$$

$$|g'_\omega| = -\frac{ik}{4\pi R}\left(1 + \frac{i}{kR}\right) e^{ikR}$$

There can also be more complex point sources, made up of assemblages of dipoles. For example, there can be a combination of four simple sources, two of strength $+Q$, placed at the points $x = \frac{1}{2}d$, $y = \frac{1}{2}d$, $z = 0$, and $(-\frac{1}{2}d, -\frac{1}{2}d, 0)$ and two of strength $-Q$ at the points $(+\frac{1}{2}d, -\frac{1}{2}d, 0)$ and $(-\frac{1}{2}d, +\frac{1}{2}d, 0)$, which correspond to two dipoles of strength Qd, one pointed along the positive x axis with its center at $(0, +\frac{1}{2}d, 0)$, the other pointed along the negative x axis with its center at $(0, -\frac{1}{2}d, 0)$. Extension of the argument resulting in Eq. (7.1.7) shows that if $d \ll \lambda$,

$$p = -ik\rho c Q_{xy} \left[\frac{\partial^2}{\partial x_0 \, \partial y_0} g_\omega(\mathbf{r} \mid \mathbf{r}_0)\right]_{r_0=0} e^{-i\omega t}$$

$$= i\rho c k^3 Q_{xy} \frac{xy}{4\pi r^3}\left[1 + \frac{3i}{kr} - 3\left(\frac{1}{kr}\right)^2\right] e^{ikr-i\omega t} \quad (7.1.9)$$

where $Q_{xy} = Qd^2$. Such a combination of sources is called a tesseral *quadrupole* of strength Q_{xy} [it has dimensions of (length)^5 per time]. There can be three such arrangements, $Q_{xy} = Q_{yx}$, $Q_{xz} = Q_{zx}$, and $Q_{yz} = Q_{zy}$, lying in the three rectangular coordinate planes and having similar expressions for p, except for the factors xy, xz, or yz and for the magnitude of the strengths.

There can also be a pair of dipoles, pointed along the x axis and spaced along the x axis, which would correspond to a simple source of strength $-2S$ at the origin and two of strength S, placed at $(+\frac{1}{2}d, 0, 0)$ and $(-\frac{1}{2}d, 0, 0)$. This would produce a pressure wave of the form

$$p = -ik\rho cQ_{xx}\left[\frac{\partial^2}{\partial {x_0}^2} g_\omega(\mathbf{r} \mid \mathbf{r}_0)\right]_{r_0=0} e^{-i\omega t}$$

$$= i\rho ck^3 Q_{xx} \frac{1}{4\pi r}\left[\left(\frac{x}{r}\right)^2 + \frac{3x^2 - r^2}{r^2}\left(\frac{i}{kr} - \frac{1}{k^2r^2}\right)\right] e^{ikr-i\omega t} \quad (7.1.10)$$

where $Q_{xx} = Sd^2$. This is called an axial *quadrupole source* of strength Q_{xx}. Similar arrangements along the other axes would produce Q_{yy} and Q_{zz} with similar pressure waves, using y^2 and z^2 instead of x^2.

We thus see that the possible quadrupole source strengths can be arranged in tensor form.

$$\mathfrak{Q}_\omega = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix}$$

There are only six independent quadrupole components, however, for the tensor is symmetric, Q_{xy} being equal to Q_{yx}, etc. As a matter of fact, there are only five distinct quadrupole waves, for a sum of three axial quadrupoles Q_{xx}, Q_{yy}, Q_{zz}, of equal magnitude, produces (outside the source region) a wave which is indistinguishable from a simple-source wave. This can be seen by adding three expressions like that of Eq. (7.1.10) and remembering that $x^2 + y^2 + z^2 = r^2$, or else by recognizing that

$$\left(\frac{\partial^2}{\partial {x_0}^2} + \frac{\partial^2}{\partial {y_0}^2} + \frac{\partial^2}{\partial {z_0}^2}\right) g_k(\mathbf{r} \mid \mathbf{r}_0) = \nabla_0^2 g_k = -k^2 g_k(\mathbf{r} \mid \mathbf{r}_0)$$

from the wave equation. Of course, g_k itself is the spatial factor for the wave from a simple source, as given by Eq. (7.1.6). Because of the extra k^2 factor, the magnitude of the simple-source wave coupled to the quadrupole wave is small if the dimensions of the quadrupole are small.

There can be poles of still higher order, though they are of less practical interest. Considering the simple source (monopole) to be a pole of zero order, a dipole to have order $m = 1$, and a quadrupole to be a pole of order $m = 2$, then a pole of order three would be called an octupole, and so on, the number of simple sources required to generate a wave of order m being 2^m.

Source impedances

To produce any of these elementary waveforms, some mechanism, of dimensions very small compared with the wavelength, must be pushing the fluid about rhythmically. To produce a simple source wave, S_ω volume units of fluid is pushed out in all directions from the source, and then back again; for the dipole the fluid is pushed back and forth along some axis; for the quadrupole there must be two opposed motions. To produce such motion requires a driving force, acting somehow on the fluid, capable of contributing energy to the wave. The exact details will depend on the particular mechanism which is producing the wave, but for each elementary type we can calculate, from the near field, a typical driving pressure which could produce such a wave. The effective driving force and responding motion can be related in several ways; an rms pressure and the corresponding flux (volume per second) or an effective force (pressure times area) and a corresponding velocity. In one case the ratio is an analogous impedance [Eq. (9.1.10)]; in the other case it is a mechanical impedance [Eq. (2.3.2)]; in either case the real part of the product (pressure or force) times flux or velocity will equal the time average of the power radiated.

For example, in the simple source the flow is uniformly radial, so that, at the surface of the sphere of radius $a \ll \lambda$, just enclosing the generating mechanism, the radial flux is the source strength S_ω times the phase factor $e^{ik(a-ct)}$, and the radial velocity U_s is equal to this flux divided by the surface area $4\pi a^2$. The driving force F_s at $r = a$ required to produce this radial velocity is the pressure P_s at $r = a$ times the surface area $4\pi a^2$. Thus, from Eq. (7.1.5), the simple-source driving force, velocity, mechanical impedance, and total power radiated are

$$\begin{aligned} F_s &= -i\rho ckaS_\omega e^{ik(a-ct)} \qquad U_s = \frac{S_\omega}{4\pi a^2}(1 - ika)e^{ik(a-ct)} \\ Z_s &= \frac{F_s}{U_s} = -ik\rho c(4\pi a^3)(1 - ika)^{-1} \\ &\simeq -i\omega(4\pi\rho a^3) + \rho c(4\pi a^2)(ka)^2 = -iX_s + R_s \\ \Pi_s &= \mathrm{Re}\,(F_s U_s^*) = R_s\,|U_s|^2 = \frac{\rho\omega^2}{4\pi c}\,|S_\omega|^2 \end{aligned} \tag{7.1.11}$$

where the approximate values are, for $ka \ll 1$, appropriate for these elementary sources. This equivalent mechanical impedance has a reactive term equivalent to the mass load three times the mass of a sphere of radius a filled with fluid. The mechanical resistance is $4\pi a^2$ times ρc, the characteristic impedance of the medium, times the small quantity $(ka)^2$, but this small term cannot be neglected, for it is the real part of the impedance and thus is responsible for the power radiated.

Alternatively, we could have used the pressure P_s and the flux S_s and obtained an analogous impedance Z_{as}.

$$P_s = -\frac{i\rho ckA}{4\pi a^2} S_\omega e^{ik(a-ct)} \qquad S_s = S_\omega(1 - ika)e^{ik(a-ct)}$$

$$Z_{as} = \frac{P_s}{S_s} = -\frac{ik\rho c}{4\pi a}(1 - ika)^{-1} \simeq -i\omega\frac{\rho}{4\pi a} + \rho c\frac{k^2}{4\pi}$$

The power radiated, $\mathrm{Re}\,(P_s S_s^*)$, is equal to the Π_s given in Eq. (7.1.11), as of course it must.

In the case of the dipole, the flow at the surface $r = a$ is back and forth in the direction of the dipole axis, so that both radial velocity and pressure are proportional to the cosine of the angle ϑ between $\mathbf{r}$ and $\mathbf{D}_\omega$. The force required to push the sphere of radius a to and fro is the integral of the component of the surface force along $\mathbf{D}_\omega$, over the sphere's surface,

$$\mathbf{F}_d = -ik\rho c\mathbf{D}_\omega(1 - ika)\tfrac{1}{2}\int_0^\pi \cos^2\vartheta \sin\vartheta\, d\vartheta\, e^{ik(a-ct)}$$
$$= -\tfrac{1}{3}ik\rho c\mathbf{D}_\omega(1 - ika)e^{ik(a-ct)}$$

and the corresponding velocity $\mathbf{U}_d$ is the maximum radial velocity at the sphere's surface (for $\vartheta = 0$). It thus is the velocity the sphere would have if it were producing the dipole by moving to and fro bodily. If the dipole is at an angle with respect to the coordinate axes, with components D_x, D_y, D_z, the pressure wave is

$$p_d = -ik\rho c\frac{D_\omega}{4\pi a^2}(1 - ika)\left(\frac{x}{r}D_x + \frac{y}{r}D_y + \frac{z}{r}D_z\right)$$

In this case we find the component along D_ω of the force acting on the sphere by multiplying p_d by the directionality factor

$$\frac{1}{D_\omega}\left(\frac{x}{r}D_x + \frac{y}{r}D_y + \frac{z}{r}D_z\right)$$

and integrating over the surface of the sphere. Since

$$D_\omega = (D_x^2 + D_y^2 + D_z^2)^{\frac{1}{2}}$$

is the magnitude of the dipole strength, we end up with the same result as given above.

Similarly with the quadrupole source. We define the magnitude of such a source as $Q_\omega = [(Q_{xy} + Q_{yx})^2 + (Q_{xz} + Q_{zx})^2 + (Q_{yz} + Q_{zy})^2 + \frac{1}{3}(2Q_{zz} - Q_{xx} - Q_{yy})^2 + (Q_{xx} - Q_{yy})^2]$ so as to combine the axial terms to eliminate the simple source part [see also Eq. (7.2.24)]. We then multiply the expression, obtained from Eqs. (7.1.9) and (7.1.10), for the total quadrupole

pressure at $r = a$,

$$-3\frac{ik\rho c}{4\pi a^3}\left\{(1 - ika - \tfrac{1}{3}k^2a^2)\left[(Q_{xy} + Q_{yx})\frac{xy}{a^2} + \cdots + \tfrac{1}{2}(Q_{xx} - Q_{yy})\frac{x^2 - y^2}{a^2}\right.\right.$$
$$\left.\left. + (Q_{zz} - \tfrac{1}{2}Q_{xx} - \tfrac{1}{2}Q_{yy})\frac{z^2 - \frac{1}{3}a^2}{a^2}\right] + \tfrac{1}{3}k^2a^2(Q_{xx} + Q_{yy} + Q_{zz})\right\}e^{ik(a-ct)}$$

