ϑ with respect to the line from the center of the sphere through the radiating element of area at point $(a,0)$. But the principle of reciprocity says that the pressure at a point Q due to a unit simple source at point P is equal to the pressure at point P due to a unit source at point Q. Consequently, the curves of Fig. 7.1 also represent the square of the pressure amplitude at a point $(a,0)$ on the surface of the sphere due to a point source of strength $\pi\Delta^2 u_0$ at the point (r,ϑ). Therefore the curves are useful as an indication of the directional properties of the ear plus head, or of a microphone in a roughly spherical housing.

Radiation from a piston set in a sphere

The other example to be worked out is that of a piston of radius $a \sin \vartheta_0$ set in the side of a rigid sphere. As long as ϑ_0 is not too large, this corresponds fairly closely to the following distribution of velocity on the surface of the sphere:

$$U(\vartheta) = \begin{cases} u_0 & 0 \leqslant \vartheta < \vartheta_0 \\ 0 & \vartheta_0 < \vartheta \leqslant \pi \end{cases}$$

The general formulas (7.2.17) can be used, with

$$U_m = (m + \tfrac{1}{2})u_0 \int_{\cos \vartheta_0}^{1} P_m(x)\, dx = \tfrac{1}{2}u_0[P_{m-1}(\cos \vartheta_0) - P_{m+1}(\cos \vartheta_0)]$$

where, for the case $m = 0$, we consider $P_{-1}(x) = 1$.

These expressions for U_m can be substituted in Eq. (7.2.17) to give series for intensity and power. Curves for I as function of ϑ are given in Fig. 7.2. When the wavelength is long compared with $2\pi a$, the pressure and intensity are those for a simple source of strength $4\pi a^2 u_0 \sin^2 (\vartheta_0/2)$.

The radiation impedance for the piston set in a sphere can be computed by integrating the expression for the pressure at $r = a$ over the surface of the piston. After quite a little algebraic juggling, we find

$$Z_p = R_p - iX_p = \rho c 4\pi a^2 \sin^2\left(\frac{\vartheta_0}{2}\right)(\theta_p - i\chi_p)$$

$$\theta_p = \frac{1}{4}\sum_{m=0}^{\infty} \frac{[P_{m-1}(\cos \vartheta_0) - P_{m+1}(\cos \vartheta_0)]^2}{\mu^2(2m + 1)B_m{}^2 \sin^2 (\vartheta_0/2)} \tag{7.2.19}$$

$$\chi_p = \frac{1}{4}\sum_{m=0}^{\infty} \frac{[P_{m-1}(\cos \vartheta_0) - P_{m+1}(\cos \vartheta_0)]^2}{(2m + 1)B_m \sin^2 (\vartheta_0/2)} \times [j_m(\mu) \sin (\delta_m) - n_m(\mu) \cos (\delta_m)]$$

where $\mu = ka = 2\pi a/\lambda$.

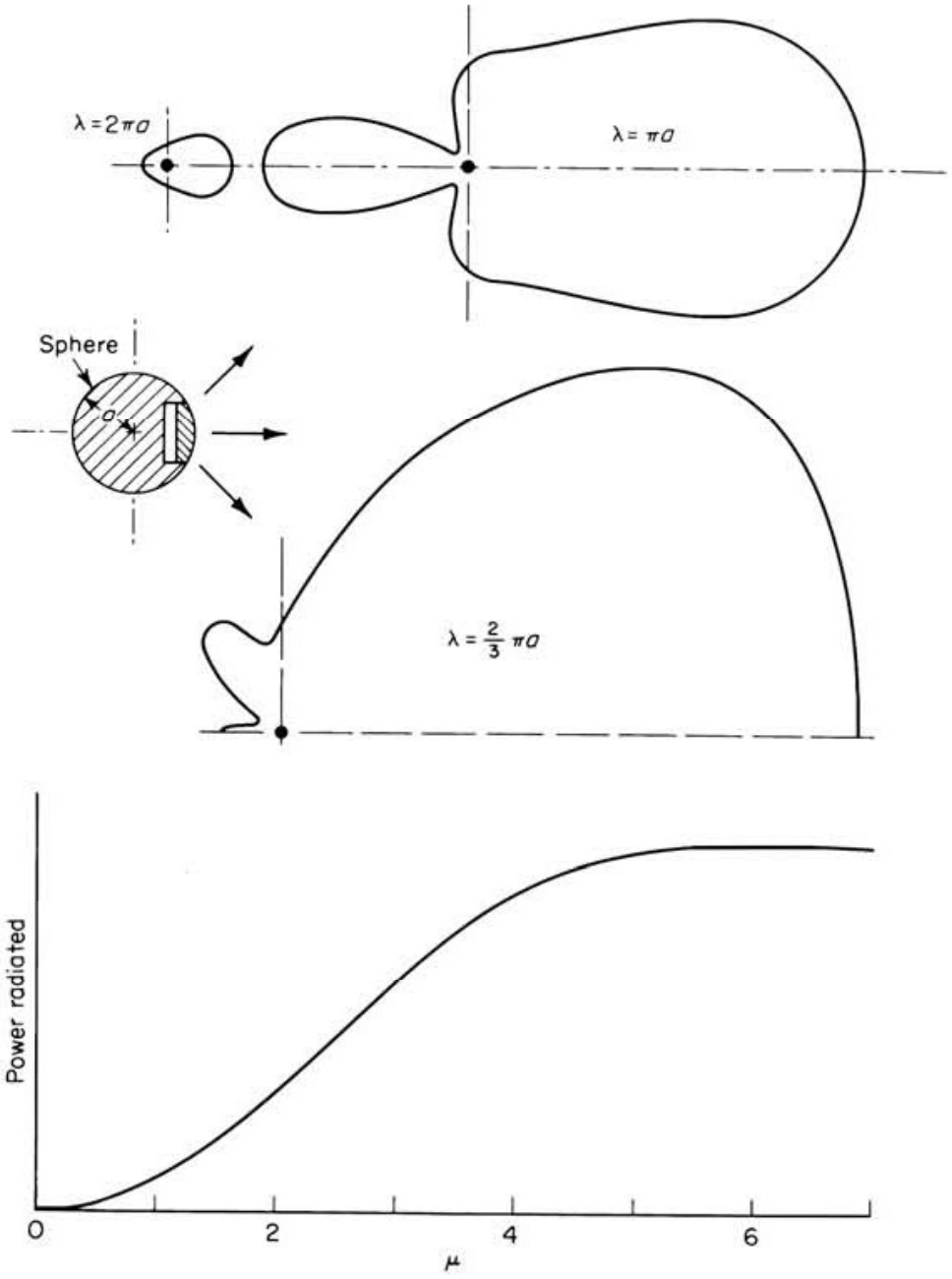


FIGURE 7.2
Distribution in angle of intensity and total power radiated from a piston set in a sphere.

Since $4\pi a^2 \sin^2(\vartheta_0/2) = \pi a_p^2$ is the area of the piston (a_p thus defined being the effective piston radius), the dimensionless quantity $\theta_p - i\chi_p$ is the impedance per unit area, divided by the characteristic impedance ρc of the medium. The specific resistance θ_p and reactance χ_p are plotted in Fig. 7.3 as functions of the ratio of equivalent diaphragm circumference to wavelength, $2\pi a_p/\lambda = (\omega a_p/c)$ $[a_p = 2a \sin(\vartheta_0/2)]$, for several different values of $\vartheta_0 = 2 \sin^{-1}(a_p/2a)$. We notice that the resistive terms all are small at low frequencies, increasing as ω^2; rising to a value of approximately unity

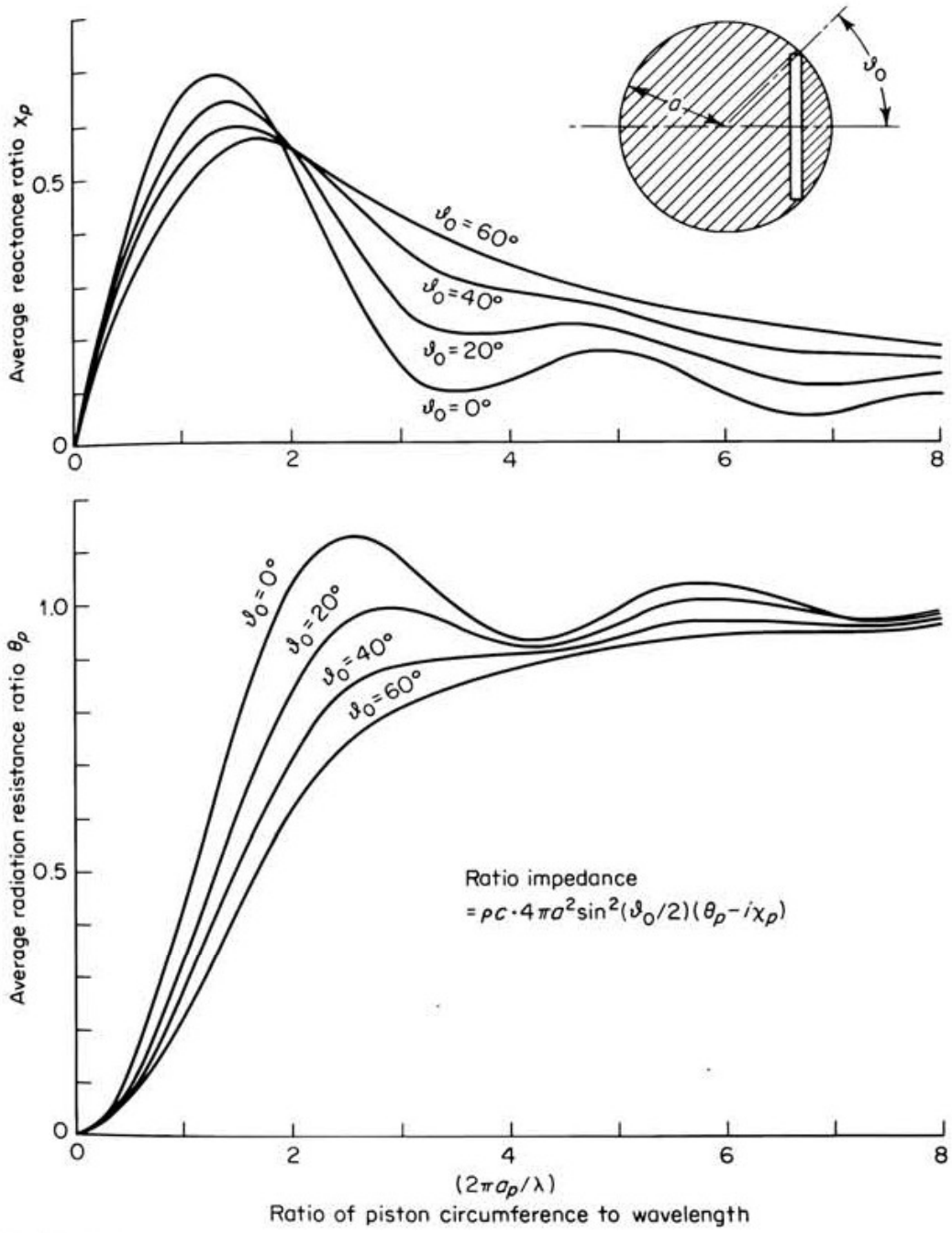


FIGURE 7.3
Values of radiation resistance and reactance ratios $(R_r, X_r)/\pi a_p^2 \rho c$, where $a_p = 2a \sin(\vartheta_0/2)$, as function of $2\pi a_p/\lambda = \omega a_p/c$ for different values of ϑ_0, for radiation of sound from a piston set in a sphere of radius a.

at wavelengths a little smaller than one-third times the equivalent diaphragm circumference; and then, for very high frequencies, approaching the usual limit of 1 ($z = \rho c$). The reactance is always positive, representing a mass load. It first increases linearly with frequency, as the reactance due to a constant mass would, but then it reaches a maximum at $\lambda \simeq 2\pi a$, and for higher frequencies it diminishes more or less rapidly.

These curves indicate the sort of radiation load one would expect on the diaphragm of a dynamic loudspeaker, set in a spherical case. The diaphragm of the dynamic speaker is not usually the surface of a sphere, but often has the shape of an inverted cone in order to increase its mechanical strength. The radiation from such a cone would naturally differ from that from a section of a sphere; but it turns out that the average radiation impedance on a piston is approximately the same, no matter what its shape, as long as its circumference is not changed and as long as the volume of the mounting case is not changed. Therefore the radiation reaction on the diaphragm of a dynamic speaker of outer circumference $2\pi a_p$, set in a cabinet of volume $V_p = 4\pi a^3/3$, is approximately given by $\pi a_p{}^2 \rho c$ times the curves of Fig. 7.3 for $\vartheta_0 = 2\sin^{-1}[a_p(\pi/6V_p)^{\frac{1}{3}}]$, with the frequency scale equal to $2\pi\nu a_p/c$.

Multipoles and Legendre functions

Since the Green's function $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$ of Eqs. (7.1.6) and (7.1.14) may be expressed in spherical coordinates, it should be possible to express it in terms of Legendre functions and spherical Hankel functions. First we discuss the derivatives of g_ω with respect to x_0, y_0, z_0 at $\mathbf{r}_0 = 0$, the basis for our earlier discussion of multipole radiation.

The Green's function itself is

$$g_\omega(\mathbf{r} \mid 0) = \frac{ik}{4\pi} h_0(kr) = \frac{1}{4\pi r} e^{ikr} \tag{7.2.20}$$

in terms of the spherical Hankel function, defined in Eqs. (7.2.9). The three dipole waves are proportional to the x, y, z components of the gradient of g_ω at $\mathbf{r}_0 = 0$.

$$\begin{aligned} g'_x(\mathbf{r} \mid 0) &= \left[\frac{\partial}{\partial x_0} g_\omega(\mathbf{r} \mid \mathbf{r}_0)\right]_{r_0=0} = \frac{ik^2}{4\pi} Y_{11}{}^1(\vartheta,\varphi) h_1(kr) \\ g'_y(\mathbf{r} \mid 0) &= \frac{ik^2}{4\pi} Y_{11}{}^{-1}(\vartheta,\varphi) h_1(kr) \qquad g'_z(\mathbf{r} \mid 0) = \frac{ik^2}{4\pi} Y_1(\vartheta) h_1(kr) \end{aligned} \tag{7.2.21}$$

involving all three of the spherical harmonics of order $m = 1$. The five spherical harmonics of order $m = 2$ all represent quadrupole radiation, as defined in Eqs. (7.1.9) and (7.1.10). Thus

$$\begin{aligned} g''_{xy}(\mathbf{r} \mid 0) &= \left[\frac{\partial^2}{\partial x_0\, \partial y_0} g_\omega(\mathbf{r} \mid \mathbf{r}_0)\right]_{r_0=0} = \frac{ik^3}{24\pi} Y_{22}{}^{-1}(\vartheta,\varphi) h_2(kr) \\ g''_{xz}(\mathbf{r} \mid 0) &= \frac{ik^3}{12\pi} Y_{21}{}^1(\vartheta,\varphi) h_2(kr) \qquad g''_{yz} = \frac{ik^3}{12\pi} Y_{21}{}^{-1}(\vartheta,\varphi) h_2(kr) \end{aligned} \tag{7.2.22}$$

as mentioned earlier. The axial solutions, as mentioned, contain a certain

amount of the $m = 0$ wave as well:

$$g''_{zz}(\mathbf{r} \mid 0) = \frac{ik^3}{24\pi}\,[4\,Y_{20}{}^1(\vartheta,\varphi)h_2(kr) - 2\,Y_{00}{}^1(\vartheta,\varphi)h_0(kr)]$$

$$g''_{xx}(\mathbf{r} \mid 0) = \frac{ik^3}{24\pi}\,[Y_{22}{}^1 h_2 - 2\,Y_{20}{}^1 h_2 - 2h_0] \tag{7.2.23}$$

$$g''_{yy}(\mathbf{r} \mid 0) = \frac{ik^3}{24\pi}\,[-Y_{22}{}^1 h_2 - 2\,Y_{20}{}^1 h_2 - 2h_0]$$

with the Y's as defined in Eqs. (7.2.7). Of course, $Y_{00}{}^1 = 1$ and $Y_{20}{}^1 = P_2(\cos\vartheta)$. Therefore the multipole fields of Eqs. (7.1.5) to (7.1.10) may be expressed in terms of spherical harmonics and spherical Hankel functions as follows:

$$p_s(\mathbf{r}) = \frac{\rho ck^2}{4\pi}\,S_\omega\,Y_{00}{}^1(\vartheta,\varphi)h_0(kr)$$

$$p_d(\mathbf{r}) = \frac{\rho ck^3}{4\pi}\,(D_x Y_{11}{}^1 + D_y Y_{11}{}^{-1} + D_z Y_{10}{}^1)h_1(kr)$$

$$\begin{aligned} p_q(\mathbf{r}) = \frac{\rho ck^4}{24\pi}\,\{&-2(Q_{xx} + Q_{yy} + Q_{zz})h_0(kr) \\ &+ [2(Q_{zz} - Q_{xx} - Q_{yy})\,Y_{20}{}^1(\vartheta,\varphi) \\ &+ 2(Q_{xz} + Q_{zx})\,Y_{21}{}^1 + 2(Q_{yz} + Q_{zy})\,Y_{21}{}^{-1} \\ &+ (Q_{xx} - Q_{yy})\,Y_{22}{}^1 + (Q_{xy} + Q_{yx})\,Y_{22}{}^{-1}]h_2(kr)\} \end{aligned} \tag{7.2.24}$$

In accord with page 314, there are only five distinct quadrupole distributions corresponding to the five different spherical harmonics of order 2.

Sphere with randomly vibrating surface

It should be evident by now that we are using the sphere as a simplified model of an object which has roughly the same width, height, and depth. The details of radiation from a more complicated shape differ somewhat from those computed for the sphere, but the general acoustical behavior will be pretty much the same. As we mentioned previously, a loudspeaker in a cubical cabinet will produce a sound field which is not markedly different from that portrayed in Figs. 7.2 and 7.3.

Another type of sound field which also can be approximated by using the spherical model is the sort of random noise which surrounds a piece of machinery in violent vibration, or the sort of roar which is generated when highly turbulent air reacts with the surface of a large object placed in it (or moving through it). In many cases the machine action (or the turbulence) produces vibrations of the outer surface of the object which are random in nature, containing components of all frequencies and having little correlation between the motion of one area of the surface and another. Such a source of sound can be approximated by a spherical surface in random oscillation.

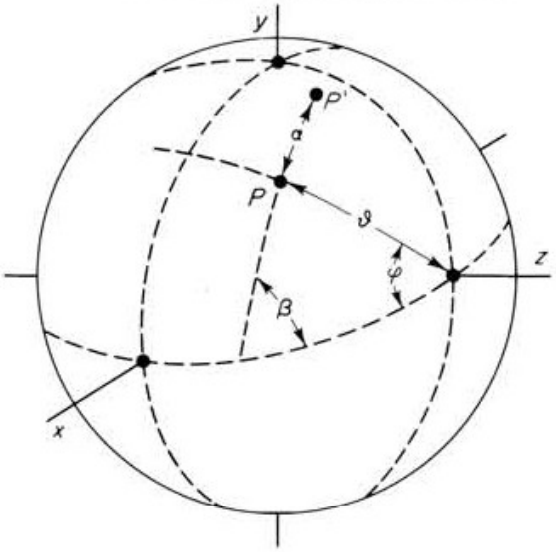


FIGURE 7.4
Angles defining the correlation function on the surface of a sphere.

To describe such motion we go back again to Eqs. (1.3.20) and (2.3.11) [see also Eqs. (7.1.33) to (7.1.38)], relating the Fourier transform of the motion and the autocorrelation function of the motion. We first take the Fourier transform, with respect to time, of the surface velocity, obtaining the component $U_\omega(\vartheta,\varphi)$, for the frequency $\omega/2\pi$, of the radial velocity of the part of the spherical surface at the point specified by the angles ϑ, φ.

Even the single-frequency motion is random in regard to its spatial distribution over the sphere, so that we use an autocorrelation function in space to describe this degree of randomness. We define this function by the integral

$$\Upsilon_\omega(\alpha) = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} U_\omega(P)U_\omega(P')\sin\vartheta\, d\vartheta \qquad (7.2.25)$$

where point P has coordinates ϑ, φ as shown in Fig. 7.4, and point P' is an angular distance α from P, in a direction at angle β with respect to the xz meridian. If the motion is isotropically random, Υ_ω will be independent of β. The value of $\Upsilon_\omega(0)$ is, of course, $|U_\omega^0|^2$, the mean-square radial velocity of the surface at frequency $\omega/2\pi$, the analogue of the spectrum density of Eq. (2.3.11). For random motion, $\Upsilon_\omega(\alpha)$ will decrease as α is increased, dropping to zero when α is larger than the "mean correlation length," divided by the radius a of the sphere [see the discussion of Eq. (2.3.13)].

A rough approximation to such a correlation function would thus be

$$\Upsilon_\omega(\alpha) = \begin{cases} (\cos\alpha)^{N^2-1}|U_\omega^0|^2 & 0 \leqslant \alpha \leqslant \tfrac{1}{2}\pi \\ 0 & \tfrac{1}{2}\pi \leqslant \alpha \leqslant \pi \end{cases} \qquad (7.2.26)$$

which equals $|U_\omega^0|^2$ at $\alpha = 0$ and drops to zero (more rapidly the larger N is), with no correlation assumed between opposite halves of the sphere. The mean angular width of the peak in α is roughly $2\pi/N$, so that the *correlation length*, the mean distance along the surface within which the motion is more or less in phase, is $w \simeq 2\pi a/N$, one-Nth of the perimeter of the sphere. We assume for the moment that N is independent of ω.

If this were a plane-surface problem [as that of Eq. (7.4.27) will be], so that the velocity and its correlation function could be expressed in terms of Fourier integrals, we could use Eq. (1.3.20) to find the transform of the velocity with respect to its wavenumber $k = 2\pi/\lambda$. Since the surface is a sphere, the analogue to the Fourier transform of $U(\vartheta,\varphi)$ is the coefficient $U_{mn}{}^{\sigma}$ in the expansion of $U(\vartheta,\varphi)$ in terms of the spherical harmonics $Y_{mn}{}^{\sigma}(\vartheta,\varphi)$, the discrete variable m being more or less the analogue of the continuous variable k. We should expect that the mean-square values of $U_{mn}{}^{\sigma}$ will be independent of n and σ, but will decrease slowly with m, corresponding to the decrease in component amplitude as the wavelength of the component decreases. The rate of this decrease should bear some relation to the correlation length w, analogous to the relationship between the dependence of Υ on τ and of $|F|^2$ on ω shown in Eq. (2.3.13).

The correlation function of Eq. (7.2.26) can be expanded in spherical harmonics; the coefficients are products of gamma functions, but when N is large, the approximate form,

$$\Upsilon_{\omega}(\alpha) \simeq \sum_{m=0}^{\infty} \frac{2\,|U_{\omega}^{0}|^2}{2m+1} N^{-2} e^{-2m(m+1)/N^2} P_m(\cos\alpha)$$

is a fairly good approximation. The coefficients for $n > 0$ do not appear because of the choice of axis centering around P for α; if the last of Eqs. (7.2.7) is used, the series can be transformed to one in $Y_{mn}{}^{\sigma}(\vartheta,\varphi)$. The coefficients of this series are the analogues of the Fourier transform of the correlation function, and thus should be proportional to the square of the radial velocity coefficient $U_{mn}{}^{\sigma}$. Consequently, the series for the radial velocity of the surface for frequency $\omega/2\pi$ is

$$U_{\omega}(\vartheta,\varphi) = \frac{U_{\omega}^{0}}{N} \sum_{m,n,\sigma} \left[\frac{2}{2m+1}\frac{(m-n)!}{(m+n)!}\epsilon_n\right]^{\frac{1}{2}} \exp\left[-\frac{m(m+1)}{N^2} + i\gamma_{mn}\right] \times Y_{mn}(\vartheta,\varphi) \quad (7.2.27)$$

where we have taken the square root of the coefficient for Υ_{ω}, multiplied by a factor depending on n, to make the mean square of the $n > 0$ terms equal to that for $n = 0$. We have also included a phase γ_{mn} to allow for the fact that the phase of each term is a random angle between 0 and 2π, corresponding to the random nature of the velocity. The series for Υ_{ω} does not specify these angles.

This series is an obvious generalization of the series of Eq. (7.2.14), and is thus ready for us to calculate the radiated intensity and power. From Eq. (7.2.17) the intensity at the point r, ϑ, φ, a considerable distance from the sphere, is

$$I_r \to \tfrac{1}{2}\rho c\,|U_{\omega}^{0}|^2 \left(\frac{a}{r}\right)^2 |\psi(\vartheta,\varphi)|^2$$

where the angle-distribution function $|\psi|^2$ contains squares and cross products

of the coefficients $U_{mn}{}^{\sigma}$. Because of the random nature of the phase relations between the $U_{mn}{}^{\sigma}$'s, and because these phase relations are changing with time (since the time variation is also random), the time average of $|\psi|^2$ will be the sum of the squared terms only.

$$|\psi|^2 \simeq (ka)^{-2} \sum_{m,n,\sigma} \frac{1}{2m+1} \frac{(m-n)!}{(m+n)!} \epsilon_n \frac{e^{-2m(m+1)N^2}}{N^2 B_m{}^2} (Y_{mn}{}^{\sigma})^2$$

$$= \frac{1}{N^2 k^2 a^2} \sum_{m=0}^{\infty} \frac{e^{-2m(m+1)/N^2}}{(2m+1) B_m{}^2} \qquad N \gg 1 \tag{7.2.28}$$

where the sum over n, σ has been accomplished by use of the last of Eqs. (7.2.7) for $\vartheta = \theta$, $\varphi = \phi$. Thus the radiated intensity is independent of angle, as it should be if the random motion is isotropic.

This averaged series can be still further simplified by referring to the approximate formulas for B_m given in Eqs. (7.2.15). When m is smaller than $ka = 2\pi a/\lambda$, $(1/B_m)^2 \simeq (ka)^2$ and, as m is made larger than ka, $(1/kaB_m)^2$ drops rapidly to zero. Consequently, an approximate series for $|\psi|^2$, valid for $N^2 \gg 1$, is

$$|\psi|^2 \simeq \frac{1}{N^2} \sum_{m=0}^{ka} \frac{e^{-2m(m+1)/N^2}}{2m+1} \simeq 1 - e^{-(ka/N)^2} \tag{7.2.29}$$

where

$$\frac{ka}{N} = \frac{2\pi a}{\lambda} \frac{w}{2\pi a} = \frac{w}{\lambda} = \frac{w\omega}{2\pi c}$$

For very small values of this parameter, $|\psi|^2$ is approximately $(ka/N)^2$; for very large values, $|\psi|^2$ is approximately 1. The last expression for $|\psi|^2$ is a very crude approximation to the earlier series, but it should be satisfactory in view of the approximations we have been making on account of the random nature of the motion. It shows that when the correlation length w is small compared with λ, the intensity radiated at this wavelength, $\frac{1}{2}\rho c\,|U_\omega|^2\,(wa/\lambda r)^2$, is as if a circular patch of the sphere's surface, of radius w, is sending energy to the point r, ϑ, φ, the radiation from all the rest of the surface interfering to zero. On the other hand, when w is large compared with λ (i.e., for the high-frequency part of the radiation), the radiated intensity $\frac{1}{2}\rho c\,|U_\omega|^2\,(a^2/r^2)$ is as if the whole spherical surface were radiating with surface velocity equal to the root-mean-square velocity U_ω. When w is small, I_r is proportional to $\pi(wa/\lambda)^2$; when w is large, I_r is proportional to the total area πa^2 visible by the observer.

To bring in the time dependence of the motion, we must invert the Fourier analysis of ψ in frequency. The surface velocity is also random in time, with a time-correlation function $\Upsilon_f(\tau)$, which decreases as τ increases, like the Υ_f of Eq. (2.3.13). The spectrum level $|F(\omega)|^2$ for the surface velocity, the square of the Fourier transform of the root-mean-square velocity of the surface, should be some function which goes to zero when ω

becomes larger than some upper-limit value ω_n. Equation (2.3.13) shows that the correlation time interval τ_n is the reciprocal of ω_n.

Suppose the time-correlation function for the surface velocity has the familiar form

$$\Upsilon_f(\tau) = |U_0|^2 e^{-\frac{1}{2}\omega_n^2\tau^2} \qquad \tau_n = \frac{1}{\omega_n}$$

where $|U_0|^2$ is the mean-square surface velocity, averaged over the surface of the sphere and also over time T. According to Eq. (1.3.20), the spectrum density $|F(\omega)|^2$ for velocity is proportional to the Fourier transform of Υ_f,

$$|F(\omega)|^2 = \frac{T}{4\pi^2}\int_{-\infty}^{\infty} \Upsilon_f(\tau)e^{i\omega\tau}\,d\tau = \frac{T|U_0|^2}{\omega_n(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(\omega/\omega_n)^2}$$

Therefore the spectrum density for intensity at r, ϑ, φ is $\frac{1}{2}\rho c(a^2/r^2)\,|F(\omega)|^2\,|\psi|^2$, where ψ is the angle-distribution factor of Eqs. (7.2.28) and (7.2.29). The mean value of the intensity, averaged over time and surface, is then [according to Eq. (1.3.19)]

$$\begin{aligned}\langle I_r\rangle &= \frac{2\pi}{T}\int_{-\infty}^{\infty} I_r(\omega)\,d\omega \\ &= -\rho c\left(\frac{a}{r}\right)^2 \frac{|U_0|^2}{\omega_n\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\omega/\omega_n)^2}[1 - e^{-(w\omega/2\pi c)^2}]\,d\omega \qquad (7.2.30) \\ &= \tfrac{1}{2}\rho c\left(\frac{a}{r}\right)^2 |U_0|^2 \frac{2w^2}{\lambda_n^2 + 2w^2} \qquad \lambda_n = \frac{2\pi c}{\omega_n}\end{aligned}$$

When the wavelength λ_n, corresponding to upper limit ω_n of the spectrum density, is small compared with the correlation length w of velocity distribution over the spherical surface, the intensity is as though the whole sphere were vibrating in phase, with a velocity amplitude equal to the root-mean-square velocity U_0. When λ_n is large compared with w, that is, when the cutoff frequency is low or the correlation length w is very small, the intensity is much smaller, as though a patch of the sphere, of radius aw/λ_n, were doing the radiating. In many cases encountered in practice, neither of these limiting cases holds, λ_n being one to four times w.

Many modifications and improvements could be made in this calculation; a different dependence on ω of the spectrum level could be used, or we could have taken into account possible variation of the correlation length on ω, or we could have used a better approximation for the factor $|\psi|^2$ of Eq. (7.2.29). The resulting formulas would then have had a somewhat different appearance, but if the surface vibration is random and isotropic, the relationship between I_r and the spatial and temporal correlation intervals, w and $1/\omega_n$, will be more or less that set forth in Eq. (7.2.30).

Green's function in spherical coordinates

To compute the radiation from a volume distribution of sources which is not small compared with the wavelength, or to develop an alternative way of computing the radiation from a vibrating sphere of moderate size, we need the expansion of the Green's function $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$ in terms of the coordinates r, ϑ, φ of the measurement point and r_0, ϑ_0, φ_0 of the source point. This expansion is

$$\begin{aligned} g_\omega(\mathbf{r} \mid \mathbf{r}_0) &= \frac{ik}{4\pi} h_0(kR) \\ &= \frac{ik}{4\pi} \sum_{m,n,\sigma} (2m+1)\epsilon_n \frac{(m-n)!}{(m+n)!} Y_{mn}{}^{\sigma}(\vartheta,\varphi) \\ &\qquad \times Y_{mn}{}^{\sigma}(\vartheta_0,\varphi_0) \begin{cases} j_m(kr_0)h_m(kr) & r > r_0 \\ j_m(kr)h_m(kr_0) & r < r_0 \end{cases} \end{aligned} \tag{7.2.31}$$

which exhibits the reciprocity between measurement and source points.

For example, if a region of violent motion is confined inside a sphere of radius a (not necessarily small compared with λ), the radiation outside $r = a$ can be obtained by inserting series (7.2.31) in Eq. (7.1.25). The required gradients of the series for g_ω can be worked out from the formulas of (7.2.5) and (7.2.13). After a lot of algebra we obtain

$$\begin{aligned} \frac{\partial}{\partial z} Y_{mn}{}^{\sigma}(\vartheta,\varphi) j_m(kr) &= \frac{k}{2m+1} [(m+n) Y^{\sigma}_{m-1,n}(\vartheta,\varphi) j_{m-1}(kr) \\ &\qquad - (m-n) Y^{\sigma}_{m+1,n}(\vartheta,\varphi) j_{m+1}(kr)] \\ \frac{\partial}{\partial x} Y_{mn}{}^{\sigma} j_m &= \frac{k}{(2m+1)\epsilon_n} [(m+n)(m+n-1) Y^{\sigma}_{m-1,n-1} j_{m-1} \\ &\qquad - Y^{\sigma}_{m-1,n+1} j_{m-1} \\ &\qquad + (m-n+1)(m-n+2) Y^{\sigma}_{m+1,n-1} j_{m+1} - Y^{\sigma}_{m+1,n+1} j_{m+1}] \\ \frac{\partial}{\partial y} Y_{mn}{}^{\sigma} j_m &= \frac{-\sigma k}{(2m+1)\epsilon_n} \{[(m+n)(m+n-1) Y^{-\sigma}_{m-1,n-1} \\ &\qquad + Y^{-\sigma}_{m-1,n+1}] j_{m-1} \\ &\qquad + [(m-n+1)(m-n+2) Y^{-\sigma}_{m+1,n-1} + Y^{-\sigma}_{m+1,n+1}] j_{m+1}\} \end{aligned} \tag{7.2.32}$$

When r is considerably larger than a, large enough so that $h_m(kr) \to (i^{-m-1}/kr)e^{ikr}$, the pressure wave can be expressed in terms of the integrals of the various source functions over the region of violent motion ($r < a$).

$$S^{\sigma}(m,n) = \frac{i}{\rho c k} \int f_\omega(\mathbf{r}) Y_{mn}{}^{\sigma}(\vartheta,\varphi) j_m(kr)\, dv$$

$$D_x{}^{\sigma}(m,n) = \frac{i}{\rho c k} \int F_x(\mathbf{r}) \frac{\partial}{\partial x} [Y_{mn}{}^{\sigma}(\vartheta,\varphi) j_m(kr)]\, dv \qquad \cdots$$

$$Q_{xy}{}^{\sigma}(m,n) = \frac{i}{\rho c k} \int T_{xy}(\mathbf{r}) \frac{\partial^2}{\partial x\, \partial y} (Y_{mn}{}^{\sigma} j_m)\, dv \qquad \cdots$$

Referring to the discussion following Eqs. (7.2.12) and (7.2.15), we see that when $ka \to 0$, all these coefficients go as $(ka)^m$ to zero, except for $S^1(0,0) \to S_\omega$, the strength of the equivalent monopole source. Also, the largest D is $D_x{}^1(0,0) \to D_x$, etc., for the three dipole components, and the largest Q's are $Q_{xy}{}^1(0,0) \to Q_{xy}$, etc. When ka is not small, more terms in the series must be used. The full series is

$$p_\omega(\mathbf{r}) \simeq -\frac{ik\rho c}{4\pi r}\, e^{ikr}[\Phi_s(\vartheta,\varphi) - ik\Phi_d(\vartheta,\varphi) - k^2\Phi_q(\vartheta,\varphi)] \qquad (7.2.33)$$

for $r > a$ and $kr \gg 1$, where

$$\Phi_s = \sum_{mn\sigma} i^{-m}\epsilon_n\,(2m+1)\,\frac{(m-n)!}{(m+n)!}\,S^\sigma(m,n)\,Y_{mn}{}^\sigma(\vartheta,\varphi)$$

The first term in the series is a spherically symmetric wave of strength $S^1(0,0)$. This is the *only* term which appears *if* $f_\omega(\mathbf{r}_0)$ is spherically symmetric, independent of ϑ_0 and φ_0. But if f_ω does depend on ϑ_0 and/or φ_0, some of the coefficients $S^\sigma(m,n)$ will differ from zero, and the radiation from the distribution of simple sources, in a region not small compared with λ, will not be spherically symmetric; some higher multipole radiation will be present. The amplitude of the mth multipole is proportional to $(ka)^m$, becoming negligible when ka is very small.