(where $x^2 + y^2 + z^2 = a^2$) by the "directionality factor"

$$\frac{1}{Q_\omega}\left[(Q_{xy} + Q_{yx})\frac{xy}{a^2} + \cdots + \tfrac{1}{2}(Q_{xx} - Q_{yy})\frac{x^2 - y^2}{a^2}\right.$$
$$\left. + (Q_{zz} - \tfrac{1}{2}Q_{xx} - \tfrac{1}{2}Q_{yy})\frac{z^2 - \frac{1}{3}a^2}{a^2}\right]$$

and integrate over the surface of the sphere, obtaining the "component of the applied force" in the direction of the quadrupole. The cross terms integrate to zero; only the squared terms remain, giving a factor $Q_\omega^2/Q_\omega = Q_\omega$. The final results for the effective force, velocity, impedance, and power radiated, for dipole and quadrupole, are

$$F_d = -\tfrac{1}{3}ik\rho cD_\omega(1 - ika)e^{ik(a-ct)}$$
$$U_d = \frac{D_\omega}{2\pi a^3}(1 - ika - \tfrac{1}{2}k^2a^2)e^{ik(a-ct)}$$
$$Z_d \simeq -i\omega\rho\frac{2\pi a^3}{3}[1 + \tfrac{1}{2}(ka)^2] + \rho c(4\pi a^2)\frac{(ka)^4}{12}$$
$$\Pi_d = R_d|U_d|^2 \simeq \frac{\rho\omega^4}{12\pi c^3}|D_\omega|^2$$
$$F_q = -i\frac{k\omega c}{5a}Q_\omega(1 - ika - \tfrac{1}{3}k^2a^2)e^{ik(a-ct)} \qquad (7.1.12)$$
$$U_q = \frac{9}{4\pi a^4}Q_\omega(1 - ika - \tfrac{4}{9}k^2a^2) + \tfrac{1}{9}ik^3a^3)e^{ik(a-ct)}$$
$$Z_q \simeq -i\omega\rho\frac{4\pi a^3}{45}[1 + \tfrac{1}{9}(ka)^2 + \tfrac{4}{81}(ka)^4] + \rho c(4\pi a^2)\frac{(ka)^6}{1{,}215}$$
$$\Pi_q \simeq \frac{\rho\omega^6}{60\pi c^5}|Q_\omega|^2$$

The reactive load for the monopole of Eq. (7.1.11) is the mass of a volume $4\pi a^3$ of the fluid, three times the volume of the sphere of radius a; that for the dipole is $\frac{1}{2}$ of the sphere's mass, and that of the quadrupole is $\frac{1}{15}$ of it. For a pole of order m, the reactive load is $3/[(m + 1)\,1 \cdot 3 \cdot 5 \cdots (2m + 1)]$ of the sphere's mass, for $ka \ll 1$ (the order of the monopole is $m = 0$, that for the dipole is $m = 1$, etc.). The resistive load, the first real term in the expansion of Z in powers of ka, diminishes much more rapidly than this; it is

$[\rho c(4\pi a^2)][(ka)^{2m}/(m+1)1\cdot 3\cdot 5\cdots(2m+1)]$. These expressions are valid when the multipoles radiate into unbounded space. If the fluid is confined within boundaries, which are a finite distance away, both the waveform and the value of the acoustic impedance are changed, as will be shown later.

Radiation from nonperiodic multipoles

Now that we have worked out the radiation from simple-harmonic multipole sources, we can use the Fourier transform to calculate the radiation from multipoles with strengths which vary in an arbitrary way with time, of the sort already mentioned for the simple source. We consider the periodic sources just discussed, with strengths S_ω, $\mathbf{D}_\omega$, etc., to be the components of the general source which has frequency $\omega/2\pi$, and we add all the components of different frequencies together to make up the general wave. In other words, we assume that the strengths S_ω, $\mathbf{D}_\omega$, etc., are the Fourier transforms of the general time-varying sources of strengths $S(t)$, $\mathbf{D}(t)$, $\mathfrak{Q}(t)$, according to the procedure of Eqs. (1.3.16).

$$S_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(t)e^{i\omega t}\,dt \qquad \mathbf{D}_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{D}(t)e^{i\omega t}\,dt \qquad \cdots$$

Having calculated the radiation from the component simple-harmonic sources, we can then obtain the complete field by taking the inverse Fourier transform of the spatial part $p(\omega)$,

$$p_s(\mathbf{r},t) = \int_{-\infty}^{\infty} p_s(\omega)e^{-i\omega t}\,d\omega \qquad p_d(\mathbf{r},t) = \int_{-\infty}^{\infty} p_d(\omega)e^{-i\omega t}\,d\omega \qquad \cdots$$

To carry this out we note that, if $F(\omega)$ is the Fourier transform of $f(t)$, as given in Eqs. (1.3.16), then the inverse transform of $-i\omega F(\omega)$ is the time derivative $f'(t)$ of f, $-\omega^2 F(\omega)$ transforms to the second time derivative $f''(t)$, and so on. From Eqs. (7.1.5) we can thus obtain

$$p_s(\mathbf{r},t) = \frac{\rho}{4\pi r}\int_{-\infty}^{\infty}(-i\omega)S_\omega \exp\left(-i\omega t + i\frac{\omega}{c}r\right)d\omega = \frac{\rho}{4\pi r}S'\left(t-\frac{r}{c}\right)$$

which corresponds to Eq. (7.1.4) for the nonperiodic simple source.

Similarly, from Eqs. (7.1.7) to (7.1.10), we can compute the pressure waves from the nonperiodic dipole and quadrupole sources.

$$p_d(\mathbf{r},t) = \frac{\rho}{4\pi c r}\frac{\mathbf{r}}{r}\cdot\left[\mathbf{D}''\left(t-\frac{r}{c}\right) + \frac{c}{r}\mathbf{D}'\left(t-\frac{r}{c}\right)\right] \tag{7.1.13}$$

$$p_q(\mathbf{r},t) = \frac{\rho}{4\pi c^2 r}\left\{\frac{\mathbf{r}}{r}\cdot\left[\mathfrak{Q}'''\left(t-\frac{r}{c}\right) + \frac{3c}{r}\mathfrak{Q}''\left(t-\frac{r}{c}\right) + 3\left(\frac{c}{r}\right)^2\mathfrak{Q}'\left(t-\frac{r}{c}\right)\right]\cdot\frac{\mathbf{r}}{r} - \left|\frac{c}{r}\mathfrak{Q}''\left(t-\frac{r}{c}\right) + \left(\frac{c}{r}\right)^2\mathfrak{Q}'\left(t-\frac{r}{c}\right)\right|\right\}$$

where vector $\mathbf{r}$ has the components (x,y,z), so that $(\mathbf{r}/r) \cdot \mathbf{D}''$, for example, is $(x/r)D_x'' + (y/r)D_y'' + (z/r)D_z''$, and

$$\frac{\mathbf{r}}{r} \cdot \mathfrak{Q} \cdot \frac{\mathbf{r}}{r} = \left[\left(\frac{xx}{r^2}\right)Q_{xx} + \left(\frac{xy}{r^2}\right)Q_{xy} + \cdots + \left(\frac{yz}{r^2}\right)Q_{yz} + \left(\frac{zz}{r^2}\right)Q_{zz}\right]$$

Q_{xx}, Q_{xy}, etc., being the elements of the matrix $\mathfrak{Q}$ of quadrupole strengths. The symbol $|\mathfrak{Q}''|$ around matrix $\mathfrak{Q}''$ denotes the sum of the diagonal elements, $Q_{xx}'' + Q_{yy}'' + Q_{zz}''$, etc.

Thus we see that all the waves are propagated with velocity c and that, far from the source, the wave is propagated with unchanged shape but with amplitude inversely proportional to r. The shape of the far field from the simple source is proportional to the first time derivative of the source strength $S(t)$, as we noted earlier. The shape of the far field from the dipole is proportional to the second time derivative of $\mathbf{D}(t)$, and that from the quadrupole is proportional to the third time derivative of the components of the matrix $\mathfrak{Q}(t)$. The near field for the dipole and quadrupole has terms proportional to the first derivative of the source strengths. The dipole field is oriented along the direction of the dipole, and the far field from the quadrupole has the angular symmetry of $\mathfrak{Q}$, but the near field has an additional isotropic term, coming from the diagonal terms. Another aspect of the property of the quadrupole field was mentioned on page 314.

The Green's function

Before applying these results to specific situations of interest, we should say more about the function $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$ of Eq. (7.1.6). In the first place, it is a solution, for the unbounded medium, of the equation

$$\nabla^2 g_\omega(\mathbf{r} \mid \mathbf{r}_0) + k^2 g_\omega(\mathbf{r} \mid \mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0) \qquad (7.1.14)$$

where $k = \omega/c$, and $\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)$ is the delta function for three dimensions [Eq. (1.3.24)]. Thus g_ω is the spatial factor for a wave from a unit, simple-harmonic, point source at $\mathbf{r}_0$. Its form, as we have mentioned earlier, is $(1/4\pi R)e^{ikR}$, where $R = |\mathbf{r} - \mathbf{r}_0|$.

To verify this we first note that g_ω is a solution of the homogeneous equation $\nabla^2 g + k^2 g = 0$ everywhere except at the point $\mathbf{r}_0$. Next we integrate both sides of Eq. (7.1.14) (in the coordinates x, y, z of $\mathbf{r}$) over the volume of a sphere of very small radius a, centered at $\mathbf{r}_0$. The integral of the right-hand side, because of the properties of the delta function, is -1. The integral of the $k^2 g_\omega$ term goes to zero as $a^3 \to 0$. The integral of the $\nabla^2 g_\omega$ term can be obtained by remembering that $\nabla^2 g_\omega$ is the divergence of the gradient of g_ω. Now the volume integral of the divergence of a vector is equal to the surface integral of the outward, normal component of the vector, over the surface bounding the volume. Therefore the integral of $\nabla^2 g_\omega$ over the volume of a sphere, with $\mathbf{r}_0$ the center, is equal to the radial gradient of g_ω, integrated over

the surface of the sphere. This radial gradient, with center at $\mathbf{r}_0$, is $\partial g_\omega/\partial R = [-(1/4\pi R^2) + (ik/4\pi R)]e^{ikR}$, which is constant in value over the spherical surface $R = a$. Thus the volume integral of $\nabla^2 g_\omega$ over the interior of the sphere is equal to $(\partial g_\omega/\partial R)_{R=a}$ times the surface area $4\pi a^2$ of the sphere. When $a \to 0$ this approaches the value -1, which thus equals the right-hand side, verifying the form of g_ω.

We should note that the most general solution of Eq. (7.1.14) is $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$ plus any solution $\chi(\mathbf{r})$ of the homogeneous equation $\nabla^2\chi + k^2\chi = 0$. We have chosen the particular form g_ω as being appropriate for outgoing radiation in an unbounded medium. If the medium is bounded by some surface on which boundary conditions are imposed, we may have to use the function

$$G_\omega(\mathbf{r} \mid \mathbf{r}_0) = g_\omega(\mathbf{r} \mid \mathbf{r}_0) + \chi(\mathbf{r}) \tag{7.1.15}$$

in order to satisfy these boundary conditions. Thus G_ω is a solution of Eq. (7.1.14) which satisfies appropriate boundary conditions along boundaries within finite distance of the source; if no such boundaries exist, $\chi = 0$. We shall later show that this more general Green's function G_ω also satisfies the principle of reciprocity, $G_\omega(\mathbf{r} \mid \mathbf{r}_0) = G_\omega(\mathbf{r}_0 \mid \mathbf{r})$.

To demonstrate the utility of the Green's function G_ω, we consider the following problem: We wish to solve the equation

$$\nabla^2 p_\omega + k^2 p_\omega = -f_\omega(\mathbf{r}) \tag{7.1.16}$$

for the spatial factor of a pressure wave from a simple-harmonic, distributed source, represented by the function $f_\omega(\mathbf{r})e^{-i\omega t}$. We multiply Eq. (7.1.14) for G_ω by p and subtract from it G_ω times the equation for p, obtaining

$$G_\omega(\mathbf{r} \mid \mathbf{r}_0)\,\nabla^2 p_\omega(\mathbf{r}) - p_\omega(\mathbf{r})\,\nabla^2 G_\omega(\mathbf{r} \mid \mathbf{r}_0) = p_\omega(\mathbf{r})\,\delta(\mathbf{r} - \mathbf{r}_0) - G_\omega(\mathbf{r} \mid \mathbf{r}_0) f_\omega(\mathbf{r})$$

Now interchange $\mathbf{r}$ and $\mathbf{r}_0$, utilize the symmetry of G_ω and $\delta(\mathbf{r} - \mathbf{r}_0)$ with respect to this interchange, and integrate with respect to x_0, y_0, z_0 over the whole volume occupied by the medium, obtaining

$$\iiint [G_\omega(\mathbf{r} \mid \mathbf{r}_0)\,\nabla_0^2 p_\omega(\mathbf{r}_0) - p_\omega(\mathbf{r}_0)\,\nabla_0^2 G_\omega(\mathbf{r} \mid \mathbf{r}_0)]\,dv_0$$

$$= \iiint p_\omega(\mathbf{r}_0)\,\delta(\mathbf{r} - \mathbf{r}_0)\,dv_0 - \iiint f_\omega(\mathbf{r}_0) G_\omega(\mathbf{r} \mid \mathbf{r}_0)\,dv_0$$

where the zero subscripts indicate differentiation and integration with respect to the x_0, y_0, z_0 coordinates.