Likewise, the radiation from the driving-force distribution $\mathbf{F}_\omega(\mathbf{r})$ will not be a pure dipole wave if ka is not very small. For example, the wave from the z component of the force, once we have inserted Eqs. (7.2.31) and (7.2.32) in the appropriate part of (7.1.25), is the combination

$$\begin{aligned}
&-\frac{ik\rho c}{4\pi r}\, e^{ikr}\sum_{mn\sigma} i^{-m}\epsilon_n\,\frac{(m-n)!}{(m+n)!}\,(2m+1)\,Y_{mn}{}^\sigma(\vartheta,\varphi)\\
&\qquad\times\iiint F_z\,\frac{\partial}{\partial z_0}\,Y_{mn}{}^\sigma(\vartheta_0,\varphi_0)j_m(kr_0)\,dv_0\\
&=\frac{ik^2\rho c}{4\pi r}\, e^{ikr}\sum i^{-m}\epsilon_n\,\frac{(m-n)!}{(m+n)!}\,Y_{mn}{}^\sigma(\vartheta,\varphi)\\
&\qquad\times\left[(m-n)\iiint F_z\,Y_{m+1,n}j_{m+1}\,dv_0 - (m+n)\iiint F_z\,Y_{m-1,n}j_{m-1}\,dv_0\right]
\end{aligned} \qquad (7.2.34)$$

If $F_z(\mathbf{r}_0)$ is independent of ϑ_0 and φ_0, only the $m = 1$, $n = 0$, $\sigma = 1$ term differs from zero, since it is the only one having the nonzero integral $-F_z Y_{00}{}^1 j_0\,dv_0$. The angle distribution of this term is $Y_{10}{}^1(\vartheta,\varphi)$, typical of a dipole pointed along the z axis. But if $F_z(\mathbf{r}_0)$ is a function of ϑ_0 and/or φ_0, other integrals can be nonzero, and the radiation will contain multipole terms. This is also true of the radiation from the matrix components of $\mathfrak{T}$. Only when all components are independent of ϑ_0 and φ_0 will the radiation be purely quadrupole if ka is not small. This result was anticipated in our discussion of Eq. (7.1.30).

Spherical-surface effects

The Green's function $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$, defined in Eq. (7.1.14) and expanded in series (7.2.31), is appropriate for calculating acoustic radiation in an unbounded medium. When there are boundaries enclosing the medium at which boundary conditions must be satisfied, it is usually convenient to modify g_ω by the addition of a free standing wave $\chi(\mathbf{r})$ of such a form that $G_\omega = g_\omega + \chi$ satisfies some boundary condition on the surface. The general equation (7.1.17), for p_ω in terms of integrals of G_ω, indicates the alternatives. As it stands, we should have to specify both the value of p and of its normal gradient on the surface S_0 in order to compute the surface integral. But if we arranged that G_ω was zero on S_0, then we should need only to specify p_ω, or alternatively, if we chose χ so that the normal gradient of G_ω was zero on S_0, we should need only to specify the normal gradient of p on S_0 in order to determine p uniquely elsewhere, in the region occupied by the medium. Since we are often interested in the effects of the motion of a part of the boundary surface, and since, for simple-harmonic motion, $\mathbf{u} = (1/ik\rho c)\,\text{grad}\,p$, it appears that we should prefer a G_ω which has a zero normal gradient on the surface. When this choice is made, Eq. (7.1.17) becomes

$$p_\omega(\mathbf{r}) = \iiint f_\omega(r_0) G_\omega(\mathbf{r} \mid \mathbf{r}_0)\, dv_0 + ik\rho c \iint u_n(\mathbf{r}_0{}^s) G_\omega(\mathbf{r} \mid \mathbf{r}_0{}^s)\, dS_0 \qquad (7.2.35)$$

where $u_n(\mathbf{r}_0{}^s) = \dfrac{1}{ik\rho c}\dfrac{\partial p_\omega}{\partial n_0}$ is the normal velocity of the boundary surface S_0 at the surface point $\mathbf{r}_0{}^s$. The positive direction of u_n is outward, away from the medium; the equation is valid for any point $\mathbf{r}$ within or on the boundary surface S_0.

To demonstrate the use of this equation, we first consider the fluid enclosed in a hollow rigid sphere of internal radius a. Here we must add to the series of Eq. (7.2.31) a solution of $(\nabla^2 + k^2)\chi = 0$ which has no poles within or on the surface of the sphere, such that the radial gradient of the sum $g_\omega + \chi$ is zero at $r = a$. Symbolically, this is not difficult to do. If $j'_m(\zeta) = dj_m/d\zeta$, the function

$$G_\omega(\mathbf{r} \mid \mathbf{r}_0) = \frac{ik}{4\pi} \sum_{m=0}^{\infty} (2m+1) \sum_{n,\sigma} \epsilon_n \frac{(m-n)!}{(m+n)!} Y_{mn}{}^{\sigma}(\vartheta,\varphi) \times Y_{mn}{}^{\sigma}(\vartheta_0,\varphi_0) \left[H_m - \frac{h'_m(ka)}{j'_m(ka)} j_m(kr) j_m(kr_0) \right] \qquad \text{for } r, r_0 \leqslant a \qquad (7.2.36)$$

where

$$H_m = \begin{cases} j_m(kr_0) h_m(kr) & r > r_0 \\ j_m(kr) h_m(kr_0) & r < r_0 \end{cases}$$

has a radial gradient zero at $r = a$ (when $r_0 \leqslant a$) and also at $r_0 = a$ (when $r \leqslant a$). The series with H_m alone is g_ω; the additional series is finite and

continuous inside r, $r_0 = a$ and modifies g_ω to make $\partial G_\omega/\partial r = 0$ at $r = a$; it represents the wave reflected by the rigid spherical surface.

To use Eq. (7.2.35), we note that S_0 is the sphere of radius a and that the positive direction of the normal velocity is outward, in the positive r direction. If the sphere is rigid, the surface integral is zero, and p_ω equals the volume integral of G_ω times the source function $f_\omega(\mathbf{r}_0)$, integrated over the interior of the sphere. We note that when ka is such that $j_m'(ka)$ is zero, the reflected wave χ becomes infinite; the source frequency has coincided with one of the resonance frequencies of the hollow sphere. Green's functions for enclosures of finite size exhibit this resonance property, as with the Green's function of Eq. (4.4.14) for the string between two rigid supports a finite distance apart.

On the other hand, if the region of interest is between the outer surface of a sphere of radius a and the sphere at infinity, we must add to g_ω a function χ such that the radial gradient of the sum is zero at $r = a$ (when $r_0 \geqslant a$). The combination which does this is

$$G_\omega = \frac{ik}{4\pi}\sum_{m=0}^{\infty}(2m+1)\sum_{n,\sigma}\epsilon_n\frac{(m-n)!}{(m+n)!}\,Y_{mn}^{\sigma}(\vartheta,\varphi) \times Y_{mn}^{\sigma}(\vartheta_0,\varphi_0)\left[H_m - \frac{j_m'(ka)}{h_m'(ka)}\,h_m(kr)h_m(kr_0)\right]$$

$$\to \frac{e^{ikr}}{4\pi r}\sum_{m=0}^{\infty}(2m+1)i^{-m}\sum_{n,\sigma}\epsilon_n\frac{(m-n)!}{(m+n)!}\,Y_{mn}^{\sigma}(\vartheta,\varphi) \times Y_{mn}^{\sigma}(\vartheta_0,\varphi_0)\left[\frac{h_m'(ka)j_m(kr_0) - j_m'(ka)h_m(kr_0)}{h_m'(ka)}\right] \qquad kr \to \infty \quad (7.2.37)$$

Since $h_m'(ka)$ has no zeros for real values of ka, this Green's function has no resonances, in contrast to that of Eq. (7.2.36).

If the only source of acoustic energy, in the region outside $r = a$, comes from the motion of the spherical surface itself, the only nonzero integral in Eq. (7.2.35) is the surface integral

$$p_\omega(\mathbf{r}) = ik\rho ca^2\int_0^{2\pi}d\varphi_0\int_0^{\pi}u_nG_\omega\sin\vartheta_0\,d\vartheta_0$$

$$= \rho ck^2a^2\sum_{mn\sigma}U_{mn}^{\sigma}\left[\frac{h_m'(ka)j_m(ka) - j_m'(ka)h_m(ka)}{h_m'(ka)}\right]Y_{mn}^{\sigma}(\vartheta,\varphi)h_m(kr)$$

where

$$U_{mn}^{\sigma} = \frac{\epsilon_n}{4\pi}(2m+1)\frac{(m-n)!}{(m+n)!}\int_0^{2\pi}d\varphi_0\int_0^{\pi}[-u_n(\vartheta_0,\varphi_0)]\,Y_{mn}^{\sigma}(\vartheta_0,\varphi_0)\sin\vartheta_0\,d\vartheta_0$$

since the normal velocity u_n away from the fluid is here pointed in toward the origin; thus $-u_n$ is the radial velocity of the surface of the sphere.

From Eq. (7.2.12) we see that $h_m' j_m - j_m' h_m = i/k^2a^2$, and from Eq. (7.2.15) we see that $h_m'(ka) = iB_m e^{i\delta_m}$, so that

$$p_\omega(\mathbf{r}) = \rho c \sum_{mn\sigma} \frac{U_{mn}{}^{\sigma}}{B_m} e^{-i\delta_m} Y_{mn}{}^{\sigma}(\vartheta,\varphi) h_m(kr)$$

$$\to -\frac{i\rho c}{kr} e^{ikr} \sum_{mn\sigma} i^{-m} \frac{U_{mn}{}^{\sigma}}{B_m} e^{-i\delta_m} Y_{mn}{}^{\sigma}(\vartheta,\varphi) \qquad kr \to \infty \quad (7.2.38)$$

which is an obvious generalization of Eq. (7.2.16). Thus the integral formula (7.2.35) can be used to compute radiation from any source, whether it be a moving surface or the motion of the fluid itself.

7.3 RADIATION FROM CYLINDERS

The other simple geometrical surface which is often encountered in acoustics is the circular cylinder. Waves of cylindrical shape are obvious generalizations, to three dimensions, of the waves on a circular membrane, already discussed in Sec. 5.2. We showed there that circular waves have a more complicated shape than plane or spherical waves; they leave a "wake" behind them as they spread out. Cylindrical waves show the same behavior. As with the rest of this chapter, we shall concentrate on simple-harmonic waves; solutions with more general time dependence can be computed by use of the Fourier transform.

The general solution

The equation $\nabla^2 p + k^2 p = 0$ $(k = \omega/c)$ can be expressed in the cylindrical coordinates w, ϕ, z by using Eq. (7.1.3). The separated equations for the factored solution $p = R(w)\Phi(\phi)Z(z)e^{-i\omega t}$ are

$$\frac{d^2Z}{dz^2} + k_z{}^2 Z = 0 \qquad \frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\frac{1}{w}\frac{d}{dw}\left(w\frac{dR}{dw}\right) + \left(k_w{}^2 - \frac{m^2}{w^2}\right)R = 0 \qquad k^2 = k_w{}^2 + k_z{}^2 \quad (7.3.1)$$

The general solution for Z is a combination of $e^{ik_z z}$ and $e^{-ik_z z}$; the solution for Φ is $\cos(m\phi)$ or $\sin(m\phi)$, where m must be an integer for the solution to be continuous at $\phi = 0 = 2\pi$. The solution of the radial equation is a combination of the Bessel function $J_m(k_w w)$, defined in Eqs. (5.2.20) and (5.2.21), and the Neumann function $N_m(k_w w)$, mentioned in Eq. (5.3.15) as being the second solution of the Bessel equation for R. They have the

following properties:

$$\frac{1}{z}\frac{d}{dz}\left(z\frac{dN_m}{dz}\right)+\left(1-\frac{m^2}{z^2}\right)N_m=0$$

$$N_m(z)\xrightarrow[z\to\infty]{}\sqrt{\frac{2}{\pi z}}\sin\left(z-\frac{2m+1}{4}\pi\right)$$

$$N_0(z)\xrightarrow[z\to 0]{}\frac{2}{\pi}\ln(0.890536z)=\frac{2}{\pi}(\ln z-0.11593) \tag{7.3.2}$$

$$N_m(z)\xrightarrow[z\to 0]{}-\frac{(m-1)!}{\pi}\left(\frac{2}{z}\right)^m \qquad m>0$$

$$N_{m-1}(z)J_m(z)-N_m(z)J_{m-1}(z)=\frac{2}{\pi z}$$

The properties given in Eqs. (5.2.21) for the functions $J_m(z)$ are also true for the corresponding functions $N_m(z)$.

We note that $N_m(k_w w)$ goes to infinity as $w\to 0$, whereas $J_m(k_w w)$ is finite at $w=0$. We also note that the combination $H_m^{(1)}(k_w w)=J_m(k_w w)+iN_m(k_w w)$ represents an outgoing wave, for

$$J_m(k_w w)+iN_m(k_w w)\to\sqrt{\frac{2}{\pi k_w w}}\exp\left[ik_w w-\tfrac{1}{2}\pi i(m+\tfrac{1}{2})\right] \qquad k_w w\gg 1$$

For $k_w w$ near zero, in the region of the near field, the combination is not just a simple exponential; in fact, the imaginary part of the combination goes to infinity as $w\to 0$. $H_m^{(1)}$ is called a *Hankel function* of mth order.

Uniform radiation

For waves spreading uniformly out from a cylinder, we use the function for $k_z=0$ and $m=0$, which represents outgoing waves,

$$p=A\left[J_0\left(\frac{2\pi\nu w}{c}\right)+iN_0\left(\frac{2\pi\nu w}{c}\right)\right]e^{-2\pi i\nu t}$$

$$\xrightarrow[w\to\infty]{}A\sqrt{\frac{2}{\pi k w}}\,e^{ik(w-ct)-i(\pi/4)} \qquad k=\frac{2\pi\nu}{c}=\frac{2\pi}{\lambda} \tag{7.3.3}$$

$$\xrightarrow[w\to 0]{}i\frac{2A}{\pi}\ln(w)e^{-2\pi i\nu t}$$

Suppose that we have a long cylinder of radius a which is expanding and contracting uniformly in such a manner that the velocity of the surface of the cylinder is $u_0=U_0e^{-2\pi i\nu t}$. The constant A, to correspond to the radiated wave, must be chosen so that the velocity of the air perpendicular

to the cylinder surface $u_w = \dfrac{1}{2\pi i\nu\rho}\dfrac{\partial p}{\partial w}$ is equal to u_0 at $w = a$. If a is small compared with the wavelength, this velocity is $(A/\pi^2\nu\rho a)e^{-2\pi i\nu t}$, so that A must equal $\pi^2\nu\rho a U_0$. The pressure and velocity at large distances from the cylinder are then

$$p \to \pi\rho a U_0 \sqrt{\frac{c\nu}{w}}\, e^{ik(w-ct)-i(\pi/4)}$$

$$u_w \to \pi a U_0 \sqrt{\frac{\nu}{cw}}\, e^{ik(w-ct)-i(\pi/4)}$$

The product of the real part of each of these expressions gives the flow of energy outward per second per square centimeter, and the average value of this,

$$I \simeq \pi^2\rho a^2 U_0^2 \frac{\nu}{w} \qquad a \ll \frac{c}{\omega} \tag{7.3.4}$$

is the intensity of the sound at a distance w from the cylinder's axis. The total energy radiated in ergs per second per centimeter length of the cylinder is

$$\Pi \simeq 2\pi^3\rho\nu a^2 U_0^2$$

Radiation from a vibrating wire

A somewhat more complicated wave is generated by a cylinder of radius a vibrating back and forth in a direction perpendicular to its axis, with a velocity $U_0 e^{-2\pi i\nu t}$. If the plane of vibration is taken as the reference plane for ϕ, the velocity of the part of the cylinder's surface at an angle ϕ from the plane of vibration has a component $U_0 \cos\phi\, e^{-2\pi i\nu t}$ perpendicular to the surface. In this case we take the radiated wave to be

$$p = A\cos\phi[J_1(kw) + iN_1(kw)]e^{-2\pi i\nu t} \qquad k = \frac{2\pi\nu}{c}$$

$$\xrightarrow[w\to 0]{} -i\frac{AC}{\pi^2\nu w}\cos\phi\, e^{-2\pi i\nu t}$$

$$\xrightarrow[w\to\infty]{} A\sqrt{\frac{c}{\pi^2\nu w}}\, e^{ik(w-ct)-i(3\pi/4)}\cos\phi$$

If a is small, the radial velocity at $w = a$ is

$$\frac{Ac}{2\pi^3\nu^2\rho a^2}\cos\phi\, e^{-2\pi i\nu t}$$

which must equal the radial velocity of the surface, so that $A = 2\pi^3\nu^2\rho a^2 U_0/c$. The radial component of the particle velocity at large distances from the cylinder is

$$u_w \xrightarrow[w\to\infty]{} \frac{A}{\rho c}\sqrt{\frac{c}{\pi^2\nu w}}\, e^{ik(w-ct)-i(3\pi/4)}\cos\phi$$

There is a component of particle velocity perpendicular to the radius w, but

it diminishes as $w^{-\frac{3}{2}}$ at large distances, and so is negligible there compared with u_w. The intensity at large distances and the total power radiated per centimeter length of vibrating wire are

$$I \simeq \frac{4\pi^4\nu^3\rho a^4 U_0^2}{c^2 r}\cos^2\phi \qquad \Pi \simeq \frac{4\pi^5\nu^3\rho a^4 U_0^2}{c^2} \tag{7.3.5}$$

The amount of sound energy radiated by a vibrating string therefore diminishes rapidly as the frequency of vibration of the string decreases and diminishes very rapidly if the string's thickness is decreased. As we mentioned at the end of Chap. 4, a vibrating string is a very inefficient radiator of sound.

The reaction of the air back on the vibrating wire is obtained from the expression for the pressure at $w = a$.

$$p \simeq -i\omega\rho a U_0 \cos\phi\, e^{-i\omega t}$$

The net reaction force on the wire per unit length in the direction of its motion is

$$a\int_0^{2\pi} p\cos\phi\, d\phi = -F_0 e^{-i\omega t} \simeq -i\omega\rho(\pi a^2)U_0 e^{-i\omega t}$$

The ratio of this to the velocity of the wire is the mechanical impedance per unit length of wire, due to sound radiation,

$$\frac{F_0}{U_0} \simeq -i\omega(\pi a^2\rho) \qquad a \ll \frac{c}{\omega}$$

which is equivalent to the reactance of a mass of air of volume equal to that of the wire. The resistive part R of this impedance is too small to be included in this approximation when a is small. We can find the resistive part from Eq. (7.3.5) for the power radiated, for of course $\Pi = RU_0^2$,

$$Z_{\text{rad}} \simeq -i\omega(\pi a^2\rho) + \frac{\pi^2\omega^3\rho a^4}{2c^2} \qquad a \ll \frac{c}{\omega} \tag{7.3.6}$$

In the discussion of Eqs. (10.1.9) and (10.1.19) it will be shown, however, that a string of finite length behaves quite differently.