The quantity in brackets is the divergence of the vector $G_\omega \operatorname{grad}_0 p_\omega - p_\omega \operatorname{grad}_0 G_\omega$, so that this volume integral can be changed to a surface integral, over the surface enclosing the medium, of the component of this vector, normal to the surface, pointing outward, *away from the medium*. Writing this normal component of grad_0 as $\partial/\partial n_0$ and utilizing Eq. (1.3.24) for the

delta function, we obtain

$$p_\omega(\mathbf{r}) = \iiint f_\omega(\mathbf{r}_0) G_\omega(\mathbf{r} \mid \mathbf{r}_0)\, dv_0 + \iint \left[G_\omega(\mathbf{r} \mid \mathbf{r}_0) \frac{\partial}{\partial n_0} p_\omega(\mathbf{r}_0) - p_\omega(\mathbf{r}_0) \frac{\partial}{\partial n_0} G_\omega(\mathbf{r} \mid \mathbf{r}_0) \right] dS_0 \quad (7.1.17)$$

which is an equation for the spatial factor $p_\omega(\mathbf{r})$ of the pressure wave *within and on the surface* bounding the medium, in terms of a volume integral of the source function f_ω over the bounded volume and a surface integral of the boundary values of p_ω and its outward-pointed normal gradient over the boundary surface. Because we integrated over the bounded volume, the equation is not valid for $\mathbf{r}$ outside this volume. This formula is not very mysterious. Since $G_\omega(\mathbf{r} \mid \mathbf{r}_0)$ is the field at $\mathbf{r}$ due to a point source at $\mathbf{r}_0$, it states that the total field at r is the summation of the fields from all the elementary sources $f_\omega\, dv_0$ plus the waves reflected by the boundary surface.

This integral expression will be used frequently in later sections when we consider the effects of boundaries on the wave. In this section we are considering waves radiated out into an unbounded medium, in which case $G_\omega = g_\omega = (1/4\pi R)e^{ikR}$, and if p_ω consists solely of outward-going waves, the surface integral vanishes. Thus, for the unbounded medium, the equation becomes

$$p_\omega(\mathbf{r}) = \iiint f_\omega(\mathbf{r}_0) g_\omega(\mathbf{r} \mid \mathbf{r}_0)\, dx_0\, dy_0\, dz_0$$

All the simple-source waves, from the various elementary sources $f_\omega(\mathbf{r}_0)\, dv_0$, add together to produce the resultant field at $\mathbf{r}$, the measurement point.

At times we may wish to calculate the radiation from a source distribution without first analyzing the frequency components f_ω of the source. We can then utilize the result of Prob. 23 of Chap. 1. If, for example, $f_\omega(\mathbf{r})$ is the Fourier transform $(1/2\pi)\int f(\mathbf{r},t)e^{i\omega t}\, dt$ of the source function $f(\mathbf{r},t)$, the equation for the total pressure wave radiated from the distributed source is

$$p(\mathbf{r},t) = \iiint dv_0 \int_{-\infty}^{\infty} f(\mathbf{r}_0,t_0) g(\mathbf{r}, t \mid \mathbf{r}_0, t_0)\, dt_0$$

where $g(\mathbf{r}, t \mid \mathbf{r}_0, t_0) = (1/2\pi)g(\mathbf{r}; \mathbf{r}_0; t - t_0)$ is $1/2\pi$ times the inverse transform of the Green's function for frequency $\omega/2\pi$.

$$\frac{1}{2\pi} g(\mathbf{r};\mathbf{r}_0;\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4\pi R} e^{i(\omega/c)R - i\omega\tau}\, d\omega = \frac{1}{4\pi R} \delta\left(\tau - \frac{R}{c}\right)$$

using Eq. (1.3.25) for the inverse transform of the delta function.

Therefore we can, if more convenient, compute the total radiation from a nonperiodic source distribution by carrying out the following integration in space *and* time:

$$p(\mathbf{r},t) = \iiiint f(\mathbf{r}_0,t_0)g(\mathbf{r}, t \mid \mathbf{r}_0, t_0)\, dx_0\, dy_0\, dz_0\, dt_0$$

$$g(\mathbf{r}, t \mid \mathbf{r}_0, t_0) = \frac{1}{4\pi R}\,\delta\left(t - t_0 - \frac{R}{c}\right) \qquad \mathbf{R} = \mathbf{r} - \mathbf{r}_0 \quad (7.1.18)$$

This time-dependent Green's function is the solution of the equation

$$\nabla^2 g - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = -\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)\,\delta(t - t_0)$$

for the pulse-wave from a point pulse source at $\mathbf{r}_0$, t_0. The pulse travels outward with velocity c. Note the relationship with Eq. (2.3.9) and the fact that Eq. (7.1.4) can be obtained directly from Eq. (7.1.18).

Radiation from a region of violent fluid motion

As a first illustration of the power of the integral equation (7.1.17), let us work out the radiation, out into unbounded space, from a small region of the fluid which is in violent motion. Inside this region the motions of the air are great enough so that the linear equations of wave motion no longer hold; energy may be fed into the medium by the introduction of new fluid, by forces acting directly on portions of the fluid, and by the nonlinear conversion of mass flow into turbulent motion. Within this small region, therefore, the equations of fluid motion must include all the nonlinear terms we neglected when we obtained the linear wave equation. It is still possible, however, to neglect the damping effects of viscosity and thermal diffusion, which were treated in Sec. 6.4, as long as rigid boundaries are not present.

The sources of sound, which feed energy into the acoustic field, are usually of three sorts, monopole, dipole, and quadrupole, corresponding to three ways of feeding energy into the field. The monopole sources correspond to the introduction of new fluid into the disturbed region, which may occur when an explosion or combustion changes a solid or liquid into a gas. Or they may correspond to the introduction of heat (as when a modulated laser or other focused radiation heats the fluid), which expands the fluid irregularly. Or else they may be caused by time or space variations of the compressibility, produced by turbulence or by irregular heating.

The equation of continuity (6.1.15) for mass flow is $(\partial\rho/\partial t) + \text{div}\,(\rho\mathbf{u}) = \rho q(\mathbf{r},t)$, where q is the rate of introduction of new fluid volume into the region, and thus ρq is the rate of introduction of new fluid mass. Using Eqs. (6.4.21), relating change of ρ to change of pressure and temperature, results in

$$\frac{\partial\rho}{\partial t} \equiv \frac{\partial\delta}{\partial t} \simeq \gamma\rho\kappa_s\frac{\partial}{\partial t}(p - \alpha\tau)$$

If heat energy $\epsilon(t)$ per unit mass is introduced into the fluid, Eq. (6.4.18) becomes

$$T\frac{\partial\sigma}{\partial t} \simeq K\nabla^2\tau + \epsilon(t) \qquad \text{where } \sigma = \frac{C_p}{T}\left(\tau - \frac{\gamma-1}{\alpha\gamma}p\right)$$

Since we can neglect thermal conduction in the main body of the fluid, we have

$$\frac{\partial}{\partial t}(p - \alpha\tau) \simeq \frac{1}{\gamma}\frac{\partial p}{\partial t} - \frac{\alpha\epsilon}{C_p}$$

and the equation of continuity for mass flow becomes

$$\rho\kappa\frac{\partial P}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = \rho q + \frac{\alpha\gamma\rho\kappa}{C_p}\epsilon \tag{7.1.19}$$

where the subscript s on κ is understood; we use only the adiabatic compressibility from now on. Also, but in this discussion only, we shall use ρ instead of $\rho + \delta$ as the actual density of the fluid, and P as its total pressure instead of $P + p$; of course, $\partial P/\partial t$ equals the $\partial p/\partial t$ used elsewhere.

By using the thermodynamic relations of Eqs. (6.4.19), the heat-energy term can be alternatively written

$$\frac{\alpha\gamma\rho\kappa}{C_p}\epsilon = \frac{\beta}{C_p}H = (\gamma-1)\frac{H}{T\beta c^2} \to -(\gamma-1)\frac{H}{c^2}$$

where H is the heat energy added per unit volume ($= \rho\epsilon$). The last expression is the limiting form for a perfect gas.

The equation of continuity (6.2.11) for momentum flow can be written as

$$\frac{\partial\rho\mathbf{u}}{\partial t} = \mathbf{F} - \operatorname{grad} P - \boldsymbol{\nabla}\cdot(\mathfrak{J} + \mathfrak{D}) \tag{7.1.20}$$

where $\mathbf{F}(\mathbf{r},t)$ is the force (electric or gravitational or otherwise) acting on a unit volume of fluid at $\mathbf{r}$, t, and $\mathfrak{J}$ is the momentum-flux tensor of Eq. (6.2.10), with typical component $J_{xy} = \rho u_x u_y$. We have separated the pressure P out of the stress tensor $\mathfrak{P}$ of Eq. (6.2.10), so that, as given in Eq. (6.4.9),

$$-\mathfrak{D} = P\mathfrak{I} - \mathfrak{P} \qquad -D_{xx} = P - P_{xx} \qquad -D_{xy} = -P_{xy} \qquad \cdots$$

since we expect to obtain an equation for P. The elements of the tensor $\mathfrak{D}$ are the components of the stress of viscosity in the violently moving fluid, discussed in Sec. 6.4. The total tensor in the last term of Eq. (7.1.20),

$$\mathfrak{T} = \mathfrak{J} + \mathfrak{D} \qquad T_{xx} = P_{xx} - P + \rho u_x u_x \qquad T_{xy} = P_{xy} + \rho u_x u_y = T_{yx} \qquad \cdots$$

is the stress-momentum tensor, exclusive of fluid pressure defined in Eqs. (6.2.11) and (7.1.20). In fluids such as air and water, its components are small whenever $|\mathbf{u}|^2$ can be neglected compared with $|\mathbf{u}|$.

As before, to obtain a wave equation for P, we differentiate Eq. (7.1.19) with respect to t, take the divergence of Eq. (7.1.20), and then eliminate the $\rho\mathbf{u}$ term between them. In a region of the sort we are discussing, neither ρ nor κ will be uniform with respect to either space or time; in fact, the composition of the fluid may be nonuniform. Therefore the time derivative of the first term of Eq. (7.1.19) is

$$\rho\kappa\frac{\partial^2 P}{\partial t^2}+\frac{\partial P}{\partial t}\left[\kappa\frac{\partial\delta}{\partial t}+\rho\left(\frac{\partial\kappa}{\partial P}\right)_T\frac{\partial}{\partial t}(p-\alpha\tau)\right]+\rho\left(\frac{\partial\kappa}{\partial t}\right)_{PT}\frac{\partial P}{\partial t}$$
$$=\rho\kappa\frac{\partial^2 P}{\partial t^2}+\frac{\rho}{\gamma}\left[\kappa^2+\left(\frac{\partial\kappa}{\partial P}\right)_T\right]\left(\frac{\partial P}{\partial t}-\frac{\alpha\gamma\epsilon}{C_p}\right)\frac{\partial P}{\partial t}+\rho\left(\frac{\partial\kappa}{\partial t}\right)_{PT}\frac{\partial P}{\partial t}$$

and the time derivative of the last term is

$$\frac{\alpha\gamma\rho\kappa}{C_p}\frac{\partial\epsilon}{\partial t}+\frac{\alpha\gamma\rho\epsilon}{C_p}\left(\frac{\partial\kappa}{\partial t}\right)_{TP}+\alpha\rho\left[\kappa^2+\left(\frac{\partial\kappa}{\partial P}\right)_T\right]\left(\frac{\partial P}{\partial t}-\frac{\alpha\gamma\epsilon}{C_p}\right)\frac{\epsilon}{C_p}$$

We have assumed that the compressibility κ at $\mathbf{r}$ may depend explicitly on time because the fluid composition at $\mathbf{r}$ may change with time, in addition to the implicit dependence of κ on t through the variation of P and T when the composition does not change.