Radiation from an element of a cylinder

To solve more complicated problems, where the velocity of the surface of the cylinder is a less simple function of ϕ than the preceding examples, it is convenient first to solve the problem where only a single line element on the surface of the cylinder does the vibrating. Suppose that the radial velocity of the surface $w = a$ is

$$u_a = \begin{cases} Ue^{-2\pi i\nu t} & -\dfrac{d\alpha}{2} < \phi < +\dfrac{d\alpha}{2} \\ 0 & +\dfrac{d\alpha}{2} < \phi < 2\pi - \dfrac{d\alpha}{2} \end{cases}$$

The Fourier-series expansion for this function of ϕ is

$$u_a = \frac{U\,d\alpha}{\pi}\left[\tfrac{1}{2} + \sum_{m=1}^{\infty} \cos(m\phi)\right] e^{-2\pi i\nu t}$$

To fit this distribution of velocity at the surface, we choose a general sort of outgoing wave.

$$p = \sum_{m=0}^{\infty} A_m \cos(m\phi)[J_m(kw) + iN_m(kw)]e^{-2\pi i\nu t} \qquad k = \frac{2\pi\nu}{c}$$

The corresponding radial particle velocity at $w = a$ is

$$u_a = \frac{1}{2\pi i\nu\rho_0}\frac{\partial p}{\partial w} = \left[\frac{A_0E_0}{2\rho_0 c}\,e^{i\gamma_0} + \sum_{m=1}^{\infty}\frac{A_mE_m}{\rho c}\,e^{i\gamma_m}\cos(m\phi)\right]e^{-2\pi i\nu t}$$

where

$$2\frac{d}{d\mu}[J_0(\mu) + iN_0(\mu)] = iE_0e^{i\gamma_0}$$

$$\frac{d}{d\mu}[J_m(\mu) + iN_m(\mu)] = iE_me^{i\gamma_m} \qquad m > 0$$

and where $\mu = ka$. Therefore, using Eqs. (5.2.20),

$$\begin{aligned} J_1(ka) &= \tfrac{1}{2}E_0\sin\gamma_0 \qquad N_1(ka) = -\tfrac{1}{2}E_0\cos\gamma_0 \\ J_{m+1}(ka) - J_{m-1}(ka) &= 2E_m\sin\gamma_m \qquad m > 0 \\ N_{m-1}(ka) - N_{m+1}(ka) &= 2E_m\cos\gamma_m \end{aligned}$$

Note the additional factor of 2 in E_0, to anticipate the factor $\frac{1}{2}$ in the sum for $m = 0$. Limiting values of the amplitudes E_m and phase angles γ_m are given by the following approximate formulas:

when

$$ka = \frac{2\pi a}{\lambda} \gg m + \tfrac{1}{2}$$

$$E_0 \simeq \sqrt{\frac{8}{\pi ka}} \qquad \gamma_0 \simeq ka - \frac{\pi}{4}$$

$$E_m \simeq \sqrt{\frac{2}{\pi ka}} \qquad \gamma_m \simeq ka - \tfrac{1}{2}\pi(m + \tfrac{1}{2}) \qquad m > 0 \tag{7.3.7}$$

when

$$ka = \frac{2\pi a}{\lambda} \ll m + \tfrac{1}{2}$$

$$E_0 \simeq \frac{4}{\pi ka} \qquad \gamma_0 \simeq \pi\left(\frac{ka}{2}\right)^2$$

$$E_m \simeq \frac{m!}{2\pi}\left(\frac{2}{ks}\right)^{m+1} \qquad \gamma_m \simeq -\frac{\pi m}{(m!)^2}\left(\frac{ka}{2}\right)^{2m} \qquad m > 0$$

Values of some of the E's and γ's are given in Table V.

To fit the expression for u_w at $w = a$ to the expression for the velocity of the cylinder u_a, we must make

$$A_m = \frac{\rho c U\, d\alpha}{\pi E_m} e^{-i\gamma_m}$$

Since, at very large distances from the cylinder,

$$J_m(kw) + iN_m(kw) \simeq \sqrt{\frac{2}{\pi kw}}\, e^{i\left[kw - \frac{\pi}{4}(2m+1)\right]}$$

we have the following expressions for the pressure, particle velocity, and intensity of sound at the point w, ϕ (when w is a large number of wavelengths) and for the total power radiated by the element per unit length of cylinder:

$$u_w \simeq U\, d\alpha \sqrt{\frac{a}{w}}\, e^{ik(w-ct)}\psi(\phi) \qquad p \simeq \rho c u$$

$$\psi(\phi) = \sqrt{\frac{2}{\pi^3 ka}} \sum_{m=0}^{\infty} \frac{\cos(m\phi)}{E_m} e^{-i\left[\gamma_m + \frac{\pi}{4}(2m+1)\right]} \tag{7.3.8}$$

$$I \simeq \frac{\rho c^2 (U\, d\alpha)^2}{\pi^4 \nu w} \sum_{m,n=0}^{\infty} \frac{\cos(m\phi)\cos(n\phi)}{E_m E_n} \cos\left[\gamma_m - \gamma_n + \tfrac{1}{2}\pi(m-n)\right]$$

$$\Pi = \frac{\rho c^2 (U\, d\phi\alpha)^2}{\pi^3 \nu} \left(\frac{2}{E_0^2} + \sum_{m=1}^{\infty} \frac{1}{E_m^2}\right)$$

To find the intensity, we have, of course, multiplied the real parts of u_w and p together and averaged over time. The total power Π per unit length of cylinder is the integral $w\int_0^{2\pi} I\, d\phi$.

Long- and short-wave limits

When the wavelength is quite long compared with $2\pi a$, we can use the second part of Eqs. (7.3.7) to compute the radiation. The largest values of $1/E_m$ are for $m = 0$, so that, to the first approximation,

$$\psi(\phi) \simeq \sqrt{\frac{ka}{8\pi}}\, e^{-i\pi/4} \tag{7.3.9}$$

$$I \simeq \frac{\rho \nu a^2}{4w} (U\, d\alpha)^2 \qquad \Pi \simeq \frac{\pi \rho \nu a^2}{2} (U\, d\alpha)^2$$

At these low frequencies the sound radiates out with equal intensity *in all directions*, and the amount radiated is small. The expression for intensity is the same as that given in Eq. (7.3.4) for a uniformly expanding cylinder, if we substitute for U_0 in the earlier expression the *average velocity* $U\, d\alpha/2\pi$ of the surface.

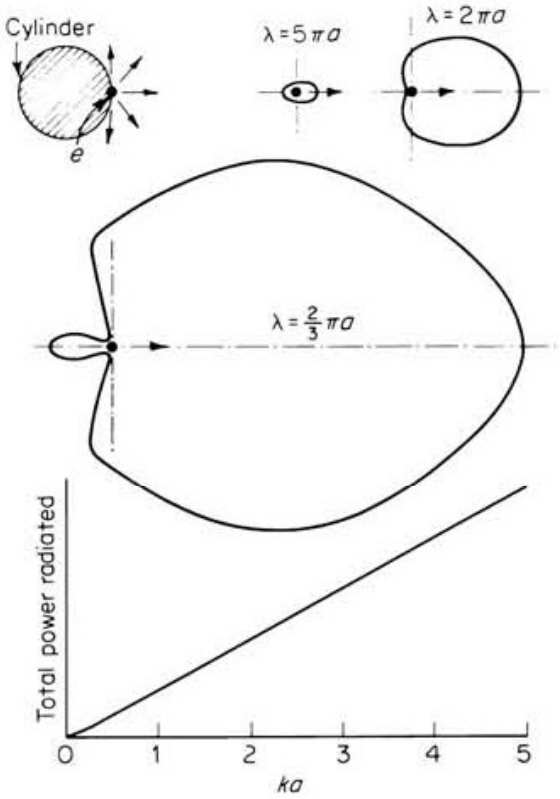


FIGURE 7.5
Power radiated and distribution-in-angle of intensity from a vibrating-line source set in a rigid cylinder, for different values of $ka = 2\pi a/\lambda$.

Values of I and Π are plotted in Fig. 7.5. Polar curves of the intensity are shown for different values of $\eta = ka = 2\pi a/\lambda$, and a curve is given for Π as a function of η. We notice that at long wavelengths very little power is radiated, and the intensity has very little directionality. As the wavelength is decreased, more power is radiated and the intensity has more directionality; the cylinder begins to cast a "shadow," and a smaller proportion of the energy is sent out on the side of the cylinder opposite the radiating element. For very short waves the intensity is large and uniform from $\phi = -(\pi/2)$ to $\phi = +(\pi/2)$ and is zero from $\phi = \pi/2$ to $\phi = 3\pi/2$, in the shadow. In the intermediate range of η, where the wavelength is about the same size as a, interference effects are noticeable. The polar curve for $\eta = 3$ shows that a fairly intense beam is sent out from the cylinder in a direction diametrically opposite to the position of the line source ($\phi = 180°$).

The general properties illustrated by this set of curves are characteristic of all wave motion when it strikes an obstacle. When the wavelength is large compared with the size of the obstacle, the wave pays hardly any attention to its presence. The first polar curve in Fig. 7.5 shows that for long waves the intensity is distributed in approximately the same manner as it would be if the line source were not in the side of a cylinder but were all by itself, radiating into free space. On the other hand, when the wavelength is very small compared with the size of the obstacle, the motion resembles the motion of particles, the waves traveling in straight lines and the obstacles casting sharp-edged shadows. Light waves have this raylike property in most cases; geometrical optics is a valid approximation because the light waves are very much shorter than the size of most of the obstacles they encounter.

When the wavelength is about the same size as the obstacle, complicated interference effects can sometimes occur, and the analysis of the behavior of the waves becomes quite involved.

Radiation from a cylindrical source of general type

If the line source is not at $\phi = 0$ on the surface of the cylinder but is at $\phi = \alpha$, the pressure and velocity at large distances are, by Eqs. (7.3.8),

$$p \to \rho c(U_0\, d\alpha)\sqrt{\frac{a}{w}}\,\psi(\phi - \alpha)e^{ik(w-ct)}$$

$$u_w \to (U_0\, d\alpha)\sqrt{\frac{a}{w}}\,\psi(\phi - \alpha)e^{ik(w-ct)}$$

the axis of the polar diagram being turned through an angle α. If several sources are distributed over the surface of the cylinder, each for a different value of α, the resulting radiated wave will be the sum of all the waves for the individual sources taken separately. This fact can be used to express the radiation from a cylinder whose surface vibrates with any arbitrary distribution of velocity amplitude. If the distribution is such that the surface at $\phi = \alpha$ has the radial velocity $U(\alpha)e^{-2\pi i\nu t}$, the wave may be considered to be the result of an infinite sequence of line sources, the one at $\phi = \alpha$ having the velocity amplitude $U(\alpha)$, etc. The pressure and radial velocity at large distances are then obtained by integrating the expressions given above for a single line source:

$$\begin{aligned} p &\simeq \rho c\sqrt{\frac{a}{w}}\,e^{ik(w-ct)}\int_0^{2\pi}\psi(\phi - \alpha)U(\alpha)\,d\alpha \\ u &\simeq \frac{p}{\rho c} \end{aligned} \tag{7.3.10}$$

For instance, if a section of the cylinder between $\alpha = -\alpha_0$ and $\alpha = +\alpha_0$ is vibrating, so that $U(\alpha) = U_0$ for $-\alpha_0 < \alpha < \alpha_0$, and is zero for the rest of the values of α, then the pressure wave at large distances from the cylinder is

$$\begin{aligned} p &= \rho c U_0\sqrt{\frac{a}{w}}\,e^{ik(w-ct)}\int_{-\alpha_0}^{\alpha_0}\psi(\phi - \alpha)\,d\alpha \\ &= \frac{2\rho c U_0}{\pi^2}\sqrt{\frac{c}{\nu w}}\,e^{ik(w-ct)}\sum_{m=0}^{\infty}\frac{\sin(m\alpha_0)\cos(m\phi)}{mE_m}\,e^{-i\left[\gamma_m + \frac{\pi}{4}(2m+1)\right]} \end{aligned} \tag{7.3.11}$$

where we use the convention that $[\sin(m\alpha_0)]/m = \alpha_0$ when $m = 0$.

The intensity and total power radiated are

$$I = \frac{4\rho c_2 U_0^2}{\pi^4 \nu w} \sum_{m,n=0}^{\infty} \frac{\sin(m\alpha_0)\sin(n\alpha_0)\cos(m\phi)\cos(n\phi)}{mnE_m E_n} \times \cos\left[\gamma_m - \gamma_n + (m-n)\frac{\pi}{2}\right]$$

$$\Pi = \frac{4\rho c^2 U_0^2}{\pi^3 \nu}\left[\frac{2\alpha_0^2}{E_0^2} + \sum_{m=1}^{\infty}\frac{\sin^2(m\alpha_0)}{m^2 E_m^2}\right] \tag{7.3.12}$$

When the frequency is very small, so that λ is much larger than a, the intensity and power radiated are

$$I \to \frac{\rho \nu a^2 U_0^2 \alpha_0^2}{w} \qquad \Pi \to 2\pi\rho\nu a^2 U_0^2 \alpha_0^2$$

to the first approximation. The expression for I is quite similar to Eq. (7.3.4) for a uniformly expanding and contracting cylinder. Even though the velocity of the surface of the cylinder, in the present case, is not symmetrical about the axis, nevertheless, the radiation at very long wavelengths is symmetrical, behaving as though it came from a uniform cylindrical source whose velocity amplitude is $U_0\alpha_0/\pi$ (the average velocity amplitude of the actual surface). This is another example of the fact, mentioned above, that wave motion is insensitive to details smaller in size than the wavelength.

The Green's function

The Green's function of Eqs. (7.1.14) and (7.2.31) can also be expressed in cylindrical coordinates. To show this most easily we first obtain a Fourier transform of the Green's function in rectangular coordinates. The equation it must satisfy is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) g_\omega = -\delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0) \tag{7.3.13}$$

so the relation between $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$ and its transform $G_\omega(\mathbf{K})$ is the three-dimensional extension of Eq. (1.3.16) (for x, y, z instead of t).

$$G_\omega = (2\pi)^{-3}\iiint g_\omega e^{i\mathbf{K}\cdot\mathbf{r}}\, dv_r \qquad g_\omega = \iiint G_\omega e^{-i\mathbf{K}\cdot\mathbf{r}}\, dv_K$$

where $\mathbf{K}$ is the vector with components K_x, K_y, K_z and volume element $dv_K = dK_x\, dK_y\, dK_z$; $\mathbf{r}$ is the vector (x,y,z) with volume element $dv_r = dx\, dy\, dz$. The transform of the term on the right of Eq. (7.3.13) is $-(2\pi)^{-3}e^{i\mathbf{K}\cdot\mathbf{r}_0}$, and the transform of the left-hand side is $(k^2 - K^2)G_\omega(\mathbf{K})$, so that the Green's function is given by the triple Fourier integral

$$g_\omega(\mathbf{r} \mid \mathbf{r}_0) = (2\pi)^{-3}\iiint \frac{\exp[i\mathbf{K}\cdot(\mathbf{r}_0 - \mathbf{r})]}{K^2 - k^2}\, dv_K \tag{7.3.14}$$

where $k = \omega/c$, and $K^2 = K_x^2 + K_y^2 + K_z^2$. The contour for integration in the complex K plane goes above the pole at $K = -k$ and below the pole at $K = +k$ in order that the wave may be an outgoing one.

To convert this integral to one in cylindrical coordinates w, ϕ, z, we first change to cylindrical coordinates μ, β, K_z in K space, where $K_x = \mu \cos \beta$, $K_y = \mu \sin \beta$, and $K^2 = K_z^2 + \mu^2$. Then $dv_K = dK_z\, d\beta \mu\, d\mu$, $\mathbf{K} \cdot \mathbf{r} = \mu w \cos(\beta - \phi) + zK_z$, and $\mathbf{K} \cdot \mathbf{r}_0 = \mu w_0 \cos(\beta - \phi_0) + z_0 K_z$. We first integrate over K_z, using Eqs. (1.2.15) to evaluate the integral in terms of the poles at $K_z = \pm\sigma = \pm\sqrt{k^2 - \mu^2}$ (going around the pole at $+\sigma$ when $z_0 - z$ is positive and around the pole at $-\sigma$ when $z - z_0$ is positive), so that

$$g_\omega(\mathbf{r} \mid \mathbf{r}_0) = -\frac{i}{8\pi^2}\int_0^{2\pi} d\beta \int_0^\infty \exp\left[i\mu w_0 \cos(\beta - \phi_0) - i\mu w \cos(\beta - \phi)\right] \frac{\mu\, d\mu}{\sigma}\, e^{i\sigma|z - z_0|}$$

Next we use the series expansion of Eq. (1.2.9) to convert the exponential into a pair of Fourier series.

$$\exp[-] = \sum_{m,n=0}^{\infty} \epsilon_m \epsilon_n i^{m-n} \cos m(\beta - \phi_0) \cos n(\beta - \phi) J_m(\mu w_0) J_n(\mu w)$$

Integration over β removes all but the terms for which $m = n$; the final result is

$$g_\omega(\mathbf{r} \mid \mathbf{r}_0) = \sum_{m=0}^{\infty} \frac{i\epsilon_m}{4\pi} \cos m(\phi - \phi_0) \int_0^\infty J_m(\mu w_0) J_m(\mu w) \frac{\mu\, d\mu}{\sigma}\, e^{i\sigma|z - z_0|} \qquad (7.3.15)$$

where $\sigma = \sqrt{k^2 - \mu^2}$ when $0 \leqslant \mu < k$ and equals $i\sqrt{\mu^2 - k^2}$ when $0 \leqslant k < \mu$.