Thus the final wave equation for P, with the thermal and viscous terms neglected (except for the heat source term ϵ), is

$$\nabla^2 P-\rho\kappa\frac{\partial^2 P}{\partial t^2}\simeq -f(\mathbf{r},t)+\operatorname{div}\mathbf{F}(\mathbf{r},t)-\boldsymbol{\nabla}\cdot\mathfrak{T}(\mathbf{r},t)\cdot\boldsymbol{\nabla}\qquad(7.1.21)$$

where the scalar quantity

$$\boldsymbol{\nabla}\cdot\mathfrak{T}\cdot\boldsymbol{\nabla}=\frac{\partial^2}{\partial x^2}T_{xx}+\frac{\partial^2}{\partial x\,\partial y}(T_{xy}+T_{yx})+\cdots$$
$$\simeq\frac{\partial^2}{\partial x^2}\rho u_x{}^2+\frac{\partial^2}{\partial x\,\partial y}(\rho u_x u_y+\rho u_y u_x)+\cdots$$

and where the monopole source term is

$$f(\mathbf{r},t)\equiv\frac{\partial}{\partial t}\rho s(\mathbf{r},t)=\frac{\partial}{\partial t}\rho q+\frac{\alpha\gamma\rho\kappa}{C_p}\frac{\partial\epsilon}{\partial t}-\rho\left(\frac{\partial\kappa}{\partial t}\right)_{PT}\left(\frac{\partial P}{\partial t}-\frac{\alpha\gamma\epsilon}{C_p}\right)$$
$$-\frac{\rho}{\gamma}\left[\kappa^2+\left(\frac{\partial\kappa}{\partial P}\right)_T\right]\left(\frac{\partial P}{\partial t}-\frac{\alpha\gamma\epsilon}{C_p}\right)^2\qquad(7.1.22)$$

The function $s(\mathbf{r},t)$ is the effective monopole source-strength density, the generalization of the point source strength S of Eq. (7.1.4) for a distributed source. The last two terms are usually small compared with the first two. We thus see that, even if no new fluid is introduced (even if q is zero), the monopole source term is not necessarily zero in a region where heat energy is being introduced or where the composition of the fluid is nonuniform in space and time.

This equation, with its higher-order terms on the right-hand side, is usually impossible to solve exactly. But for our purposes here, we can consider the three terms on the right of Eq. (7.1.21), the monopole-source term f, the force term $\mathbf{F}$, and the stress-momentum term in $\mathfrak{T}$, as source terms, either determined by measurement or by calculation. They are non-negligible only in the region of violent fluid motion; in all the rest of the infinite region they are zero, and Eq. (7.1.21) is the usual linear wave equation for the acoustic pressure p. Here we shall assume that the values of the functions on the right-hand side are known; they are the source of new energy being injected into the sound field.

Radiation from a small region

To solve for the radiation field outside the source region, we first take the Fourier transform of Eq. (7.1.21), defining

$$p_\omega(\mathbf{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\mathbf{r},t)e^{i\omega t}\,dt \qquad P(\mathbf{r},t) = \int_{-\infty}^{\infty} p_\omega(\mathbf{r})e^{-i\omega t}\,d\omega$$

with similar relationships between $f_\omega(\mathbf{r})$ and $f(\mathbf{r},t)$, $\mathbf{F}_\omega(\mathbf{r})$ and $\mathbf{F}(\mathbf{r},t)$, and $\mathfrak{T}_\omega$ and $\mathfrak{T}$, according to Eq. (1.3.6). Then the equation for the Fourier transform of P, the quantity which becomes the acoustic pressure outside the source region, is

$$\nabla^2 p_\omega + k^2 p_\omega = -f_\omega + \operatorname{div}\mathbf{F}_\omega - \boldsymbol{\nabla}\cdot\mathfrak{T}_\omega\cdot\boldsymbol{\nabla} \qquad k = \frac{\omega}{c} \tag{7.1.23}$$

where $f_\omega = -i\omega\rho s_\omega$ is defined in Eq. (7.1.22). The source terms, on the right-hand side of Eq. (7.1.23), are assumed to be known; they are zero except within the small region of violent motion, which we have called the source region.

This equation has the same form as Eq. (7.1.16) and thus has the solution given in Eq. (7.1.17) for an unbounded medium,

$$p_\omega(\mathbf{r}) = \iiint [f_\omega(\mathbf{r}_0) - \operatorname{div}_0\mathbf{F}_\omega(\mathbf{r}_0) + \boldsymbol{\nabla}_0\cdot\mathfrak{T}_\omega\cdot\boldsymbol{\nabla}_0]g_\omega(\mathbf{r}\mid\mathbf{r}_0)\,dv_0 \tag{7.1.24}$$

where g_ω is the Green's function $(1/4\pi R)e^{ikR}$, and the source terms in the brackets represent the injection of energy by the injection of heat and the creation of fluid, or by forces acting on the fluid, or by the nonlinear effects of fluid motion (Bernoulli effect) and viscosity. Each of these terms sends a wave of different form out into the surrounding medium, as can be most easily seen if we work out the radiation from a source region small compared with the wavelength $2\pi/k$.

In the first place, we can integrate the second and third source terms in Eq. (7.1.24) by parts. For example,

$$g_\omega \operatorname{div}_0\mathbf{F}_\omega = -\mathbf{F}_\omega\cdot\operatorname{grad}_0 g_\omega + \operatorname{div}_0(g_\omega\mathbf{F}_\omega)$$

and since $\mathbf{F}_\omega$ is zero outside the source region, the integral of $g_\omega \operatorname{div}_0 \mathbf{F}_\omega$ over the whole source region is equal to minus the integral of $\mathbf{F}_\omega \cdot \operatorname{grad}_0 g_\omega$ over the source region. Thus we can rewrite Eq. (7.1.24) in the more useful form

$$p_\omega(\mathbf{r}) = \iiint [f_\omega(\mathbf{r}_0) g_\omega(\mathbf{r} \mid \mathbf{r}_0) + \mathbf{F}_\omega(\mathbf{r}_0) \cdot \boldsymbol{\nabla}_0 g_\omega(\mathbf{r} \mid \mathbf{r}_0) + \mathfrak{T}_\omega : (\boldsymbol{\nabla}_0 g_\omega \boldsymbol{\nabla}_0)] \, dv_0 \quad (7.1.25)$$

where the scalar products of the gradient and tensor gradient of g_ω are

$$\mathbf{F}_\omega \cdot \boldsymbol{\nabla} g_\omega = F_x \frac{\partial}{\partial x} g_\omega + F_y \frac{\partial}{\partial y} g_\omega + F_z \frac{\partial}{\partial z} g_\omega$$

and

$$\mathfrak{T}_\omega : \boldsymbol{\nabla} g_\omega \boldsymbol{\nabla} = T_{xx}{}^\omega \frac{\partial^2}{\partial x^2} g_\omega + T_{xy}{}^\omega \frac{\partial^2}{\partial x \, \partial y} g_\omega + \cdots$$

where

$$T_{xy}{}^\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho u_x u_y e^{i\omega t} \, dt \quad \cdots$$

We now take the integral of Eq. (7.1.25) over a sphere of radius a, which is just large enough to enclose the whole source region (we are assuming that $ka \ll 1$) and set the origin of coordinates at the center of the sphere. In this case we are making little error if we assume that g_ω and its derivatives are nearly constant throughout the sphere, being equal to their values at $\mathbf{r}_0 = 0$. Thus if, and only if, $ka \ll 1$, the g factor can be taken outside the integral, in which case the resulting field takes on a particularly simple form.

For example, the first source term, that due to the introduction of new fluid, produces a spherically symmetric wave outside the source region.

$$p_s(\mathbf{r}) \simeq g_\omega(\mathbf{r} \mid 0) \iiint f_\omega(\mathbf{r}_0) \, dv_0 = -ik \frac{\rho c}{4\pi r} \left(\iiint s_\omega \, dv_0 \right) e^{ikr} \qquad ka \ll 1 \quad (7.1.26)$$

Comparison with Eq. (7.1.5) shows that the integral in parentheses equals S_ω, the strength of the monopole source produced by the periodic injection or expansion of fluid in the region $r < a$.

The second term in the integral of Eq. (7.1.25) becomes, when $ka \ll 1$, the product of the integral of vector $\mathbf{F}_\omega$ and the gradient of g_ω.

$$p_d(\mathbf{r}) \simeq \left[\iiint \mathbf{F}_\omega(\mathbf{r}_0) \, dv_0 \right] \cdot [\operatorname{grad}_0 g_\omega(\mathbf{r} \mid \mathbf{r}_0)]_{r_0=0}$$
$$= -k^2 D_\omega \frac{\rho c}{4\pi r} \left(1 + \frac{i}{kr} \right) e^{ikr} \cos \vartheta \quad (7.1.27)$$

where $\mathbf{D}_\omega = (i/k\rho c)\int \mathbf{F}_\omega \, dv_0$ is the vector strength of the dipole source produced by the forces acting on the fluid in the source region, and ϑ is the angle between $\mathbf{D}_\omega$ and the radius vector $\mathbf{r}$ to the measurement point.

The third term in the integral of Eq. (7.1.25) results in a quadrupole wave when $ka \ll 1$.

$$p_q(\mathbf{r}) \simeq -ik\rho c\left\{Q_{xx}\left[\frac{\partial^2}{\partial x_0^2} g_\omega(\mathbf{r} \mid \mathbf{r}_0)\right]_{r_0=0} + (Q_{xy} + Q_{yx})\left[\frac{\partial^2}{\partial x_0 \, \partial y_0} g_\omega(\mathbf{r} \mid \mathbf{r}_0)\right]_{r_0=0} + \cdots\right\} \quad (7.1.28)$$

where

$$Q_{xx} = \frac{i}{\rho ck}\iiint T_{xx}(\mathbf{r}_0)\, dv_0 = \frac{i}{\rho ck}\iiint (P_{xx} - P + \rho u_x^2)\, dv_0$$

and the form of the wave is given in Eq. (7.1.10), where

$$Q_{xy} = \frac{i}{\rho ck}\iiint (P_{xy} + \rho u_x u_y)\, dv_0$$

and the waveform is that given in Eq. (7.1.9), and so on. Thus the integral, over the source region, of the tensor

$$\frac{i}{\rho ck}\,\mathfrak{T} = \frac{i}{\rho ck}\,(\mathfrak{P} - P\mathscr{I} + \mathfrak{J})$$

is the tensor strength of the quadrupole source produced by the nonlinear effects of the fluid motion in the source region, if this region is small enough.

The complete wave radiated from the small region of violent fluid motion,

$$p_\omega(\mathbf{r}) = p_s(\mathbf{r}) + p_d(\mathbf{r}) + p_q(\mathbf{r}) \quad (7.1.29)$$

is thus a combination of monopole, dipole, and quadrupole waves, with relative strengths determined by the relative magnitudes of the fluid injection, fluid force, and stress-momentum characteristics of the small source region. Corresponding expressions for nonperiodic sources can be obtained from Eq. (7.1.13).