This expansion for the free-field Green's function in cylindrical coordinates w, ϕ, z will be useful later in this chapter. When $w_0 = 0$, this function must become the spherically symmetric function $g_\omega(\mathbf{r} \mid 0)$ of Eq. (7.1.6). Since $J_0(0) = 1$ and $J_m(0) = 0$ for $m > 0$, the series reduces to

$$g_\omega(\mathbf{r} \mid 0) = \frac{e^{ikr}}{4\pi r} = \frac{i}{4\pi}\int_0^\infty J_0(\mu w) e^{i|z|\sqrt{k^2 - \mu^2}} \frac{\mu\, d\mu}{\sqrt{k^2 - \mu^2}} \qquad (7.3.16)$$

where of course $r^2 = w^2 + z^2$. Thus series (7.3.15) is identical with the function defined in Eq. (7.1.6).

We shall return to other acoustical situations which require the use of cylindrical coordinates, later in this chapter, and also in Chap. 9, when we take up the generation and transmission of sound inside a hollow cylinder.

Occasionally, we shall have use for the Green's function in two dimensions, the solution of the equation

$$\nabla^2 g_\omega + k^2 g_\omega = -\delta(x - x_0)\, \delta(y - y_0)$$

which can also be considered to be the three-dimensional wave generated by a unit line source at $x = x_0$, $y = y_0$, $-\infty < z_0 < +\infty$. By changing to coordinates centered on the source and by methods paralleling the discussion of Eq. (7.1.14), we can see that the solution of

$$\frac{1}{W}\left[\frac{\partial}{\partial W}\left(W\frac{\partial g}{\partial W}\right)\right] + k^2 g = \frac{\delta(W)}{W}$$

which represents an outgoing wave, is

$$g_\omega(x,y \mid x_0 y_0) = \frac{i}{4}\left[J_0(kW) + iN_0(kW)\right] = \frac{i}{4}H_0^{(1)}(kW) \qquad (7.3.17)$$

where $W^2 = (x - x_0)^2 + (y - y_0)^2$. By using the Fourier transform it is then possible to show that this function can be expanded in the polar coordinates w, ϕ.

$$g_\omega(w,\phi \mid w_0,\phi_0) = \frac{i}{4}\sum_{m-0}^{\infty} \epsilon_m \cos m(\phi - \phi_0)\begin{cases} J_m(kw)H_m^{(1)}(kw_0) & w < w_0 \\ H_m^{(1)}(kw)J_m(kw_0) & w > w_0 \end{cases} \qquad (7.3.18)$$

where the function $H_m^{(1)}$ is the outgoing-wave solution of the Bessel equation (1.2.4).

$$H_m^{(1)}(z) = J_m(z) + iN_m(z) \to i^{-m}\sqrt{\frac{2}{\pi i z}}\, e^{iz} \qquad z \to \infty$$

7.4 RADIATION AND REFLECTION FROM A PLANE SURFACE

So far we have been considering the radiation of sound into an infinite medium, unbounded by any barriers. This is a rather unrealistic arrangement, though the computed results do approximate the sound field generated by aircraft or by a source in an anechoic chamber. The arrangements discussed in this section, sources in the presence of an infinite plane barrier, so that the medium is confined to one side of the plane, are somewhat more realistic. They will serve to introduce the subject of the effects of boundaries which generate, reflect, or absorb sound energy.

Image sources

Suppose the plane is perfectly rigid. Then the boundary condition at its surface is that the normal fluid velocity is zero there, and thus [by Eq. (6.2.7)] that the normal gradient of the pressure is zero at the surface. This can most easily be accomplished by imagining that the boundary plane is replaced by a continuation of the medium into the region back of the boundary. Since the gradient of the field is still to be zero at the plane where the boundary had been, the field must be symmetric with respect to this plane; its value on one

side of the plane should be the reflection of its value on the other side. Thus we may replace the effect of the boundary plane by a set of image sources, symmetrically placed with respect to the boundary plane, with both source and its image radiating into unbounded space. Of course, only the region outside the boundary plane contains the medium and carries acoustic energy. The part of the field from the original source, which is inside the boundary, has no physical reality (as is the case with the wave in the bounded string of Fig. 4.7). Conversely, the portion of the wave from the image source, outside

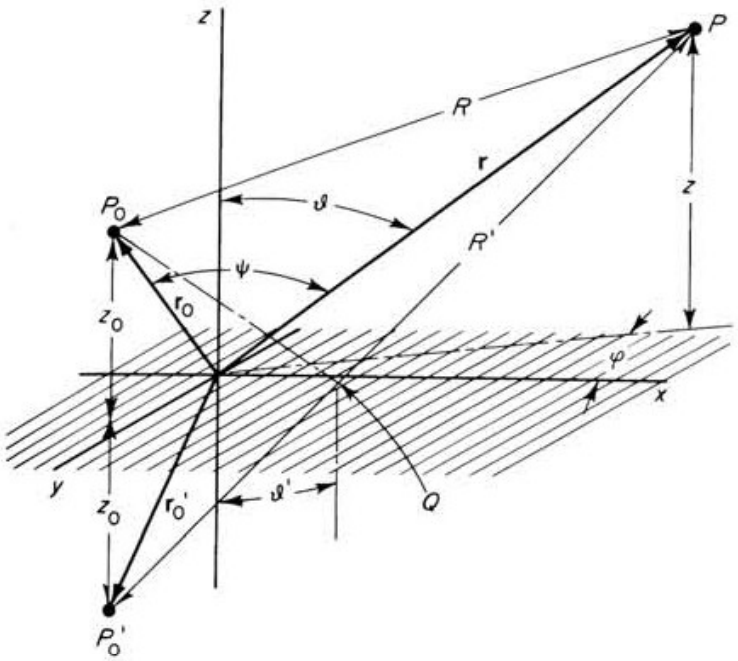


FIGURE 7.6
Angles and distances for the field at P from a source at P_0 and image, in the xy plane, at P_0'.

the boundary, does have physical reality. It constitutes the reflected wave and carries the energy reflected from the boundary surface.

For example, suppose the boundary surface is the xy plane, with the medium in the region $z > 0$. If the source is a simple source at point P_0 shown in Fig. 7.6, located by vector $\mathbf{r}_0$ from the origin, the image source will also be a simple source, of equal strength and phase, at point P_0', a vertical distance $2z_0$ below P_0, located by vector $\mathbf{r}_0'$. The acoustic pressure at P, located by vector $\mathbf{r}$, above the xy plane, for a simple-harmonic source, is then

$$-i\rho ckS_\omega\left(\frac{1}{4\pi R}\,e^{ikR}+\frac{1}{4\pi R'}\,e^{ikR'}\right)e^{-i\omega t} \tag{7.4.1}$$

This represents, in the region $z > 0$, the original wave plus a reflected wave, which automatically satisfies the boundary conditions at $z = 0$. Along the boundary surface, $R' = R$ and the pressure is just twice what it would have been if the boundary had not been present.

At distances R from the source, which are much larger than the source is from the origin (i.e., for $R \gg r_0$), the two waves combine to form what appears to be a single nonsimple source at the origin. To see this we note that when $r \gg r_0$,

$$R = |\mathbf{r} - \mathbf{r}_0| \simeq r - \left(\frac{\mathbf{r}}{r}\right) \cdot \mathbf{r}_0 = r - r_0 \cos \psi$$

and

$$R' \simeq r + 2z_0 \cos \vartheta - r_0 \cos \psi$$

Thus, neglecting higher-order terms in the small quantity r_0/r, the pressure field at $\mathbf{r}$ is

$$p \simeq -\frac{i\rho ck}{2\pi r} S_\omega \cos (kz_0 \cos \vartheta) \exp [ik(r - r_0 \cos \psi + z_0 \cos \vartheta) - i\omega t] \tag{7.4.2}$$

If kz_0 and $kr_0 \ll 1$, that is, if the source is considerably less than a wavelength above the boundary surface, the far field is very weakly dependent on ϑ or ψ, thus resembling the far field from a simple source, at the origin, of strength $2S_\omega$. The effective strength is doubled because the reflected wave adds to the initial wave in this case; the source and image are close enough together to be considered as a single source of double strength. On the other hand, when $kz_0 = \frac{1}{2}\pi$, when the source is a quarter wavelength above the surface, the source and its image are a half wavelength apart, and there is destructive interference between the two waves. Directly above the source ($\vartheta = 0$), the interference is complete and the far-field pressure is zero; near the surface ($\vartheta = \frac{1}{2}\pi$), the source and image still add together. Thus the effect of reflection, when $z_0 > 1/k$, is to provide the radiated pressure wave with interference bands.

Comparison with Eq. (7.1.6) indicates that the appropriate Green's function for a rigid boundary at the xy plane is

$$G_\omega(\mathbf{r} \mid \mathbf{r}_0) = g_\omega(\mathbf{r} \mid \mathbf{r}_0) + g_\omega(\mathbf{r} \mid \mathbf{r}_0') \tag{7.4.3}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{r}_0' = (x_0, y_0, -z_0)$.

If the source at P_0 is a dipole of strength $\mathbf{D}_\omega$, inclined at the angles θ, ϕ with respect to the axes related to the boundary plane, the mirror image will have the same x and y components as the source, but will have a D_z opposite in sign from that of the source. Therefore the field is

$$\begin{aligned} p &= -ik\rho c\mathbf{D}_\omega \cdot [\operatorname{grad}_0 g_\omega(\mathbf{r} \mid \mathbf{r}_0) + \operatorname{grad}_0 g_\omega(\mathbf{r} \mid \mathbf{r}_0')]e^{-i\omega t} \\ &\simeq -k^2 \frac{\rho c}{2\pi r} \left[\left(\frac{x}{r} D_x + \frac{y}{r} D_y \right) \cos (kz_0 \cos \vartheta) \right. \\ &\qquad \left. + i \frac{z}{r} D_z \sin (kz_0 \cos \vartheta) \right] e^{ik(r - r_0 \cos \psi - z_0 \cos \vartheta) - i\omega t} \end{aligned} \tag{7.4.4}$$

the last expression being the far field, for $r \gg r_0$. The source and its image thus combine, at large distances and for long wavelengths, into a dipole of components $2D_x$ and $2D_y$, parallel to the xy plane, plus an axial quadrupole, oriented along the z axis, of strength $Q_{zz} = z_0 D_z$. As kz_0 is increased, interference between the waves from the horizontal and vertical components appears. At no time does the far field appear to be that from a single isolated dipole (except when $z_0 = 0$).

The Green's function

It might be well to examine more carefully the reasons for our choice of the Green's function of Eq. (7.4.3). It is, of course, to be inserted in Eq. (7.1.17) to compute the radiation in the presence of a rigid boundary in the xy plane. The boundary condition in this case is that $\partial p/\partial n_0$ is zero at $z_0 = 0$. The reason we choose G_ω to satisfy the same boundary condition is that, if this is done, the surface integral of Eq. (7.1.17) vanishes and the pressure wave in the medium ($z > 0$) is

$$p_\omega(\mathbf{r}) = \iiint f_\omega(\mathbf{r}_0) G_\omega(\mathbf{r} \mid \mathbf{r}_0)\, dv_0 \tag{7.4.5}$$

where f_ω is the source function in the region outside the boundary surface. The difference between this solution and that of Eq. (7.1.18), for the unbounded medium, is that function G_ω, in contrast to g_ω, contains a term representing the wave reflected from the boundary plane.

As we have already shown under the preceding heading, the term to add to $g_\omega(\mathbf{r} \mid \mathbf{r}_0)$ to make $\partial G_\omega/\partial n = 0$ at $z = 0$ is one containing a wave from an image source, $g_\omega(\mathbf{r} \mid \mathbf{r}_0')$. Since the image part comes from a source inside the surface, this term is a solution of the homogeneous wave equation $(\nabla^2 + k^2)\chi = 0$ everywhere outside the surface, in the medium. Thus, when the rigid boundary is the xy plane,

$$G_\omega(\mathbf{r} \mid \mathbf{r}_0) = \frac{1}{4\pi R} e^{ikR} + \frac{1}{4\pi R'} e^{ikR'} \tag{7.4.6}$$

where

$$R = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{\frac{1}{2}}$$

and

$$R' = [(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{\frac{1}{2}}$$

The geometry of this function is shown in Fig. 7.6. The first term is the pressure at point P, $\mathbf{r} = (x,y,z)$, caused by a unit source at P_0, $\mathbf{r}_0 = (x_0,y_0,z_0)$, in the absence of a boundary. The second term is the pressure at P, caused by an image source at P_0', $\mathbf{r}_0' = (x_0, y_0, -z_0)$, outside the limits of the medium. The first term has a singularity at $\mathbf{r} = \mathbf{r}_0$, but the second term has no singularity in the region $z > 0$, occupied by the medium. This second term, of course, represents the wave reflected by the rigid boundary at $z = 0$. Function G_ω satisfies Eq. (7.1.14) when z and z_0 are both positive.

It will occasionally be convenient to write this Green's function in other forms, related to those of Eqs. (7.3.14) and (7.3.15) for g_ω. For example, the Fourier integral expression for G_ω of Eq. (7.4.6) is

$$G_\omega(x,y,z \mid x_0,y_0,z_0) = \frac{1}{4\pi^3}\int_{-\infty}^{\infty}\!\!\int\!\!\int \exp\left[iK_x(x_0 - x) + iK_y(y_0 - y) + iK_z z\right] \times \frac{\cos(K_z z_0)\, dK_x\, dK_y\, dK_z}{K_x^2 + K_y^2 + K_z^2 - k^2} \tag{7.4.7}$$

It can also be expressed in terms of cylindrical coordinates w, ϕ, z, with the cylindrical axis perpendicular to the boundary plane.

$$G_\omega = \sum_{m=0}^{\infty} \frac{i\epsilon_m}{2\pi}\cos m(\phi - \phi_0)\int_0^\infty J_m(\mu w)J_m(\mu w_0)\frac{\mu\, d\mu}{\kappa}\begin{cases} e^{i\kappa z_0}\cos(\kappa z) \\ e^{i\kappa z}\cos(\kappa z_0)\end{cases} \tag{7.4.8}$$

where $\kappa^2 = k^2 - \mu^2$, and the upper term after the brace is used when $0 \leqslant z \leqslant z_0$ and the lower term is used when $0 \leqslant z_0 \leqslant z$. This is to be compared with the integral of Eq. (7.3.15).

The effect of boundary impedance

If the boundary surface at $z = 0$ is not perfectly rigid, but is still passive, yielding somewhat to the incident acoustic pressure, the reflected wave is modified, so that the image term of Eq. (7.4.6) must be modified. If the surface is one of local reaction (see page 260), the formulas of Eqs. (6.3.5) and (6.3.6), relating the reflected wave to the acoustic impedance $z(\omega) = p/u_n$ of the surface, may be used to find the reflected wave. The most convenient form to use is the Fourier integral of Eq. (7.3.14). The integrand of this expression represents a plane wave which strikes the surface $z = 0$ at an angle of incidence $\cos^{-1}(K_z/K)$. Using Eq. (6.3.7), we see that the presence of the plane surface of impedance $z(\omega)$ at $z = 0$ introduces an extra term into the integrand, corresponding to a reflected wave

$$\exp\left[iK_x(x_0 - x) + iK_y(y_0 - y) + iK_z(z_0 + z)\right]$$

with a factor

$$C_r = \frac{(K_z/K) - [\rho c/z(\omega)]}{(K_z/K) + [\rho c/z(\omega)]}$$

to take into account the change of phase and amplitude on reflection.

In the discussion of this and succeeding chapters it will be convenient to deal with the *acoustic admittance* of the surface, the reciprocal of the impedance. We define the dimensionless parameter, the *specific admittance* of the surface, by

$$\beta = \frac{\rho c}{z(\omega)} = \xi - i\sigma \qquad z(\omega) = \frac{p}{u_n} = R - iX$$
$$\xi = \rho c\frac{R}{R^2 + X^2} \qquad \sigma = -\rho c\frac{X}{R^2 + X^2} \tag{7.4.9}$$

The parameter ξ is the acoustic conductance of the surface, in units of the characteristic admittance $1/\rho c$ of the medium; σ is the corresponding susceptance. If the wall is perfectly rigid, $\beta = 0$, $C_r = 1$, and the integral reduces to that of Eq. (7.4.7). If the wall is quite soft, β is large, and the integral differs appreciably from that of Eq. (7.4.7).

Assuming that β is uniform over the whole boundary surface $z = 0$, we can transform the Fourier integral into one in the cylindrical coordinates w, ϕ, z [a natural extension of Eq. (7.4.8)],

$$G_\omega = \sum_{m=0}^{\infty} \frac{i\epsilon_m}{4\pi} \cos m(\phi - \phi_0) \int_0^\infty J_m(\mu w) J_m(\mu w_0) \times \left[e^{i\kappa|z-z_0|} + \frac{\kappa - k\beta}{\kappa + k\beta} e^{i\kappa(z+z_0)} \right] \frac{\mu \, d\mu}{\kappa} \quad (7.4.10)$$

which will be of use later in this section.

An exact representation of this Green's function in the form of Eq. (7.4.6) cannot be obtained, but a very good approximation, valid as long as the measurement point P is not closer than a half wavelength to the boundary surface, is

$$G_\omega \simeq \frac{e^{ikR}}{4\pi R} + \frac{\cos\vartheta' - \beta}{\cos\vartheta' + \beta} \frac{e^{ikR'}}{4\pi R'} \quad (7.4.11)$$

where the quantities involved are defined in Fig. 7.6. To this approximation there is an image at the point P_0' (as with the rigid boundary), but the wave it radiates has an amplitude which depends on the angle ϑ' which the line R' makes with the z axis, normal to the boundary. The physical significance of the dependence is easily seen if we realize that the ray of acoustic radiation which reaches the measurement point P from the source point P_0, by reflection, strikes the reflecting surface at Q at an angle ϑ' of incidence and reflection. Thus, according to Eq. (6.3.5), its reflection factor C_r should be the one given in Eq. (7.4.11). The wave which goes directly from P_0 to P without reflection, given by the first term, is independent of the inclination of R; the reflected wave, given by the second term, depends on the angle of reflection.

From these expressions, by methods which are by now familiar, we can compute the radiation from a multipole source in the presence of a plane surface of specific admittance β. For example, the counterpart to Eq. (7.4.2), for the far field from a simple-harmonic, simple source of strength S_ω, is

$$p \simeq -\frac{i\rho ck}{2\pi r} S_\omega \frac{\cos\vartheta \cos(kz_0 \cos\vartheta) - i\beta \sin(kz_0 \cos\vartheta)}{\cos\vartheta + \beta} \times \exp[ik(r - r_0 \cos\psi + z_0 \cos\vartheta) - i\omega t] \quad (7.4.12)$$

which reduces to that of Eq. (7.4.2) when $\beta = 0$. A parallel formula can be worked out for a dipole source, the generalization of Eq. (7.4.4).