Random source functions

It should be pointed out that when the disturbed region is not small in dimensions compared with a wavelength, the angular distribution of the radiation is not as simply organized as this. Sound generated by the various portions of the region can interfere destructively in some directions and constructively in others. It is possible to have a distribution of simple sources which radiates like a dipole or quadrupole, or even a combination with higher multipoles. For example, if the radiating region is a sphere of radius a and the simple-source function $s_\omega(\mathbf{r}_0)$ of Eq. (7.1.26) is distributed

within the sphere so that s_ω is positive for $x_0 > 0$ and negative for $x_0 < 0$, then, when ka is not negligibly small, part of the radiation will be dipole in character. Likewise, a quadrupole source distribution can produce some simple-source and some dipole radiation if the radiating region is large enough. We shall have more to say about this under the next heading and in Chap. 11.

To simplify Eq. (7.1.29), we calculate the acoustic field at large distance r from a region, inside a sphere of radius a which is small compared with r but not small compared with λ, where the monopole and quadrupole source densities $s(\mathbf{r},t)$ and $\mathfrak{T}(\mathbf{r},t)$ fluctuate in a random manner with respect to both space and time. In this case the first and last terms in the integral of Eq. (7.1.25) are important, but there is no fluctuating gravitational or electric field; so the middle term is missing.

To compute the far field we note that, when $r \gg r_0$, the Green's function approaches the simple form

$$g_\omega(\mathbf{r} \mid \mathbf{r}_0) \to \frac{e^{ikr}}{4\pi r} e^{-i\mathbf{k}\cdot\mathbf{r}_0}$$

where vector $\mathbf{k}$, with components $k_x = k \sin\vartheta\cos\varphi$, $k_y = k \sin\vartheta\cos\varphi$, $k_z = k\cos\vartheta$, has magnitude k and is pointed in the direction of the radius vector $\mathbf{r}$ to the measurement point. Inserting this in Eq. (7.1.25) and omitting the term in $\mathbf{F}_\omega$, we obtain

$$p_\omega(\mathbf{r}) \to \frac{e^{ikr}}{r} \psi_\omega(\vartheta,\varphi)$$

where

$$\psi_\omega(\vartheta,\varphi) = \frac{1}{4\pi}\iiint [-ik\rho c s_\omega(\mathbf{r}_0) - \mathbf{k}\cdot\mathfrak{T}_\omega(\mathbf{r}_0)\cdot\mathbf{k}] e^{-i\mathbf{k}\cdot\mathbf{r}_0}\, dv_0 \qquad (7.1.30)$$

is the function determining the angle distribution of the component of the wave having frequency $\omega/2\pi$.

If the region of disturbance is very small compared with the wavelength $\lambda = 2\pi/k$, then the exponential $e^{i\mathbf{k}\cdot\mathbf{r}_0}$ will be practically equal to 1 over the region of integration, and the wave radiated will be a combination of simple-source and quadrupole waves. As pointed out earlier, if we define the constants

$$S^0_\omega = \iiint s_\omega(\mathbf{r}_0)\, dV_0 \qquad Q^0_{\omega xx} = \frac{i}{\rho c k}\iiint T_{\omega xx}\, dv_0 \qquad \cdots$$

the long-wavelength radiation has the simple angle distribution

$$\psi_\omega \to \frac{-ik\rho c}{4\pi} S^0_\omega + \frac{ik^3\rho c}{4\pi}(Q^0_{\omega xx}\sin^2\vartheta\cos^2\varphi + Q^0_{\omega xy}\sin^2\vartheta\cos\varphi\sin\varphi + \cdots)$$

a combination of a pure simple source and a pure quadrupole.

But for somewhat higher frequencies, when the exponential must be written $e^{-i\mathbf{k}\cdot\mathbf{r}_0} \simeq 1 - i(k_x x_0 + k_y y_y + k_z z_0)$, integrals like

$$D_{\omega x}{}^{s} = \iiint s_\omega(\mathbf{r}_0) x_0 \, dv_0 \qquad O^{q}_{\omega xyz} = \frac{i}{\rho c k} \iiint T_{\omega xy}(\mathbf{r}_0) z_0 \, dv_0 \qquad \cdots$$

also enter. The fluid-injection distribution s_ω (if it is not symmetrically distributed) will contribute some dipole radiation to ψ_ω of the form $-(k^2\rho c/4\pi)(D_{\omega x}{}^{s} \sin\vartheta\cos\varphi + D_{\omega y}{}^{s} \sin\vartheta \sin\varphi + D_{\omega z}{}^{s} \cos\vartheta)$, and the momentum-tensor terms will contribute octupole radiation with terms in ψ_ω such as $(k^4\rho c/4\pi) O^{q}_{\omega xyz} \sin^2\vartheta \cos\vartheta \sin\varphi \cos\varphi$, etc. At still higher frequencies the radiation becomes quite complex, and there is no particular advantage in separating it into monopole, dipole, etc., components. A more useful separation is into various spherical harmonics, as will be discussed in Sec. 7.2.

The autocorrelation function

In such a disturbed region, if both monopole and quadrupole source densities fluctuate in a highly random manner, both in space and time, the methods of Sec. 1.3 are most appropriate for their description. The mean rate of introduction of heat, for example, averaged over a long interval of time T and over the whole volume V of the active region, produces no noise, but the difference between the actual rate of production at point $\mathbf{r}_0$ at time t and this constant mean value may be a highly fluctuating quantity, containing a wide spread of frequencies and wavelengths. In accord with Eq. (1.3.18), we call $s(\mathbf{r}_0,t)$ this difference between actual and average source density, taken only in the time interval $-\frac{1}{2}T < t < \frac{1}{2}T$ and in the volume V of the active region. By an obvious generalization of Eq. (1.3.20) it can be described in terms of its multiple Fourier transform

$$S(\mathbf{K},\omega) = \frac{1}{16\pi^4} \int\cdots\int s(\mathbf{r}_0,t) e^{i\omega t - i\mathbf{K}\cdot\mathbf{r}_0} \, dv_0 \, dt \tag{7.1.31}$$

where the integration over t is from $-\frac{1}{2}T$ to $+\frac{1}{2}T$, and the integration over x_0, y_0, z_0 is over volume V. We note that, according to Eq. (7.1.22),

$$s_\omega = \frac{1}{2\pi} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} s(r_0,t) e^{i\omega t} \, dt$$

But we can also describe the monopole source density in terms of the autocorrelation function

$$\Upsilon_s(\mathbf{d},\tau) = \frac{1}{VT} \int\cdots\int s(\mathbf{r},t) s(\mathbf{r}+\mathbf{d}, t+\tau) \, dv \, dt \tag{7.1.32}$$

where the integration is again over T and V. In accord with Eq. (1.3.20), we see that the Fourier transform of Υ_s is proportional to the square of the Fourier transform of s.

$$|S(\mathbf{K},\omega)|^2 = \frac{VT}{(2\pi)^8}\int\cdots\int \Upsilon_s(\mathbf{d},\tau)e^{i\omega\tau - i\mathbf{K}\cdot\mathbf{d}}\,d\tau\,dv_d \tag{7.1.33}$$

The intensity of sound of frequency $\omega/2\pi$, caused by gas expansion or production, measured at $\mathbf{r}$, if $r \gg a$, the "radius" of the disturbed region, is

$$\begin{aligned} I_{s\omega} &= \frac{1}{\rho c r^2}|\psi_s(\vartheta,\varphi)|^2 = \frac{k^2\rho c}{64\pi^4 r^2}\left|\int\cdots\int s(\mathbf{r}_0,t)e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}_0}\,dv_0\,dt\right|^2 \\ &= \frac{4\pi^4 k^2\rho c}{r^2}|S(\mathbf{k},\omega)|^2 \\ &= \frac{VT\rho c k^2}{64\pi^4 r^2}\int\cdots\int \Upsilon_s(\mathbf{d},\tau)e^{i\omega\tau - i\mathbf{k}\cdot\mathbf{d}}\,d\tau\,dv_d \end{aligned} \tag{7.1.34}$$

which enables us to compute the total radiation produced by the monopole source in the active region, averaged over the time interval T.

The correlation function Υ_s will have its maximum value at $\mathbf{d}$ and $\tau = 0$, where its value will be the mean square of $s(\mathbf{r},t)$, as shown by Eq. (7.1.32). This difference $s(\mathbf{r},t)$ between the value of the $s(\mathbf{r},t)$ of Eq. (7.1.22) and its average value has dimensions volume per second per volume $= \text{sec}^{-1}$; its mean square, the *variance* of this density, averaged over V and T, written $\langle\Delta_s^2\rangle$, has dimensions sec^{-2}. According to the discussion of Eq. (2.3.12), if all frequencies are present in the fluctuation between 0 and some cutoff frequency $\omega_s/2\pi = 1/2\pi\tau_s$, Υ_s will be fairly constant for $0 < \tau < \tau_s$ and then will fall off to zero as τ is further increased. Thus there is little time correlation over a longer time interval than τ_s, called the *time correlation interval.* Likewise, there will be a *spatial correlation distance* w_s; for d^2 larger than w_s^2, function Υ_s will drop rapidly to zero. An obvious generalization of Eq. (2.3.13) would lead us to expect that

$$\Upsilon_s(\mathbf{d},\tau) = \langle\Delta_s^2\rangle \exp\left[-\frac{1}{2}\left(\frac{d}{w_s}\right)^2 - \tfrac{1}{2}(\omega_s\tau)^2\right]$$

would be a good approximation to the correlation function if the monopole fluctuations are isotropically random.

Inserting this in Eq. (7.1.34), we find for the intensity at frequency $\omega/2\pi$

$$I_{s\omega} = \rho c\langle\Delta_s^2\rangle\frac{k^2VT}{16\pi^2r^2}\frac{w_s^3}{\omega_s}\exp\left\{-\tfrac{1}{2}\omega^2\left[\left(\frac{w_s}{c}\right)^2 + \left(\frac{1}{\omega_s}\right)^2\right]\right\} \tag{7.1.35}$$

This increases proportionally to the square of the frequency below the limiting value $(\omega_s/2\pi)(1 + k_s^2 w_s^2)^{-\frac{1}{2}}$, where $k_s = \omega_s/c = 2\pi/\lambda_s$, λ_s being the wavelength corresponding to the cutoff frequency $\omega_s/2\pi$. For frequencies greater than this, the intensity at $\mathbf{r}$ falls off to zero.

To obtain the total intensity, for all frequencies, we use Eq. (1.3.17) to convert an average over the time interval T into an integral over all frequencies. Since

$$\frac{1}{\rho c T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} |p(\mathbf{r},t)|^2\, dt = \frac{2\pi}{\rho c T}\int_{-\infty}^{\infty} |p_\omega(\mathbf{r})|^2\, d\omega$$

the total intensity at $\mathbf{r}$, averaged over interval T, is

$$I_s = \frac{\rho c V}{\sqrt{32\pi}\, r^2}\langle \Delta_s^2\rangle \frac{k_s^2 w_s^3}{(1 + k_s^2 w_s^2)^{\frac{3}{2}}} \tag{7.1.36}$$

When the space-correlation distance w_s for the monopole fluctuations is larger than c times the time-correlation interval τ_s of these fluctuations, the total radiated intensity is about $1/40r^2$ of the mean variance $\langle \Delta_s^2\rangle$ of these fluctuations, times the volume V of the active region, times the cutoff wavelength λ_s, times ρc. When w_s is smaller than $c\tau_s = \lambda_s/2\pi$, the resulting intensity is reduced by the factor $(2\pi w_s/\lambda_s)^3$.

Exactly the same sort of calculation can be carried out for the radiation from turbulent motion, corresponding to the second term in Eq. (7.1.30). When the gas has low viscosity, the largest part of each term in the matrix $\mathfrak{T}$ is the kinetic-energy term $\rho u_x u_y$, etc.; these exhibit large fluctuations in a region of turbulence, as will be shown in Sec. 11.4. The term in the integral of Eq. (7.1.30) is the component of this matrix in the direction of $\mathbf{k}$, that is, in the direction of the measurement point. Thus $\mathbf{k}\cdot\mathfrak{T}(\mathbf{r}_0,t)\cdot\mathbf{k}$ is equal to k^2 times twice the kinetic energy of turbulent motion in the $\mathbf{k}$ direction (which is $\frac{1}{3}$ of the mean kinetic energy if the turbulence is isotropic). The mean value of this, averaged over time and over V, produces no radiation, but the deviation from this mean, $\Delta_e = \frac{1}{2}T_{kk}(\mathbf{r}_0,t) - \frac{1}{2}\langle T_{kk}\rangle$, is responsible for the sound of turbulence at $\mathbf{r}$. It can be described in terms of its Fourier transform $\Delta_e(\mathbf{K},\omega)$ or in terms of its correlation function

$$\Upsilon_e(\mathbf{d},\tau) = \langle \Delta_e^2\rangle \exp\left[-\frac{1}{2}\left(\frac{d}{w_e}\right)^2 - \tfrac{1}{2}(\omega_e t)^2\right]$$

having a Fourier transform proportional to $|\Delta_e(\mathbf{k},\omega)|^2$ [see Eq. (11.4.15) for further discussion]. Constant w_e is the spatial correlation distance for the turbulent fluctuations, and $\tau_e = 1/\omega_e = 1/ck_e = \lambda_e/2\pi c$ is the time-correlation interval.

Thus, as with the monopole sources, we obtain an expression for the intensity of the turbulent sound at point $\mathbf{r}$, over all frequencies.

$$I_e = \frac{3V\langle \Delta_e^2\rangle}{\sqrt{2\pi}\,\rho c r^2}\frac{k_e^4 w_e^3}{(1 + k_e^2 w_e^2)^{\frac{5}{2}}} \tag{7.1.37}$$

If the turbulence has some orientation along an axis, $k_e w_e$ probably is independent of angle, but $k_e\langle\Delta_e^2\rangle$ has the general form $A + B\cos^2\vartheta$, where ϑ is the angle between $\mathbf{r}$ and the axis of symmetry. Further discussion of this radiation will be found in Sec. 11.4.

The total intensity from both effects is the sum of the two, if the fluctuations are statistically independent. Then

$$I_r = \frac{V}{\sqrt{2\pi}\, r^2}\left\{\frac{\rho c\langle\Delta_s^2\rangle}{4k_s}\frac{(k_s w_s)^3}{[1+(k_s w_s)^2]^{\frac{3}{2}}} + \frac{3\langle\Delta_e^2\rangle}{\rho c w_e}\frac{(k_e w_e)^4}{[1+(k_e w_e)^2]^{\frac{5}{2}}}\right\} \tag{7.1.38}$$

When the products kw are small, i.e., when the spatial correlation is more restricted than c times the time-correlation interval τ, then the total radiation is weak and the simple-source term predominates. The intensity at $\mathbf{r}$ is equal to that of a simple source of strength roughly equal to $(2\pi)^{\frac{3}{4}}(w_s^3 V\langle\Delta_s^2\rangle)^{\frac{1}{2}}$ unit volumes per second and frequency $\omega_s/2\pi$ cps. When the products kw are both large, i.e., when the spatial correlation is more widespread than the temporal correlation times c, then the turbulence term is more important, and the intensity is roughly equal to that of a source of radius w_e and frequency $\omega_e/2\pi$, with driving force F_s [Eq. (7.1.11)] of magnitude $(288\pi)^{\frac{1}{4}}(V\lambda_e\langle\Delta_e^2\rangle)]^{\frac{1}{2}}$ newtons. Translational motion of the active region with respect to the surrounding air distorts this wave in a manner which will next be discussed in Chap. 11.

7.2 RADIATION FROM SPHERES

Sound waves can also be generated by the motion of solid bodies, of course. To correlate with the discussion of the preceding section, we shall first discuss the sound waves caused by the motion of a sphere, centered at the origin. Such waves can most easily be described in terms of spherical coordinates. Using Eqs. (7.1.4) and (7.1.3), we see that the first-order wave equation in these coordinates is (omitting dependence on φ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial p}{\partial\vartheta}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

If p is set equal to the factored quantity $p = R(r)P(\vartheta)e^{-i\omega t}$, the equation can be separated, becoming

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2 = -\frac{1}{P}\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{dP}{d\vartheta}\right) \qquad k = \frac{\omega}{c}$$

Each side of this equation must equal the same constant, C. On the other hand, if we wish to include dependence also on the axial angle φ, we use the

factored form $p = R(r)P(\vartheta)\Phi(\varphi)e^{-i\omega t}$, giving

$$\frac{d^2\Phi}{d\varphi^2} + n^2\Phi = 0 \qquad \Phi = \cos(n\varphi) \text{ or } \sin(n\varphi)$$

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP}{d\vartheta}\right) + \left(C - \frac{n^2}{\sin^2\vartheta}\right)P = 0 \tag{7.2.1}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left(k^2 - \frac{C}{r^2}\right)R = 0$$

where k^2, n^2, and C are the separation constants.

Legendre functions

The dependence on the axial coordinate φ is sinusoidal, as it is for the circular membrane of Eq. (5.2.18). In order that Φ have no discontinuity at $\varphi = 0, 2\pi$, n must be an integer. In a large number of cases there is axial symmetry, so that the wave is independent of φ, in which case $n = 0$ and the equation for the ϑ factor is

$$(1 - \eta^2)\frac{d^2P}{d\eta^2} - 2\eta\frac{dP}{d\eta} + CP = 0 \tag{7.2.2}$$

where we have made the substitution $\eta = \cos\vartheta$ for the independent variable. This solution for $n = 0$ is doubly important, for the solutions for $n \neq 0$ can be expressed in its terms. In fact, it is not difficult to show that the first two equations of (7.2.1) are satisfied by the solutions

$$\text{cosine or sine of } (n\varphi) \text{ times } (1 - \eta^2)^{\frac{1}{2}n}\frac{d^n}{d\eta^n}P(\eta) \tag{7.2.3}$$

where $P(\eta) = P(\cos\vartheta)$ is a solution of Eq. (7.2.2). Therefore we concentrate first on solving this equation, for the axially symmetric solutions.

We solve it by setting P equal to an arbitrary power series in η, substituting this in Eq. (7.2.2), and solving for the coefficients. The result is

$$P = a_0\left[1 - \frac{C}{2!}\eta^2 - \frac{C(6-C)}{4!}\eta^4 - \frac{C(6-C)(20-C)}{6!}\eta^6 - \cdots\right]$$
$$+ a_1\left[\eta + \frac{2-C}{3!}\eta^3 + \frac{(2-C)(12-C)}{5!}\eta^5 + \cdots\right]$$

These two series diverge at $\eta = \pm 1$ if the series are infinite ones, with all the powers of η. The only way to avoid this is to have one or the other series

break off at a finite number of terms, i.e., to have $C = 0$ or $C = 6$ or $C = 20$, etc., and have $a_1 = 0$, or else to have $C = 2$ or $C = 12$, etc., and have $a_0 = 0$. Thus the allowed values of C are

$$C = m(m + 1) \qquad m = 0, 1, 2, 3, \ldots$$

The solution of Eq. (7.2.2) which is finite over the range of x from -1 to $+1$, corresponding to $C = m(m + 1)$, is labeled $P_m(\eta)$. It can be obtained by substituting the proper value for C in the series given above, making one a equal to zero and giving the other a the value that makes $P_m(1) = 1$. The resulting solutions are

$$\begin{aligned}
m = 0, \quad C = 0: \quad & P_0(\eta) = 1 \\
& P_0(\cos\vartheta) = 1 \\
m = 1, \quad C = 2: \quad & P_1(\eta) = \eta \\
& P_1(\cos\vartheta) = \cos\vartheta \\
m = 2, \quad C = 6: \quad & P_2(\eta) = \tfrac{1}{2}(3\eta^2 - 1) \\
& P_2(\cos\vartheta) = \tfrac{1}{4}(3\cos 2\vartheta + 1) \\
m = 3, \quad C = 12: \quad & P_3(\eta) = \tfrac{1}{2}(5\eta^3 - 3\eta) \\
& P_3(\cos\vartheta) = \tfrac{1}{8}(5\cos 3\vartheta + 3\cos\vartheta) \\
& \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{7.2.4}$$

The function P_m is called a *Legendre function* of order m. It can be shown to have the following properties:

$$(\eta^2 - 1)\frac{d^2P_m}{d\eta^2} + 2\eta\frac{dP_m}{d\eta} - m(m + 1)P_m = 0 \qquad \eta = \cos\vartheta$$

$$P_m(\eta) = \frac{1}{2^m m!}\frac{d^m}{d\eta^m}(\eta^2 - 1)^m$$

$$(2m + 1)\eta P_m(\eta) = (m + 1)P_{m+1}(\eta) + mP_{m-1}(\eta) \tag{7.2.5}$$

$$(2m + 1)P_m(\eta) = \frac{d}{d\eta}[P_{m+1}(\eta) - P_{m-1}(\eta)]$$

$$\int_{-1}^{1} P_n(\eta)P_m(\eta)\,d\eta = \begin{cases} 0 & n \neq m \\ \dfrac{2}{2m + 1} & n = m \end{cases}$$

Values of some of these functions are given in Table VI.

The last equation of (7.2.5) shows that the functions $P_m(\eta)$ constitute a set of orthogonal characteristic functions. Any function of η in the range from

$\eta = 1$ to $\eta = -1$ can be expanded in terms of a series of these functions:

$$F(\eta) = \sum_{m=0}^{\infty} B_m P_m(\eta) \qquad B_m = (m + \tfrac{1}{2}) \int_{-1}^{1} F(\eta) P_m(\eta)\, d\eta \tag{7.2.6}$$

The expression for the coefficients B_m can be obtained by the method discussed on page 123.

Having defined the axially symmetric angle factors for waves from spheres, we now turn to Eq. (7.2.3) to obtain the solutions which depend on φ. We define the *spherical harmonics*

$$Y_m(\vartheta) = Y_{m0}{}^{1}(\vartheta,\varphi) = P_m(\eta) \qquad Y_{m0}{}^{-1} = 0; \qquad \eta = \cos\vartheta$$

$$Y_{mn}{}^{1}(\vartheta,\varphi) = \cos(n\varphi)(1-\eta^2)^{\frac{1}{2}n} \frac{d^n}{d\eta^n} P_m(\eta)$$

$$0 < n \leqslant m$$

$$Y_{mn}{}^{-1}(\vartheta,\varphi) = \sin(n\varphi)(1-\eta^2)^{\frac{1}{2}n} \frac{d^n}{d\eta^n} P_m(\eta)$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} Y_m(\vartheta)\, Y_l(\vartheta) \sin\vartheta\, d\vartheta = \frac{4\pi}{2m+1}\, \delta_{ml} \tag{7.2.7}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} Y_{mn}{}^{\sigma}(\vartheta,\varphi)\, Y_{lk}{}^{\tau}(\vartheta,\varphi) \sin\vartheta\, d\vartheta = \frac{2\pi}{2m+1} \frac{(m+n)!}{(m-n)!}\, \delta_{ml}\, \delta_{nk}\, \delta_{\sigma\tau}$$

$$P_m[\cos\vartheta\cos\theta - \sin\vartheta\sin\theta\cos(\varphi-\phi)] = \sum_{m,n,o} \epsilon_n \frac{(m-n)!}{(m+n)!}\, Y_{mn}{}^{\sigma}(\vartheta,\varphi)\, Y_{mn}{}^{\sigma}(\theta,\phi)$$

where σ, τ have the values $+1$ or -1; m, n are positive integers or zero; and $Y_{mn} = 0$ if $n > m$. Also, the coefficient δ_{ml} equals 1 if $m = l$ and equals zero if $m \neq l$, and coefficient ϵ_n equals 1 if $n = 0$ and equals 2 if $n > 0$. The last equation gives the relation between a Legendre function of one side of a spherical triangle in terms of spherical harmonics of the other two sides and the angle between them; the order m of the harmonic is not changed by a change of axis. The integrals show that any function of the angles ϑ, φ on the surface of a sphere can be expressed in terms of a generalization of series (7.2.6), using the spherical harmonics $Y_{mn}{}^{\sigma}(\vartheta,\varphi)$.