Boundary effects on multipole impedances

To calculate the effects of the boundary on the radiation load of a multipole source, we must compute the distortion of the near field produced by the boundary. For ease of computation, let us place the origin at the center of the radiating multipole. The boundary surface, of specific acoustic admittance β, is then the plane $z = -z_0$, and the image source is at $z = -2z_0$. Using the expansion of the Green's function in spherical harmonics, given in Eq. (7.2.31), we have

$$G_\omega = \frac{ik}{4\pi}\left[h_0(kr) + C_r \sum_{m=0}^{\infty} (-1)^m (2m+1) P_m(\cos\vartheta) j_m(kr) h_m(2kz_0)\right] \quad (7.4.13)$$

where we have assumed that $r < 2z_0$. For the case under consideration, actually, r is considerably smaller than z_0, so that the $\cos\vartheta'$ in the formula for C_r differs from unity only in terms of the second order in r/z_0. Therefore, to compute pressures and velocities at the surface of the sphere $r = a$, we can consider C_r to be the constant

$$C_r \simeq \frac{1-\beta}{1+\beta} = \frac{z(\omega) - \rho c}{z(\omega) + \rho c}$$

A simple source at the origin would produce a pressure wave $-ik\rho c S_\omega G_\omega e^{-i\omega t}$. The integral of the pressure over the surface of the small sphere $r = a$, just outside the source, the driving force for the source, is

$$F_s = -i\rho c k a S_\omega \left[e^{ika} + C_r \frac{\sin(ka)}{k} \frac{e^{2ikz_0}}{2z_0}\right] e^{-i\omega t}$$

$$\rightarrow -i\rho c k a S_\omega \left[1 + C_r \frac{a}{2z_0} e^{2ikz_0}\right] e^{-i\omega t} \qquad ka \rightarrow 0$$

where we have substituted from Eqs. (7.2.9) and (7.2.11) for the various spherical Bessel functions, and we have taken advantage of the fact that the integral over the sphere of $P_m(\cos\vartheta)$ is zero except for $m = 0$. Comparison with Eq. (7.1.11) shows that the presence of the boundary has distorted the near field. As long as ka is small, the distortion is not large, however. The radial velocity $\dfrac{1}{ik\rho c}\dfrac{\partial p}{\partial r}$ at $r = a$ is also somewhat nonisotropic, but its mean value is

$$U_s \rightarrow \frac{S_\omega}{4\pi a^2}\left(1 - ika + \tfrac{1}{3} C_r k^2 a^2 \frac{a}{2z_0} e^{2ikz_0}\right) e^{-i\omega t}$$

the distorting effect being smaller by a factor k^2a^2 than the correction term for the pressure.

Therefore the mechanical impedance of a simple source a distance z_0 away from a plane wall of acoustic impedance $\rho c/\beta$ is [see Eq. (7.1.11)]

$$Z_s = -i\omega\rho(4\pi a^3)\left\{1 + \frac{a}{2z_0}[R_r \cos(2kz_0) - X_r \sin(2kz_0)]\right\}$$
$$+ \rho c(4\pi a^2)(ka)^2\left\{1 + \frac{1}{2kz_0}[R_r \sin(2kz_0) + X_r \cos(2kz_0)]\right\}$$

where

$$R_r = \text{Re } C_r \to 1 - 2\xi \qquad X_r = \text{Im } C_r \to 2\sigma \qquad |\beta| \ll 1$$

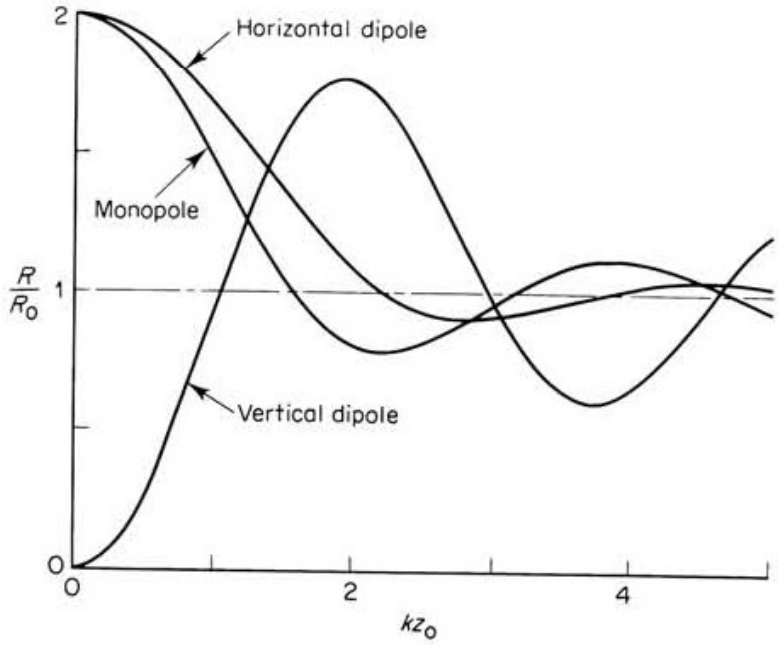


FIGURE 7.7
Ratio of effective radiation resistance R, for a multipole of frequency $\omega/2\pi = kc/2\pi$, a distance z_0 above a rigid plane, to the effective radiation resistance R_0 in the absence of the plane.

The wall correction to the mechanical reactance is small because a/z_0 is small; this correction can usually be neglected. But the correction to the small resistive term is proportionally larger and, if $\beta \to 0$, can be equal to the resistive term itself. In fact, for a rigid wall ($\beta = 0$), the impedance becomes

$$Z_s \simeq -i\omega\rho(4\pi a^3) + \rho c(4\pi a^2)(ka)^2\left[1 + \frac{\sin(2kz_0)}{2kz_0}\right] \qquad (7.4.14)$$

The real part of this impedance is a measure of the power radiated by the monopole. We note that when $kz_0 = 2\pi z_0/\lambda$ is small (either the source is close to the wall or the wavelength is long), this resistive term is twice that for a source in an unbounded medium. At these long wavelengths the image source reinforces the primary source. As kz_0 is increased, the factor in brackets approaches 1 in an oscillatory manner, as shown in Fig. 7.7. For $\lambda \to 0$ the source radiates as though the boundary were not present.

To find the corresponding change in the dipole impedance we need to compute $\text{grad}_0 G_\omega$, but in Eq. (7.4.13) we have removed the r_0 coordinates by placing the origin at the source point. However, since $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ in the first term, we can set grad_0 equal to $-\text{grad}$; in the second term we remove the subscript zero by changing the sign of the x and y derivatives but keeping $\partial/\partial z_0 = \partial/\partial z$. This, together with Eqs. (7.2.32) (which hold for Hankel as well as Bessel functions), results in

$$p = -i\rho ck\Bigg(\frac{1 - ikr}{4\pi r^2}\, e^{ikr}(D_x Y_{11}^{\,1} + D_y Y_{11}^{\,-1} + D_z Y_{10}^{\,1}) + C_r \frac{ik^2}{4\pi} \sum_{m=0}^{\infty} (-1)^m h_m(2kz_0)\{D_x(Y^1_{m-1,1} j_{m-1} + Y^1_{m+1,1} j_{m+1}) + D_y(Y^{-1}_{m-1,1} j_{m-1} + Y^{-1}_{m+1,1} j_{m+1}) + D_z[m Y_{m-1,0} j_{m-1} - (m+1) Y_{m+1,0} j_{m+1}]\}\Bigg) e^{-i\omega t}$$

where the spherical harmonics Y are functions of the spherical angles ϑ, φ of the radius vector $\mathbf{r}$, and the Bessel functions j are functions of kr.

Both direction and magnitude of the net force on the dipole are modified by the presence of the boundary surface. If the components of the dipole would be D_x, D_y, D_z if the plane were not present, the components in the presence of the plane are $1 + \frac{1}{3}k^3a^3(h_0 + h_2)$ times D_x and D_y and $1 + \frac{1}{3}k^3a^3(-h_0 + 2h_2)$ times D_z, where the Hankel functions are functions of $2kz_0$. If the direction angles of the undistorted dipole are θ, ϕ, so that $D_x = D_\omega \sin\theta\cos\phi$, $D_y = D_\omega \sin\theta \sin\phi$, and $D_z = D_\omega \cos\theta$, the magnitude of the driving force is

$$F_d = -\tfrac{1}{3}ik\rho c D_\omega\{(1 - ika)e^{ika} - \tfrac{1}{3}ik^3a^3[(h_0 - \tfrac{1}{2}h_2)\cos 2\theta + \tfrac{1}{2}h_2]\}e^{-i\omega t}$$

for the case $\beta = 0$ ($C_r = 1$), and the radial fluid velocity in the dipole direction is

$$U_d = \frac{D_\omega}{2\pi a^3}\{(1 - ika - \tfrac{1}{2}k^2a^2)e^{ika} + \tfrac{1}{6}ik^3a^3[(h_0 - \tfrac{1}{2}h_2)\cos 2\theta + \tfrac{1}{2}h_2]\}e^{-i\omega t}$$

The ratio between these is the effective mechanical impedance of the dipole in the presence of the rigid wall. As with the monopole source, the effect on the reactive term is negligible, but the effect of the presence of the wall on the resistive term is of the same order in ka as is the unmodified term. This part of the correction term involves the real part of the Hankel functions [Eq. (7.2.10)], so that, for $C_r = 1$,

$$Z_a \simeq -i\omega\rho\frac{2\pi a^3}{3}(1 + \tfrac{1}{2}k^2a^2) + \rho c(4\pi a^2)\frac{k^4a^4}{12}\{1 + [\tfrac{1}{2}j_2(2kz_0) - j_0(2kz_0)]\cos 2\theta + \tfrac{3}{2}j_2(2kz_0)\} \quad (7.4.15)$$

is the effective impedance of a dipole placed a distance z_0 from a rigid plane wall and inclined at angle θ to its normal. The modification required for a nonrigid wall is not difficult and will be left for a problem.

We see that the interference effects, in the resistance term, caused by the presence of the wall, depend on the orientation of the dipole. If the dipole is vertical ($\theta = 0$ or π), the image dipole is opposed and long-wavelength interference is destructive; the factor in braces goes to zero for $kz_0 \to 0$. On the other hand, if the dipole is parallel to the wall, the interference is constructive; the factor goes to 2 as $kz_0 \to 0$. The behavior of this factor, and that for the monopole, are shown in Fig. 7.7.

Radiation from the boundary

Sound waves in the bounded medium can also be produced by motion of the boundary surface itself, the energy entering the medium from some area of its surface. With a plane rigid boundary at $z = 0$, some portion of the plane may be moving in the z direction. If the motion is simple-harmonic $u_z = u_\omega e^{-i\omega t}$, the boundary condition in this area is that the normal gradient of the pressure is $ik\rho c u_\omega(x,y)e^{-i\omega t}$, whereas, for the rest of the plane, $\partial p/\partial z$ is zero.

An obvious means of finding the pressure field is to apply the Green's function of Eq. (7.1.17) again, this time using the surface integral; since there is no sound source within the medium, the volume integral is zero. The surface integral involves both p and its outward-pointing normal gradient (that is, $-\partial p/\partial z$) at $z = 0$, but our description of the physical situation has specified only the normal gradient. However, if we use the Green's function of Eqs. (7.4.6) to (7.4.8), which has been adjusted so that its normal gradient is zero at $z = 0$, the resulting surface integral then only includes the specified normal gradient of p, and we have

$$\begin{aligned} p_\omega(x,y,z) &= -ik\rho c \iint u_\omega(x_0,y_0) G_\omega(x,y,z \mid x_0,y_0,0)\, dx_0\, dy_0 \\ &= -2ik\rho c \iint u_\omega(x_0,y_0) g_\omega(x,y,z \mid x_0,y_0,0)\, dx_0\, dy_0 \end{aligned} \tag{7.4.16}$$

since, at $z = 0$, $R' = R$, and thus $G_\omega = 2g_\omega$.

Each moving element of area of the plane boundary acts like a simple source of *twice* the strength $u_\omega\, dx_0\, dy_0$ it would have if the rest of the boundary were not there. The reason is that this element radiates only into the half space $z > 0$; the source and its image have coalesced to form a source of double strength.

If the boundary plane, at $z = 0$, is not completely rigid, but reacts to an incident pressure wave with point impedance $z(\omega) = \rho c/\beta$, then every portion of the plane is moving, and we must devise a more sophisticated method of distinguishing the active regions of the boundary, where energy

is fed into the medium, from the passive areas. For the passive areas the relationship between p and its normal derivative follows the definition of the acoustic impedance, $-u_z = -\dfrac{1}{ik\rho c}\dfrac{\partial p}{\partial z} = \dfrac{\beta}{\rho c}p$, at $z = 0$. For the active regions the normal fluid velocity, plus $(\beta/\rho c)p$, at $z = 0$, must differ from zero; we shall call this difference the *driving velocity* u_ω,

$$\left(u_z + \frac{\beta}{\rho c}p\right)_{z=0} = \left(\frac{1}{ik\rho c}\frac{\partial p}{\partial z} + \frac{\beta}{\rho c}p\right)_{z=0} = u_\omega e^{-i\omega t} \tag{7.4.17}$$

To simplify the form of the surface integral of Eq. (7.1.17), we choose a Green's function such that $\partial G_\omega/\partial z_0 = -ik\beta G_\omega$. For then the surface integral over the passive area of the boundary is zero, and the integral over the active area is, with $\partial/\partial n_0 = -\partial/\partial z_0$,

$$\begin{aligned} p_\omega &= -\iint\left(\frac{\partial p}{\partial z_0}G_\omega - p_\omega\frac{\partial G_\omega}{\partial z_0}\right)_{z_0=0} dx_0\,dy_0 \\ &= -\iint\left(\frac{\partial p_\omega}{\partial z_0} + ik\beta p_\omega\right)G_\omega\,dx_0\,dy_0 \\ &= -ik\rho c\iint u_\omega(x_0,y_0)G_\omega(x,y,z \mid x_0,y_0,0)\,dx_0\,dy_0 \end{aligned} \tag{7.4.18}$$

which has the same form as before. But now the Green's function, satisfying these requirements at $z = 0$, is the one we already wrote down in Eqs. (7.4.10) and (7.4.11) for reflection from a plane surface of impedance $\rho c/\beta$. The equation for the pressure wave, in cylindrical and spherical coordinates, is

$$\begin{aligned} p_\omega(\mathbf{r}) &= \frac{k\rho c}{2\pi}\sum_{m=0}^{\infty}\epsilon_m\int_0^{2\pi}\cos m(\phi-\phi_0)\,d\phi_0\int_0^a u_\omega(w_0,\phi_0)w_0\,dw_0 \\ &\qquad\times\int_0^\infty J_m(\mu w)J_m(\mu w_0)\frac{\mu\,d\mu}{\kappa+k\beta}e^{i\kappa z} \\ &\simeq -\frac{ik\rho c}{2\pi}\iint u_\omega(x_0,y_0)\frac{\cos\vartheta'}{\cos\vartheta'+\beta}\frac{e^{ikR}}{R}\,dx_0\,dy_0 \end{aligned} \tag{7.4.19}$$

where $\kappa^2 = k^2 - \mu^2$, $R^2 = (x-x_0)^2 + (y-y_0)^2 + z^2$, and $\cos\vartheta' = z/R$. The integral is, of course, over the active area A of the boundary, which is assumed to be confined within a circle of radius a about the origin. When $\beta = 0$ the boundary is rigid and the formulas reduce to those of Eq. (7.4.16). When $\beta > 0$ the second expression is not accurate for the near field, but is valid for the far field.

The far field

When the distance r, from the source region on the boundary, is much larger than a, the radius of the circle enclosing the source area, the last integral expression of Eq. (7.4.19) can be further simplified. Neglecting terms of order a/r and smaller, we have $\cos\vartheta' \simeq \cos\vartheta = z/r$.

$$R \simeq r - (x_0 \cos\varphi + y_0 \sin\varphi)\sin\vartheta \qquad \frac{1}{R} \simeq \frac{1}{r}$$

where $\tan\varphi = y/x$. Therefore

$$p_\omega(\mathbf{r}) \simeq -ik\rho c \frac{\cos\vartheta}{\beta + \cos\vartheta} \frac{e^{ikr}}{2\pi r} \iint u_\omega(x_0,y_0) e^{-i\mathbf{k}\cdot\mathbf{w}_0 \sin\vartheta}\, dx_0\, dy_0$$

where vector $\mathbf{w}_0$ lies in the xy plane, having components x_0, y_0; and vector $\mathbf{k}$ points along $\mathbf{r}$, having components $k_x = k\cos\varphi\sin\vartheta$, $k_y = k\sin\varphi\sin\vartheta$, $k_z = k\cos\vartheta$.

This is a very interesting formula, for it shows that the angular distribution of the far field is proportional to the two-dimensional Fourier transform of the distribution of driving velocity in the active region A of the boundary plane. By analogy with Eqs. (1.3.16), we define

$$\begin{aligned} U_\omega(K_x,K_y) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int u_\omega(x_0,y_0) e^{-iK_x x_0 - iK_y y_0}\, dx_0\, dy_0 \\ u_\omega(x_0,y_0) &= \int_{-\infty}^{\infty}\int U_\omega(K_x,K_y) e^{iK_x x_0 + iK_y y_0}\, dK_x\, dK_y \end{aligned} \tag{7.4.20}$$

In terms of this Fourier transform, the far field can be written in the particularly simple form

$$p_\omega(\mathbf{r}) \simeq -ik\rho c \frac{e^{ikr}}{4\pi r} f_\omega(\vartheta,\varphi)$$

$$f_\omega(\vartheta,\varphi) = \frac{8\pi^2\cos\vartheta}{\beta + \cos\vartheta} U_\omega(k\cos\varphi\sin\vartheta,\, k\sin\varphi\sin\vartheta) \tag{7.4.21}$$

where $k = \omega/c$. The angle-distribution factor $f_\omega(\vartheta,\varphi)$ measures the relative amplitude of the acoustic radiation in the direction ϑ, φ from the center of the active area of the boundary, at great distances from this area. It is the strength of the radiation in the direction ϑ, φ from the radiating area, measured in equivalent simple-source strengths.