The radial factor

We must now solve the equation for the radial factor R.

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + k^2 r^2 R = CR = m(m+1)R \qquad k = \frac{\omega}{c}$$

where m must be zero or a positive integer (the order of the harmonic) for the angle factor Y to be finite everywhere. Changing scale, to get rid of k, we set

$\zeta = kr$ and obtain

$$\frac{d^2R}{d\zeta^2} + \frac{2}{\zeta}\frac{dR}{d\zeta} + [\zeta^2 + m(m+1)]\frac{R}{\zeta^2} = 0 \tag{7.2.8}$$

This equation looks very much like the Bessel equation (1.2.4); in fact, solutions of the equation are $(1/\sqrt{\zeta})J_{m+\frac{1}{2}}(\zeta)$. The solutions (for m an integer) can be expressed more simply in terms of the trigonometric functions, however. We have already shown, in Eq. (7.1.5), that a solution for $m = 0$ is $R = e^{ikr}/r$. We therefore try the expression $R = (e^{i\zeta}/\zeta)F_m(\zeta)$ and obtain the equation

$$\frac{d^2F_m}{d\zeta^2} + \left(2 - \frac{2m}{\zeta}\right)\frac{dF_m}{d\zeta} - \frac{2m}{\zeta}F_m = 0$$

Assuming a power series for F_m soon demonstrates that it must be a polynomial in powers of $1/\zeta$ when m is an integer. Direct substitution demonstrates that

$$F_m(\zeta) = \sum_{s=0}^{m} \frac{(m+s)!}{s!\,(m-s)!}\left(\frac{i}{2\zeta}\right)^s \to \begin{cases} 1 & \zeta \to \infty \\ \dfrac{(2m)!}{m!}\left(\dfrac{i}{2\zeta}\right)^m & \zeta \to 0 \end{cases}$$

Therefore we define the solution of Eq. (7.2.8), the radial-wave factor to use for outgoing waves, as

$$h_m(\zeta) = \frac{i^{-m}}{i\zeta}\sum_{s=0}^{m} \frac{(m+s)!}{s!\,(m-s)!}\left(\frac{i}{2\zeta}\right)^s e^{i\zeta} \to \begin{cases} \dfrac{(2m)!}{im!\,\zeta}\left(\dfrac{1}{2\zeta}\right)^m & \zeta \to 0 \\ \dfrac{i^{-m-1}}{\zeta}e^{i\zeta} & \zeta \to \infty \end{cases} \tag{7.2.9}$$

This function is called the *spherical Hankel function* of order m. The first few such functions are

$$h_0(\zeta) = \frac{e^{i\zeta}}{i\zeta} \qquad h_1(\zeta) = -\frac{e^{i\zeta}}{\zeta}\left(1 + \frac{i}{\zeta}\right)$$

$$h_2(\zeta) = \frac{ie^{i\zeta}}{\zeta}\left(1 + \frac{3i}{\zeta} - \frac{3}{\zeta^2}\right) \qquad \cdots \qquad \text{where} \quad \zeta = kr$$

We note that the simple source of Eq. (7.1.5) has $h_0(kr)$ for its radial factor and that $h_1(kr)$ is the radial factor for the dipole radiation of Eq. (7.1.7). In fact, the three dipole components, along the x, y, and z coordinates, can be expressed in terms of the three solutions for $m = 1$: $Y_{11}{}^1h_1 = \cos\varphi \sin\vartheta\, P_1'(\cos\vartheta)h_1(kr)$, $Y_{11}{}^{-1}h_1 = \sin\varphi \sin\vartheta\, P_1'h_1(kr)$, $Y_{10}{}^1h_1 = P_1(\cos\vartheta)h_1(kr)$, where $P_1'(\eta) = dP_1/d\eta = 1$, since $P_1(\eta) = \eta$. Thus the three possible solutions for $m = 1$ correspond to the three orientations of dipole radiation.

Similarly, the set for $m = 2$ corresponds to quadrupole radiation. For example, the xy component of Eq. (7.1.9) is proportional to

$$2 \cos \varphi \sin \varphi \sin^2 \vartheta P_2''(\cos \vartheta) h_2(kr) = 3 \sin (2\varphi) \sin^2 \vartheta \, h_2(kr)$$

which is $Y_{22}^{-1}(\vartheta,\varphi)h_2(kr)$, and the xz term is proportional to $Y_{21}^{1}(\vartheta,\varphi)h_2(kr)$. As noted earlier, one combination of the axial quadrupole waves yields the simple source wave $h_0(kr)$. Two others, however, are waves of order $m = 2$, for

$$\frac{1}{2r^2}(3z^2 - r^2)h_2(kr) = P_2(\cos \vartheta)h_2(kr) = Y_{20}^{1}h_2$$

$$\frac{3}{r^2}(x^2 - y^2)h_2(kr) = Y_{22}^{1}(\vartheta,\varphi)h_2(kr)$$

Thus the five independent quadrupole waves correspond to the five possible spherical-wave solutions for $m = 2$, one for $n = 0$, two for $n = 1$, and two for $n = 2$.

Spherical Bessel functions

The spherical Hankel functions $h_m(kr)$ correspond to outgoing waves, appropriate for situations where acoustic energy is radiating outward into an unbounded medium. Such a wave goes to infinity at $r = 0$ as r^{-m-1}, which does not preclude using it for this purpose since no physical source has zero radius, and the point $r = 0$ is not included in a function valid only outside the source. It does preclude our using the spherical Hankel function for a wave in the interior of a spherical enclosure, where r does go to zero; so we must find other solutions of Eq. (7.2.8) which are finite at $r = 0$. These are obtained by taking the real part of $h_m(kr)$. We define the *spherical Bessel function* of order m, $j_m(\zeta)$, as the real part of $h_m(\zeta)$, and while we are at it, we define the *spherical Neumann function* of order m, $n_m(\zeta)$, as the imaginary part of h_m.

$$h_m(\zeta) = j_m(\zeta) + i n_m(\zeta) \qquad j_m, n_m \text{ real} \tag{7.2.10}$$

It turns out that j_m is finite at $r = 0$ but n_m is not. Since h_m is a solution of Eq. (7.2.8), so also are both j_m and n_m. Their specific expressions, for the first values of m, are

$$j_0(z) = \frac{\sin z}{z} \qquad n_0(z) = -\frac{\cos z}{z}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \qquad n_1(z) = -\frac{\sin z}{z} - \frac{\cos z}{z^2} \tag{7.2.11}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z \qquad n_2(z) = -\frac{3}{z^2}\sin z - \left(\frac{3}{z^3} - \frac{1}{z}\right)\cos z$$

. . . . . .

These functions have the following properties:

$$j_m(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, J_{m+\frac{1}{2}}(\zeta) \qquad n_m(\zeta) = \sqrt{\frac{\pi}{2\zeta}}\, N_{m+\frac{1}{2}}(\zeta)$$

$$j_m(\zeta) \xrightarrow[\zeta\to 0]{} \frac{\zeta^m}{1\cdot 3\cdot 5\cdot\cdots\cdot(2m+1)}$$

$$n_m(\zeta) \xrightarrow[\zeta\to 0]{} -\frac{1\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1)}{\zeta^{m+1}}$$

$$j_m(\zeta) \xrightarrow[\zeta\to\infty]{} \frac{1}{\zeta}\cos\left(\zeta - \frac{m+1}{2}\pi\right) \qquad n_m(\zeta) \xrightarrow[\zeta\to\infty]{} \frac{1}{\zeta}\sin\left(\zeta - \frac{m+1}{2}\pi\right) \tag{7.2.12}$$

$$\int j_0^2(\zeta)\zeta^2\, d\zeta = \frac{\zeta^3}{2}\left[j_0^2(\zeta) + n_0(\zeta)\, j_1(\zeta)\right]$$

$$\int n_0^2(\zeta)\zeta^2\, d\zeta = \frac{\zeta^3}{2}\left[n_0^2(\zeta) - j_0(\zeta) n_1(\zeta)\right]$$

$$h_m'(\zeta)\, j_m(\zeta) - j_m'(\zeta) h_m(\zeta) = \frac{i}{\zeta^2} \qquad j' = \frac{dj}{d\zeta}$$

and have the following properties, for either j_m, n_m, or h_m:

$$j_{m-1}(\zeta) + j_{m+1}(\zeta) = \frac{2m+1}{\zeta} j_m(\zeta)$$

$$\frac{d}{d\zeta} j_m(\zeta) = \frac{1}{2m+1}\left[m j_{m-1}(\zeta) - (m+1) j_{m+1}(\zeta)\right]$$

$$\frac{d}{d\zeta}\left[\zeta^{m+1} j_m(\zeta)\right] = \zeta^{m+1} j_{m-1}(\zeta) \qquad \frac{d}{d\zeta}\left[\zeta^{-m} j_m(\zeta)\right] = -\zeta^{-m} j_{m+1}(\zeta) \tag{7.2.13}$$

$$\int j_1(z)\, dz = -j_0(z) \qquad \int j_0(z) z^2\, dz = z^2 j_1(z)$$

$$\int j_m^2(z) z^2\, dz = \frac{z^3}{2}\left[j_m^2(z) - j_{m-1}(z)\, j_{m+1}(z)\right] \qquad m > 0$$

In all Eqs. (7.2.13) the function n can be substituted for j. Values of some of the functions j and n are given in Table VII.

Radiation from a general spherical source

We take next the general case of a sphere, not necessarily small, whose surface vibrates with a velocity $U(\vartheta)e^{-2\pi i\nu t}$, where U is any sort of function of ϑ. We first express the velocity amplitude $U(\vartheta)$ in terms of a series of Legendre functions

$$U(\vartheta) = \sum_{m=0}^{\infty} U_m P_m(\cos\vartheta)$$

$$U_m = (m+\tfrac{1}{2})\int_0^{\pi} U(\vartheta) P_m(\cos\vartheta)\sin\vartheta\, d\vartheta \tag{7.2.14}$$

To correspond to this we also express the radiated pressure wave in a series

$$p = \sum_{m=0}^{\infty} A_m P_m(\cos \vartheta) h_m(kr) e^{-i\omega t}$$

where the values of the coefficients A_m must be determined in terms of the known coefficients U_m. The radial velocity of the air at the surface of the sphere is

$$u_a = \frac{1}{\rho c} \sum_{m=0}^{\infty} A_m B_m P_m(\cos \vartheta) e^{i\delta_m - i\omega t}$$

where

$$\frac{d}{d\zeta} h_m(\zeta) = iB_m(\zeta) e^{i\delta_m(\zeta)} \qquad \zeta = ka = \frac{2\pi a}{\lambda}$$

Therefore, using Eqs. (7.2.12) and (7.2.13),

$$mn_{m-1}(ka) - (m+1)n_{m+1}(ka) = (2m+1)B_m \cos \delta_m$$
$$(m+1)j_{m+1}(ka) - mj_{m-1}(ka) = (2m+1)B_m \sin \delta_m$$

When $ka \gg m + \frac{1}{2}$,

$$B_m \simeq \left(\frac{1}{ka}\right) \qquad \delta_m \simeq ka - \tfrac{1}{2}\pi(m+1)$$

When $ka \ll m + \frac{1}{2}$, (7.2.15)

$$B_0 \simeq \left(\frac{1}{ka}\right)^2 \qquad \delta_0 \simeq \tfrac{1}{3}(ka)^3$$

$$B_m \simeq \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)(m+1)}{(ka)^{m+2}} \qquad m > 0$$

$$\delta_m \simeq \frac{-m(ka)^{2m+1}}{1^2 \cdot 3^2 \cdot 5^2 \cdot \cdots \cdot (2m-1)^2(2m+1)(m+1)}$$

Values of the amplitudes B_m and the phase angles δ_m are given in Table VIII.