The formula is useful in a number of ways. For example, for wavelengths long enough so that $ka \ll 1$, $U(K_x,K_y) \simeq (A_r/4\pi^2)\langle u_\omega\rangle$, where A_r is the area of the active portion of the boundary and $\langle u_\omega\rangle$ is the mean value of the driving velocity of that portion. We can write $A_r\langle u_\omega\rangle = S_\omega$ as the strength

of the simple source equivalent to the active region and write

$$f_\omega(\vartheta,\varphi) \simeq \frac{2\cos\vartheta}{\beta+\cos\vartheta} S_\omega \qquad ka \ll 1$$

When $\beta \ll 1$ the angle-distribution function f_ω is practically independent of ϑ, until ϑ is close enough to $\frac{1}{2}\pi$ so that $\cos\vartheta = |\beta|$; from that value to $\vartheta = \frac{1}{2}\pi$ it drops rapidly to zero. Thus, for a hard, but not perfectly rigid, wall, a low-frequency wave spreads out isotropically from the radiating area, except for angles closer than about $|\beta|$ radians to the grazing angle $\frac{1}{2}\pi$. That part of the wave which stays close to the boundary is absorbed by the boundary, unless it is perfectly rigid.

Another way of looking at Eq. (7.4.21) is afforded by rotating the coordinate axes on the surface from x_0, y_0 to X_0, Y_0, where $x_0 = X_0\cos\varphi - Y_0\sin\varphi$ and $y_0 = X_0\sin\varphi + Y_0\cos\varphi$; so that the X_0 axis is the intersection of the rz plane with the xy plane, and the Y_0 axis is in the xy plane, perpendicular to the rz plane. We then see that

$$U_\omega(k\cos\varphi\sin\vartheta, k\sin\varphi\sin\vartheta) = \frac{1}{4\pi^2}\int e^{-ikX_0\sin\vartheta}\,dX_0 \int u_\omega(X_0,Y_0)\,dY_0 \tag{7.4.22}$$

is the one-dimensional Fourier transform, in the X_0 direction, of the average $u_\varphi(\omega,X_0) = (1/2\pi)\int u_\omega(X_0,Y_0)\,dY_0$ of the velocity $u_\omega(X_0,Y_0)$ in the Y_0 direction. This quantity $u_\varphi(\omega,X_0)$ is a function of but one surface coordinate, X_0, the distance in the direction of the measurement point (i.e., along the rz plane); it is also a function of φ, the direction of the rz plane, and of the frequency $\omega/2\pi$.

From this expression we can go on to find the far field from a region of the boundary which is moving with a normal velocity $u_z(x,y,t)$, an arbitrary function of x, y and of time t. We first note that the $u_\omega(x,y)$ of Eqs. (7.4.20) and (7.4.22) is the time-dimension Fourier transform of $u_z(x,y,t)$; next we note [Eq. (2.3.5)] that $-i\omega u_\omega(x,y)$ is the Fourier transform of $\partial u_z/\partial t$. Thus the pressure field at the point $r = (r,\vartheta,\varphi)$ (when $r \to \infty$) is the inverse transform of p_ω of Eq. (7.4.21).

$$p(\mathbf{r},t) \simeq \frac{\rho}{4\pi r}\int_{-\infty}^{\infty} f_\omega(\vartheta,\varphi)\exp\left[-i\omega\left(t-\frac{r}{c}\right)\right]d\omega$$

This integral is tedious to compute if the boundary admittance β is strongly dependent on ω.

But if β can be considered to be constant over the range of ω for which f_ω is nonnegligible, the far field from an arbitrary velocity distribution $u_z(x,y,t)$ of the boundary can be expressed in simple form. First we rotate coordinate axes to the X_0, Y_0 just defined, expressing u_z in these coordinates as $u_z(X_0,Y_0,t)$. Then, from the definitions (1.3.16) of the Fourier transform,

we have

$$p(\mathbf{r},t) \simeq \frac{\rho}{2\pi r}\frac{\cos\vartheta}{\beta+\cos\vartheta}\int_{-\infty}^{\infty} -i\omega e^{-i\omega\left(t-\frac{r}{c}\right)}d\omega\int_{-\infty}^{\infty} e^{i(\omega/c)X_0\sin\vartheta}dX_0$$
$$\times\int_{-\infty}^{\infty} dY_0\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}u(X_0,Y_0,\tau)\,d\tau$$
$$=\frac{\rho}{2\pi r}\frac{\cos\vartheta}{\beta+\cos\vartheta}\frac{\partial}{\partial t}\left[\int dX_0\int dY_0\,u\left(X_0,\,Y_0,\,t+\frac{1}{c}\,X_0\sin\vartheta-\frac{r}{c}\right)\right] \tag{7.4.23}$$

This result demonstrates several properties of the far field. First, the field depends on an appropriate summation of the normal *accelerations* of the various portions of the active area of the boundary surface. Second, the contribution to the pressure at $\mathbf{r}$ at time t, from the line X_0, $-\infty < Y_0 < \infty$, depends on the normal accelerations along this line at time $t + (1/c)X_0\sin\vartheta - (r/c)$, a time $(r/c) - (X_0/c)\sin\vartheta$ earlier, this being the time required for the wave to travel from the line to the point $\mathbf{r}$. The dependence of $p(\mathbf{r},t)$ on φ is inherent in the choice of the coordinates X_0, Y_0; its dependence on ϑ comes from the $(X_0/c)\sin\vartheta$ term, and also from the boundary-admittance term outside the integral (for example, see Fig. 7.10). A typical modification of this result, when β depends strongly on ω, is given as a problem.

Since, for the far field, $p \simeq \rho c u_r$, the intensity of the acoustic radiation at great distances from the active area is all radial, with time-averaged magnitude $(|p|^2/\rho c)$.

Far field from a randomly vibrating area

The formulas derived in the last subsection can be used to compute the mean intensity at r, ϑ, φ produced by an area A of the boundary which is vibrating with a random distribution of velocity. Motion of this sort could be caused by the impinging of turbulent air on the surface, or it could be that a portion of the boundary is attached to some vibrating machinery. If we consider the velocity distribution $u_z(x,y,t)$ of the active area as a function of position as well as of time, u_z can be described either in terms of its auto-correlation function Υ or its spectrum density $|U_\omega|^2$, where

$$\Upsilon(\xi,\eta,\tau) = \frac{1}{AT}\iiint u_z^*(x,y,t)u_z(x+\xi,\,y+\eta,\,t+\tau)\,dx\,dy\,dt$$
$$U_\omega(K_x,K_y) = \frac{1}{8\pi^3}\iiint u_z(x,y,t)e^{i\omega t-iK_x x-iK_y y}\,dx\,dy\,dt \tag{7.4.24}$$

where the integration in x, y is over the active area A; in t, is over the long interval of time T. As we have seen, a simple extension of Eq. (1.3.20) shows that the space-time spectrum level $|U_\omega|^2$ is the Fourier transform of

the autocorrelation function:

$$|U_\omega(K_x,K_y)|^2 = \frac{AT}{64\pi^6}\int\int\int_{-\infty}^{\infty}\Upsilon(\xi,\eta,\tau)e^{i\omega\tau-iK_x\xi-iK_y\eta}\,d\xi\,d\eta\,d\tau \qquad (7.4.25)$$

However, we have already encountered the velocity transform U_ω in the expression (7.4.21) for the far field. In fact, the time-averaged intensity for the radiation of frequency $\omega/2\pi$, at large distances from the radiating area,

$$I_r(\omega) \to \frac{|p|^2}{\rho c} \simeq \frac{\rho c}{r^2}\frac{4\pi^2k^2\cos^2\vartheta}{|\beta(\omega)+\cos\vartheta|^2}|U_\omega(k\sin\vartheta\cos\varphi,\, k\sin\vartheta\sin\varphi)|^2 \qquad (7.4.26)$$

is proportional to the spectrum density, and so can be expressed in terms of the threefold autocorrelation function of the surface velocity. The total intensity at r, ϑ, φ, averaged over the time interval T, is $2\pi/T$ times the integral of $I_r(\omega)$ over ω, in accord with Eq. (1.3.19).

If the motion of the active area is completely random, we should expect that the autocorrelation function Υ falls off smoothly as τ, ξ, or η is made larger than zero; so that a reasonable assumption for its form would be

$$\Upsilon(\xi,\eta,\tau) = |U_0|^2\exp\left[-\frac{1}{2w_c^2}(\xi^2+\eta^2) - \tfrac{1}{2}(\omega_n\tau)^2\right] \qquad (7.4.27)$$

where $|U_0|^2$ is the mean-square velocity of the active area, averaged over time interval T and area A; w_c is the correlation distance along the surface, as in Eq. (7.2.29); and $1/\omega_n$ is the correlation time interval, as in Eq. (2.3.13). Therefore $\omega_n/2\pi$ is the upper frequency limit of the power spectrum, below which all frequencies are represented and above which $|U_\omega|^2$ drops rapidly to zero.

If this is a good approximation for Υ, the corresponding power spectrum is

$$I_r(\omega) \simeq \rho c\frac{AT}{r^2}(2\pi)^{-\frac{5}{2}}\frac{\omega^2w_c^2\,|U_0|^2\cos^2\vartheta}{c^2\omega_n\,|\beta+\cos\vartheta|^2}\exp\left[-\frac{1}{2}\left(\frac{\omega w_c}{c}\right)^2\sin^2\vartheta - \frac{1}{2}\left(\frac{\omega}{\omega_n}\right)^2\right]$$

To find the total intensity at $\mathbf{r}$ we must integrate this over ω and multiply by $2\pi/T$. The integrand has a maximum at $\omega = \sqrt{2}\,\omega_n(1+k_n^2w_c^2\sin^2\vartheta)^{-\frac{1}{2}}$, where $k_n = \omega_n/c = 2\pi/\lambda_n$, λ_n being the lower limit of wavelength in the power spectrum. If $\beta(\omega)$, the specific acoustic admittance of the boundary surface, does not change much with ω near this maximal frequency, then β can be taken outside the integral, and

$$I_r(\mathbf{r}) = \frac{2\pi}{T}\int_{-\infty}^{\infty}I_r(\omega)\,d\omega \simeq \rho c\frac{A}{2\pi r^2}\frac{|U_0|^2\cos^2\vartheta}{|\beta+\cos\vartheta|^2}\frac{(k_nw_c)^2}{[1+(k_nw_c)^2\sin^2\vartheta]^{\frac{3}{2}}} \qquad (7.4.28)$$

where β is evaluated at $\omega = \sqrt{2}\,\omega_n(1+k_n^2w_c^2\sin^2\vartheta)^{-\frac{1}{2}}$.

This is an interesting, and fairly simple, formula. The mean intensity at $\mathbf{r}$ is proportional to ρc, the characteristic impedance of the medium, times $U_0|^2$, the mean-square velocity of the active area, times the ratio between the area A which is active and the area of the half sphere of radius r which is in the medium, times an angle-distribution factor, which depends in part on the ratio between the spatial-correlation length w_c and the wavelength λ_n of the cutoff frequency of the power spectrum, and in part on the admittance of the boundary surface. If the motion is caused by turbulence of the medium close to the surface, then $k_n w_c$ is of the order of unity; if the motion is caused by vibrations transmitted through a stiff boundary wall, then $k_n w_c$ may be larger than unity. In the latter case the intensity has a fairly strong preference for the normal direction, $\vartheta = 0$. If the wall is stiff, so that β is small, the factor involving β is nearly unity, except for near-grazing directions, $\vartheta \to \frac{1}{2}\pi$.

Radiation from a circular piston

As an interesting example of the radiation from a moving portion of a plane boundary, we shall compute the far field from a circular piston of radius a, set in the boundary plane $z = 0$. This is, of course, a special case of the formulas we have been examining in this section; it is worth carrying through in detail because of its practical interest. A cone loudspeaker, set in a plane baffle, radiates sound roughly like a plane piston set in a plane baffle of infinite extent. We shall take as our model the case where its normal velocity is $u_\omega e^{-i\omega t}$ inside a circle of radius a, centered at the origin, and is zero over the rest of the plane $z = 0$ [if the surface has admittance β, the difference $u_z + (\beta/\rho c)p$ at the surface is $u_\omega e^{-i\omega t}$ for $r < a$].

Returning to Eq. (7.4.21), changing from coordinates x_0, y_0 to polar coordinates w_0, ϕ_0 in the integral for U_ω, we find, using Eqs. (1.3.11) and (5.2.21),

$$p_\omega(r) \simeq -ik\rho c \frac{e^{ikr}}{2\pi r} \frac{\cos\vartheta}{\beta + \cos\vartheta} u_\omega \int_0^{2\pi} d\phi_0 \int_0^a w_0\, dw_0 \times \exp\left[-ikw_0 \cos(\phi_0 - \phi)\sin\vartheta\right]$$

$$= -ik\rho c \frac{e^{ikr}}{4\pi r} f_\omega(\vartheta) \qquad f_\omega = 2u_\omega \pi a^2 \left[\frac{2J_1(ka\sin\vartheta)}{ka\sin\vartheta} \frac{\cos\vartheta}{\beta + \cos\vartheta}\right] \tag{7.4.29}$$

The equivalent strength in direction ϑ is thus equal to twice the total outflow amplitude $\pi a^2 u_\omega$ times the factor in brackets, which is nearly unity at $\vartheta = 0$, when $\beta \ll 1$.

The corresponding intensity at $\mathbf{r}$ is

$$I_r \simeq \frac{|p|^2}{\rho c} = \tfrac{1}{4}\rho c\, |u_\omega|^2 k^2 a^2 \left(\frac{a}{r}\right)^2 \frac{\cos^2\vartheta}{|\beta + \cos\vartheta|^2} \left[\frac{2J_1(ka\sin\vartheta)}{ka\sin\vartheta}\right]^2 \tag{7.4.30}$$

Values of $[2J_1(x)/x]$ can be obtained from Table IX. It is unity when x is zero, remains nearly unity until x is about $\pi/2$, goes to zero at x about 1.2π, falls to about -0.13 at x about 1.7π, goes to zero again at x about 2.2π, and so on, having a sequence of maxima and minima which diminishes in size.

When the wavelength λ of the sound radiated is longer than the circumference $2\pi a$ of the piston, the value of $\mu \sin \vartheta$ is less than $\pi/2$ even for

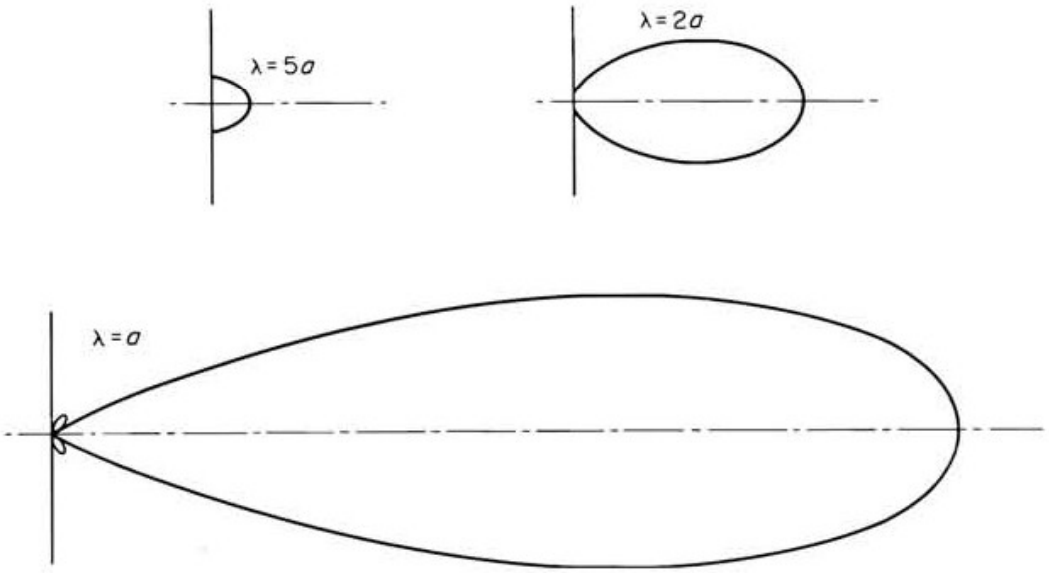


FIGURE 7.8
Polar diagrams of distribution-in-angle of radiated intensity from a piston set in a plane wall, for different ratios between the wavelength λ and the radius of the piston a. In the curve for $\lambda = a$, the small loops at the side are diffraction rings.

$\vartheta = 90°$ (i.e., even along the wall), and the term in brackets is practically unity for all values of ϑ. Therefore long-wavelength sound spreads out uniformly in all directions from the piston, with an intensity four times that due to a simple source of strength $\pi a^2 u_0$. If the wall were not present and the "piston" were the open end of a pipe, this end would act like a simple source of strength $\pi a^2 u_0$ for long wavelengths, so that the wall, or baffle plate, produces a fourfold increase in intensity. The sound reflected from the baffle reinforces the sound radiated outward, thereby doubling the amplitude of the wave, and thus quadrupling the intensity, which depends on the square of the amplitude. Of course, to have the baffle give this considerable increase in amplitude, it must be considerably larger than the wavelength of the sound radiated, so that it will act as though it were infinite in extent.

If λ is smaller than $2\pi a$, then the reflected sound still reinforces the sound radiated straight ahead, and the intensity has its maximum value

$\rho v^2(\pi a^2 u_0)^2/2cr^2$ at $\vartheta = 0$. At points off the axis, however, the reflected sound interferes with that radiated directly, and I diminishes in value as ϑ increases, falling to zero when $\sin\vartheta$ is about equal to $0.6(\lambda/a)$; then rises to a secondary maximum (where there is a little reinforcement) of about 0.02 times the intensity for $\vartheta = 0$; then goes again to zero; and so on. Therefore high-frequency sound is chiefly sent out perpendicular to the wall, with little spread of the beam. Most of the intensity is inside a cone whose axis is along the axis of the piston and whose angle is about $\sin^{-1}(\lambda/2a)$. This main beam is surrounded by *diffraction rings*, secondary intensity maxima, whose magnitude diminishes rapidly as we go from one maximum to the next away from the main beam.