The radial velocity of the air at $r = a$ must equal that of the surface of the sphere, and equating coefficients of the two series, term by term, we obtain equations for the coefficients A_m in terms of U_m:

$$A_m = \frac{\rho c U_m}{B_m} e^{-i\delta_m}$$

The pressure and radial velocity at large distances (i.e., many wavelengths) from the sphere can then be expressed

$$u_r \simeq U_0 \frac{a}{r} e^{ik(r-ct)} \psi(\vartheta) \qquad p \simeq \rho c u_r$$

(7.2.16)

$$\psi(\vartheta) = \frac{1}{ka} \sum_{m=0}^{\infty} \frac{U_m}{U_0 B_m} P_m(\cos \vartheta) e^{-i\delta_m - \frac{1}{2}i\pi(m+1)}$$

where we have multiplied and divided by U_0, the average velocity of the

surface of the sphere. The air velocity near the sphere is, of course, not in phase with the pressure, nor is it entirely radial; but far from the sphere the velocity is radial and in phase with the pressure.

When kr is very large, the intensity at the point (r,ϑ) and the total power radiated are

$$I_r \simeq \rho c U_0^2 \frac{a^2}{r^2} F_r(\vartheta) \qquad F_r(\vartheta) = |\psi(\vartheta)|^2$$

$$F_r(\vartheta) = \left(\frac{1}{kaU_0}\right)^2 \sum_{m,n=0}^{\infty} \frac{U_m U_n}{B_m B_n} P_m(\cos\vartheta) P_n(\cos\vartheta) \times \cos\left[\delta_m - \delta_n + \tfrac{1}{2}\pi(m-n)\right] \tag{7.2.17}$$

$$\Pi = \int_0^{2\pi} d\phi \int_0^{\pi} r^2 I_r \sin\vartheta \, d\vartheta = \frac{\rho c^3}{4\pi\nu^2} \sum_{m=0}^{\infty} \frac{U_m^2}{(2m+1)B_m^2}$$

The function $F_r(\vartheta)$ is called the *angle-distribution function* for radiation from a sphere.

When $ka = 2\pi a/\lambda$ is quite small, all but the first terms in these expansions can be neglected, since B_m then increases quite rapidly with m. Consequently, the pressure and intensity a distance r from the center of the sphere, at any angle, are

$$p \simeq -\frac{ik\rho c}{4\pi r}(4\pi a^2 U_0) e^{ik(r-ct)} \qquad I_r \simeq \rho c\,|U_0|^2 \frac{k^2 a^4}{r^2}$$

equivalent to those of a simple source of strength $S_\omega = 4\pi a^2 U_0$.

On the other hand, when ka is quite large, B_m is equal to $1/ka$ for all values of m from zero up to a value roughly equal to ka, then drops sharply in value for $m > ka$. Suppose the series (7.2.14), for the surface velocity $U(\vartheta)$, includes terms for m up to some value M, but all coefficients U_m for $m > M$ are negligible. Then, when $ka > M$ $(2\pi a > M\lambda)$, all the non-negligible terms in the series for the angle-distribution factor ψ are simply proportional to U_m and $\psi \simeq (1/U_0)U(\vartheta)e^{-ika}$. In this case the pressure and the radiated intensity at great distances r from the sphere are

$$p(r,\vartheta,\varphi) \simeq \rho c U(\varphi)\frac{a}{r} e^{ik(r-a-ct)} \qquad I_r \simeq \rho c\,|U(\vartheta)|^2 \left(\frac{a}{r}\right)^2 \qquad r \gg a$$

A spherical harmonic Y_{mn} of order m has m nodes and loops on the sphere's surface, and therefore a "wavelength" of about $2\pi a/m$. What we have just said is that when the wavelength of the radiation is smaller than the "fine structure" of the velocity distribution on the surface of the sphere, the pressure at r, ϑ is proportional to the velocity at a, ϑ, as though the sound traveled radially outward from the surface, the intensity above a region of large motion being greater than that above a quiet region. At $r = a$ the impedance opposing the surface motion is ρc and the intensity is $\rho c\,|U(\vartheta)|^2$. This is the region of "geometrical acoustics." At longer wavelengths the pattern of the velocity distribution $U(\vartheta)$ becomes blurred out in the radiated

wave and the intensity decreases, until, for long wavelengths, the sphere radiates like a simple source.

Of course, if $U_0 = 0$, this limiting expression is invalid. For example, suppose the wave is caused by a rigid sphere, of radius a, moving to and fro along the z axis with velocity $U_1 e^{-i\omega t}$. The radial velocity of the fluid at the surface $r = a$ is then $U_1 \cos\vartheta e^{-i\omega t}$, and the resulting pressure wave is $(\rho c U_1/B_1)e^{-i\delta_1}\cos\vartheta\, h_1(kr)e^{-i\omega t}$. This is, of course, a dipole wave, of dipole strength $D_\omega = (4\pi/k^3 B_1)U_1$; in the limit of $ka \ll 1$, this strength becomes $2\pi a^3 U_1$, the cross section πa^2 of the sphere times $2aU_1$. In this case, since there is no $m = 0$ term present, the limiting form of the wave is that given in Eq. (7.1.7).

Radiation from a point source on a sphere

To show graphically how the radiation changes from symmetrical to directional as the frequency of the radiated sound increases, we shall work out the details of the radiation problem for two cases.

The first case is that of a point source at the point $\vartheta = 0$ on the surface; i.e., the surface velocity of the sphere is zero except for a small circular area of radius Δ around $\vartheta = 0$. The definition of $U(\vartheta)$ is

$$U(\vartheta) = \begin{cases} u_0 & 0 \leqslant \vartheta < \dfrac{\Delta}{a} \\ 0 & \dfrac{\Delta}{a} < \vartheta \leqslant \pi \end{cases}$$

and the coefficients U_m are

$$U_m = (m + \tfrac{1}{2})u_0 \int_0^{\Delta/a} P_m(\cos\vartheta)\sin\vartheta\, d\vartheta \xrightarrow[\Delta\to 0]{} \tfrac{1}{2}(m + \tfrac{1}{2})u_0\left(\frac{\Delta}{a}\right)^2$$

since $P_m(1) = 1$. The intensity and power radiated can be obtained from Eqs. (7.2.17):

$$I_r \simeq \rho c {u_0}^2 \frac{\Delta^4}{16a^2r^2} F_r(\vartheta) \tag{7.2.18}$$

$$F_r(\vartheta) = \frac{1}{\mu^2}\sum_{m,n=0}^{\infty} \frac{(2m+1)(2n+1)}{B_m B_n} P_m(\cos\vartheta)P_n(\cos\vartheta) \times \cos[\delta_m - \delta_n + \tfrac{1}{2}\pi(m-n)]$$

$$\Pi = \rho c {u_0}^2 \frac{\Delta^4}{16a^2}\frac{4\pi}{\mu^2}\sum_{m=0}^{\infty}\frac{2m+1}{{B_m}^2} \qquad \mu = \frac{2\pi a}{\lambda}$$

At very low frequencies, only the first terms in these series are important, and the pressure wave again has the familiar form for the simple source

$$p \simeq -i\omega\frac{\rho}{4\pi r}(\pi\Delta^2 u_0)e^{ik(r-ct)}$$

with a strength equal to the velocity u_0 times $\pi\Delta^2$, the area of the radiating element.

Figure 7.1 gives curves for the distribution in angle of the intensity radiated from a point source on a sphere for different ratios of wavelength to sphere circumference. We see again the gradual change from radiation in all directions to a sharply directional pattern as the frequency is increased.

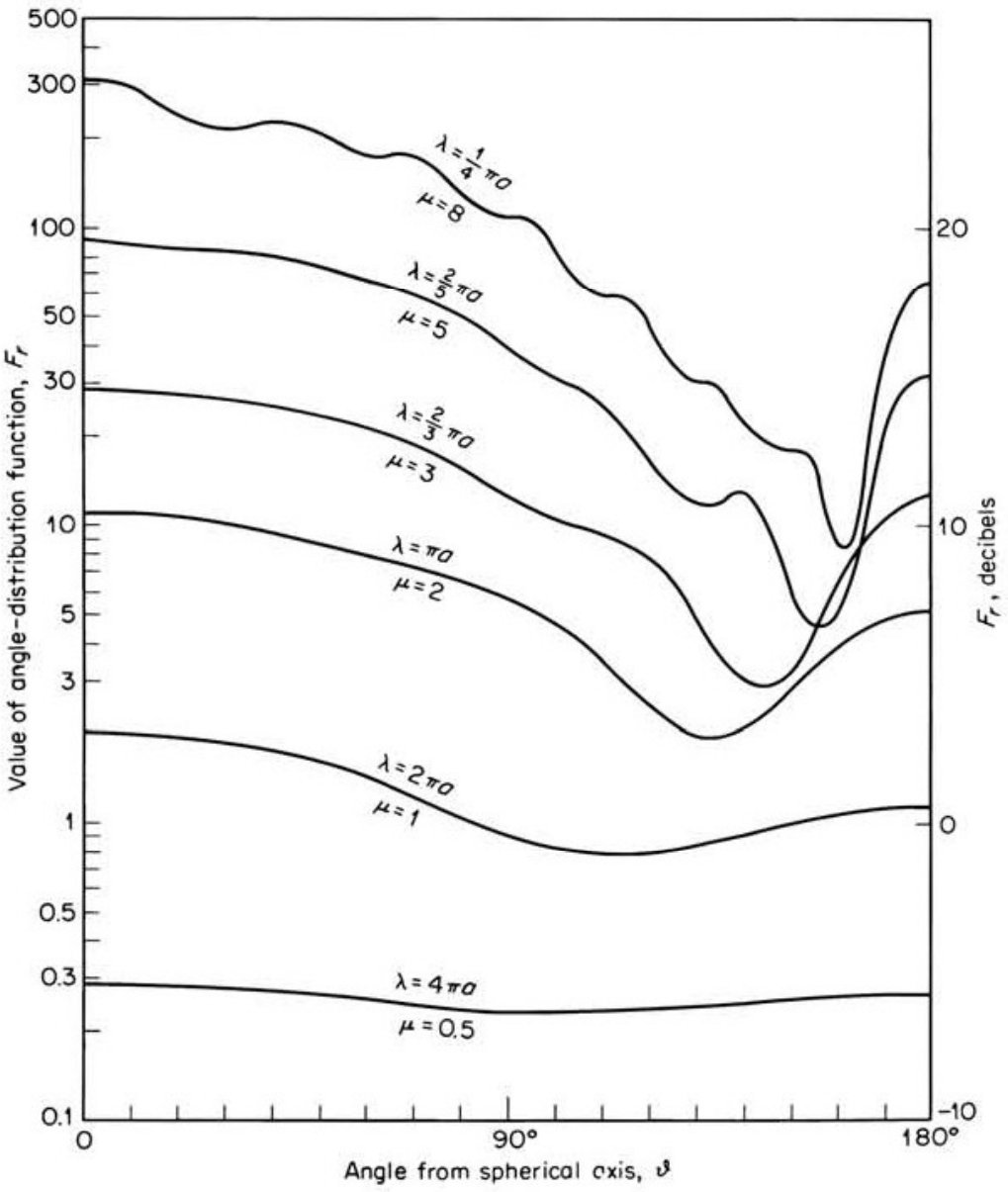

FIGURE 7.1
Distribution in angle of intensity radiated from a point source set in the surface of a sphere of radius a, for different values of $\mu = 2\pi a/\lambda$. Curves also give mean-square pressure at point (a,ϑ) on surface of sphere due to incident plane wave traveling in negative x direction.

These curves are of particular interest because of their dual role, as required by the *principle of reciprocity*, mentioned in Sec. 7.1. As computed, the curves give intensity or pressure amplitude squared at a point (r,ϑ), a considerable distance r from the sphere (when $kr \gg 1$, $|p| = \sqrt{\rho c I_r}$) at an angle