Values of the intensity as a function of ϑ are given in polar diagrams in Fig. 7.8, for different values of $\mu = 2\pi va/c$. The increased directionality with increase in frequency is apparent. It is this directionality that makes it difficult to use a single loudspeaker in outdoor public-address systems: although the low-frequency sound is spread out in all directions, only the people standing directly in front of the loudspeaker will hear the high-frequency sound. In rooms of small size, the sound is scattered so much by the walls that the directionality does not matter particularly (unless it is very marked), and one loudspeaker is usually adequate.

Radiation impedance of the piston

To find the reaction of the fluid back on the driving piston we must calculate the near field. For this purpose the first of Eqs. (7.4.19) is useful. Setting u_ω constant for $r < a$ and then setting $z = 0$ gives

$$p_\omega(\omega) = \rho ck u_\omega \int_0^a w_0\, dw_0 \int_0^\infty J_0(\mu w_0) J_0(\mu w) \frac{\mu\, d\mu}{k\beta + \sqrt{k^2 - \mu^2}}$$

$$= \rho ck a u_\omega \int_0^\infty J_0(\mu w) J_1(\mu a) \frac{d\mu}{k\beta + \sqrt{k^2 - \mu^2}}$$

since u_ω is independent of w_0 and ϕ_0. This pressure is not constant across the face of the piston. But since the piston is presumed rigid, the quantity of practical interest is the total reaction force on the whole piston. In the case of $\beta = 0$, this can be expressed in closed form.

$$F_\omega = 2\pi \int_0^a p_\omega(w) w\, dw = \pi \rho ck a^2 u_\omega \int_0^\infty [J_1(\mu a)]^2 \frac{d\mu}{\mu\sqrt{k^2 - \mu^2}}$$

$$= \pi \rho ck a^2 u_\omega \left\{ \int_0^k [J_1(\mu a)]^2 \frac{d\mu}{\mu\sqrt{k^2 - \mu^2}} - i \int_k^\infty [J_1(\mu a)]^2 \frac{d\mu}{\mu\sqrt{\mu^2 - k^2}} \right\}$$

and thus the mechanical impedance of the radiation load on the piston is

$$Z_r = \frac{F_\omega}{u_\omega} = \pi a^2 \rho c(\theta_0 - i\chi_0) = \pi a^2(R - iX)$$

$$\theta_0(\gamma) = 1 - \frac{2}{\gamma} J_1(\gamma) \qquad \gamma = 2ka = \frac{4\pi a}{\lambda} = \frac{2\omega a}{c}$$

$$\chi_0(\gamma) = M_1(\gamma) = \frac{4}{\pi}\int_0^{\pi/2} \sin(\gamma \cos\alpha) \sin^2\alpha \, d\alpha \tag{7.4.31}$$

$$\theta_0 \to \begin{cases} \frac{1}{2}(ka)^2 & ka \to 0 \\ 1 & ka \to \infty \end{cases} \qquad \chi_0 \to \begin{cases} \dfrac{8ka}{3\pi} & ka \to 0 \\ \dfrac{2}{\pi ka} & ka \to \infty \end{cases}$$

See Table IX for values of the functions θ_0 and χ_0. Curves are shown in Fig. 7.9. Comparison with Fig. 7.3 shows that change from plane to spherical baffle makes little change in the average impedance load on the piston, though comparison between Figs. 7.8 and 7.2 indicates that the directionality pattern does differ considerably. Sound is sent out in different directions from the two sources, but its reaction back on the piston depends more on $2\pi a/\lambda$ than on baffle shape. At high frequencies the impedance is resistive, equal to piston area times the characteristic impedance ρc.

We might here say a few things about the radiation reaction to the general motion $u_\omega(x_0,y_0)$, which produces the far field of Eq. (7.4.21). In this case the pressure at the point w, ϕ on the surface of the moving boundary is

$$p_\omega(w,\phi) = \frac{k\rho c}{2\pi} \sum_{m=0}^{\infty} \epsilon_m \int_0^{2\pi} \cos m(\phi_0 - \phi)\, d\phi_0 \times \int_0^\infty J_m(\mu w) \frac{\mu\, d\mu}{k\beta + \sqrt{k^2 - \mu^2}} \int_0^a u_\omega(w_0,\phi_0) J_m(\mu w_0) w_0\, dw_0$$

where we have expressed $u_\omega(x_0,y_0)$ as a function of the polar coordinates w_0, ϕ_0. This set of integrals must be worked out numerically in most cases, but in the two limiting cases $ka \ll 1$ and $ka \gg 1$, approximate formulas can be obtained.

In the first place, reference to Eq. (1.3.27) indicates that the last integral is the Hankel transform of u_ω, so that

$$u_\omega(w,\phi) = \sum_{m=0}^{\infty} \frac{\epsilon_m}{2\pi} \int_0^{2\pi} \cos m(\phi_0 - \phi)\, d\phi_0 \int_0^\infty J_m(\mu w) U_{m\omega}(\mu,\phi_0) \mu\, d\mu$$

$$U_{m\omega}(\mu,\phi_0) = \int_0^a u_\omega(w_0,\phi_0) J_m(\mu w_0) w_0\, dw_0$$

where the upper limit to the integral for the transform $U_{m\omega}$ could just as well have been ∞, since u_ω is zero for $w_0 > a$. The quantity $U_{m\omega}$, as a function of μ, goes to zero as $\mu \to 0$, for $m > 0$, because $J_m(z) \to z^m/2^m m!$ as $z \to 0$.

For $m = 0$ the limiting value

$$U_{0\omega}(\mu,\phi_0) \to \int_0^a u_\omega(w_0,\phi_0) w_0 dw_0 \qquad \mu a \ll 1$$

is proportional to the mean value of u_ω over the interior of the circle $w = a$, in the ϕ_0 direction. It also can be demonstrated that $U_{m\omega} \to 0$ for all m when μ becomes considerably larger than $1/a$.

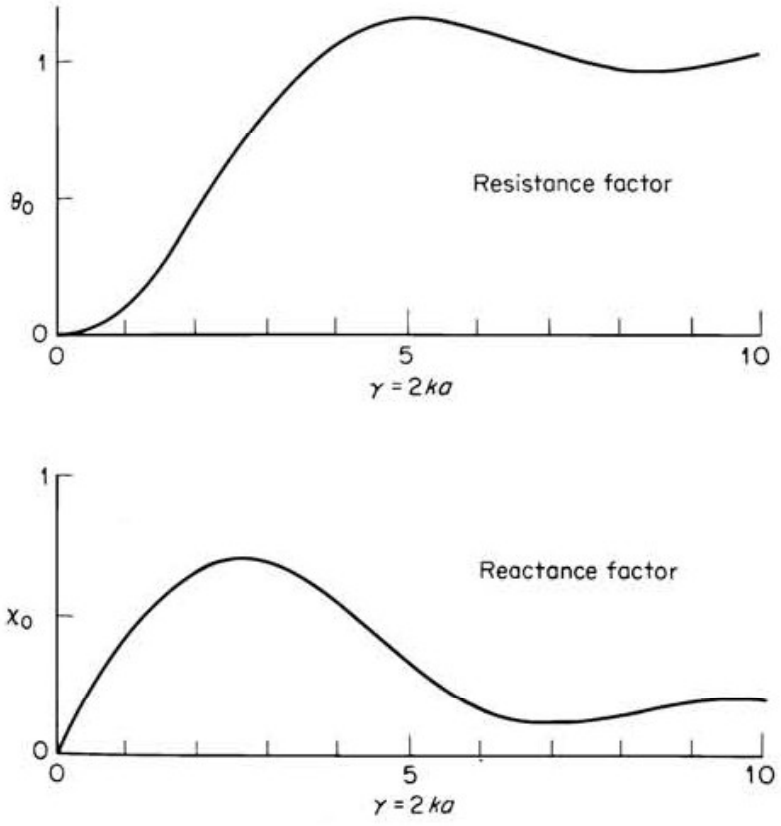


FIGURE 7.9
Radiation impedance of a circular piston in a plane baffle equals area of piston times $(\theta_0 - i\chi_0)\rho c$. Resistance and reactance factors plotted as a function of frequency $\omega/2\pi = c\gamma/4\pi a$.

Now if $ka \gg 1$, so that the radiated wavelength is small compared with a, and also small enough so that u_ω does not change appreciably in a distance of a wavelength, then, over the range of integration of μ within which $U_{m\omega}$ differs appreciably from zero, the denominator of the integral over μ is approximately $k(1 + \beta)$. In this case the pressure at point w, ϕ is

$$p_\omega(w,\phi) \simeq \frac{\rho c}{1+\beta} \sum_{m=0}^{\infty} \frac{\epsilon_m}{2\pi} \int_0^{2\pi} \cos m(\phi - \phi_0)\, d\phi_0 \int_0^{\infty} J_m(\mu w) U_{m\omega}(\mu,\phi_0) \mu\, d\mu$$

$$\to \rho c \frac{u_\omega(w,\phi)}{1+\beta} \qquad ka \gg 1 \tag{7.4.32}$$

Therefore, for very short wavelength radiation from the surface, the pressure at any point on the active part of the surface equals the normal velocity there times $\rho c/(1+\beta) = [(1/\rho c) + (1/z)]^{-1}$, where $z(\omega) = \rho c/\beta$ is the acoustic impedance of the surface, and ρc is the characteristic impedance of the medium. The radiation load on each unit area of the active surface, for $ka \gg 1$, is thus equivalent to the characteristic impedance of the medium, in parallel with the acoustic impedance of the surface. At these short wavelengths each portion of the surface radiates separately and is separately loaded.

On the other hand, if $ka \ll 1$, the denominator of the integral over μ for p_ω is approximately $i\mu$ in the range $1/a > \mu > k$ ($\ll 1/a$), where $U_{m\omega}$ is large. In this range $J_0 \to 1$ and $J_m(\mu w) \to 0$ $(m > 0)$ for $w < a$. Therefore

$$p_\omega \simeq -\frac{ik\rho c}{2\pi}\int_0^{2\pi} d\phi_0 \int_0^{1/a} d\mu \int_0^a u_\omega(w_0,\phi_0) w_0\, dw_0 = -\tfrac{1}{2} i\rho c k a \langle u_\omega \rangle \qquad ka \ll 1$$

where

$$\langle u_\omega \rangle = \frac{S_\omega}{\pi a^2} = \frac{1}{\pi a^2}\int_0^{2\pi} d\phi_0 \int_0^a u_\omega(w_0,\phi_0) w_0\, dw_0$$

is the mean velocity of the active area of the boundary, and S_ω is the amplitude of the total outflow from the surface (the equivalent strength of the surface as a simple source).

Thus, at long wavelengths, the pressure is uniform across the active area, being proportional to the mean velocity amplitude $\langle u_\omega \rangle$ of the area. To this approximation the ratio between pressure and mean velocity is imaginary, indicating that the power radiated is proportional to a higher power of $ka \ll 1$. The real part of this ratio can be obtained by using Eq. (7.4.21) to compute the total power radiated. Since, for $ka \ll 1$, the factor U_ω in f_ω is just S_ω, a constant, the far field and the total power are

$$p_\omega \simeq -ik\rho c \frac{e^{ikr}}{2\pi r} \frac{\cos\vartheta}{\beta + \cos\vartheta} S_\omega \qquad \beta = \xi - i\sigma$$

$$\Pi \to \frac{1}{\rho c}\int_0^{2\pi} d\varphi \int_0^{\pi} |p_\omega|^2 \sin\vartheta\, d\vartheta = \frac{\rho c k^2}{2\pi} |S_\omega|^2\, \Omega(\beta)$$

where

$$\Omega = 1 + (\xi^2 + \sigma^2)\tan^{-1}\left(\frac{\sigma}{1 + \xi^2 + \sigma^2}\right) - \xi \ln \frac{(1+\xi)^2 + \sigma^2}{\xi^2 + \sigma^2}$$

But the power radiated must equal the power leaving the active area, which is the pressure (constant over the area at these wavelengths) p_ω times the part of the mean outflow S_ω which is in phase with p_ω; we already know the part out of phase with p_ω. Therefore the pressure at $z = 0$, $w < a$, for $ka \ll 1$, is, approximately,

$$p_\omega \simeq \tfrac{1}{2}\rho c k a(-i + ka\Omega)\langle u_\omega \rangle = \frac{1}{2}\left(-i\omega\frac{\rho}{\pi a} + \rho c \frac{k^2 a^2}{\pi}\Omega\right) S_\omega \quad (7.4.33)$$

For long wavelengths the active area becomes the equivalent of twice a simple source of strength S_ω (when $\beta = 0$) radiating uniformly into the half space $z > 0$, and the pressure over the active area is constant, proportional to the mean velocity of the active area. The area radiates as a whole, in contrast to the situation for $ka \gg 1$.

Transient radiation from a piston

The general equation (7.4.23) for radiation from an elementary source can be used to compute the transient pressure wave from a piston in a plane wall. Suppose that the velocity of the piston is $U(t)$ and its acceleration is

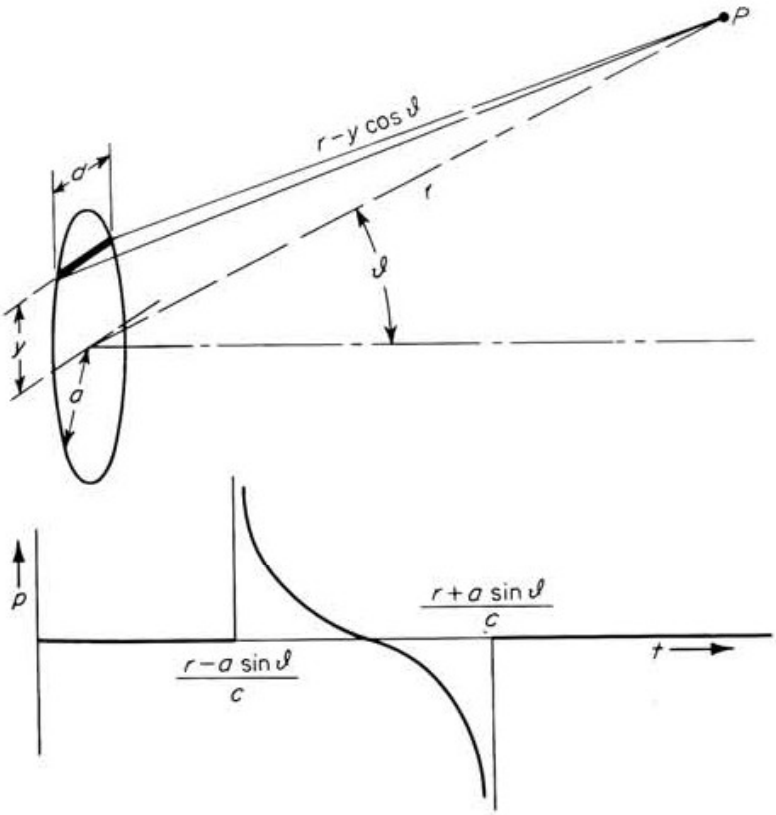


FIGURE 7.10
Radiation of a transient from a piston. Sound reaching point P came from strip d a time $(1/c)(r - y \sin \vartheta)$ earlier. Lower curve shows pressure fluctuation at P due to a velocity pulse $U = \delta(t)$ of the piston.

$A(t) = dU/dt$. Study of Fig. 7.10 shows that a part of the pressure wave arriving at the point (r,ϑ) at the time t is that due to the acceleration of the strip of length $d = 2\sqrt{a^2 - y^2}$ and width dy a time $(1/c)(r - y \sin \vartheta)$ earlier (if r is much larger than a). The amplitude of this component of the pressure pulse is, from Eq. (7.4.23),

$$dp \simeq \frac{\rho}{2\pi r} \sqrt{a^2 - y^2}\, A\left[t - \frac{1}{c}(r - y \sin \vartheta)\right] dy$$

The total pressure is obtained by integrating this over the surface of the piston, from $-a$ to $+a$. Change of integration variable and integration by parts yields the general formulas

$$p = \frac{\rho c}{2\pi r \sin^2 \vartheta} \int_e^f \sqrt{a^2 \sin^2 \vartheta - (c\tau + r - ct)^2}\, A(\tau)\, d\tau$$

$$= \frac{\rho c^2}{2\pi r \sin^2 \vartheta} \int_e^f \frac{(r + c\tau - ct) U(\tau)\, d\tau}{\sqrt{a^2 \sin^2 \vartheta - (r + c\tau - ct)^2}}$$

where the limits of integration are $e = t - (1/c)(r + a \sin \vartheta)$ and $f = t - (1/c)(r - a \sin \vartheta)$.

This is a very interesting formula, for it shows, perhaps more clearly than Eq. (7.4.30), the dependence of the pressure wave on the angle ϑ. Directly ahead ($\vartheta = 0$) the pressure wave reproduces the piston *acceleration* exactly:

$$p = \frac{\rho a^2}{4r} A\left(t - \frac{r}{c}\right) \qquad \vartheta = 0$$

The force on a microphone diaphragm directly ahead of the piston is therefore proportional to the piston acceleration; and if the diaphragm is mass-controlled, its acceleration is proportional to the force, so that the accelerations, velocities, and displacements of piston and diaphragm are proportional. As ϑ is increased, however, the integral covers a larger and larger interval of time τ, so that more and more of the piston motion gets blurred together in the pressure wave arriving at P.

A simple example of this is for the case when the piston suddenly moves outward a distance Δ. In this case the displacement of the piston is $\Delta u(t)$, where $u(t)$ is the step function defined in Eq. (1.3.22); the velocity is $\Delta\delta(t)$, proportional to the impulse function; and the acceleration $\Delta\delta'(t)$ is formally proportional to the derivative of the impulse function, a "pathological" function going first to plus infinity and then to minus infinity in an infinitesimal period of time. Its integral properties are $\int_{-\infty}^{t} \delta'(\tau - a) f(\tau)\, d\tau = -\{[df(a)/da]u(t - a)\}$. The resulting pressure at (r,ϑ) is

$$p = \begin{cases} 0 & ct < r - a \sin \vartheta \\ \dfrac{\rho c^2 \Delta}{2\pi r \sin^2 \vartheta} \dfrac{r - ct}{\sqrt{a^2 \sin^2 \vartheta - (r - ct)^2}} & r - a \sin \vartheta < ct < r + a \sin \vartheta \\ 0 & ct > r + a \sin \vartheta \end{cases}$$

This pulse, shown in Fig. 7.10, is a stretched-out version of the δ' function; the greater the angle ϑ, the greater the stretch. Only directly ahead of the piston, at $\vartheta = 0$, is the pressure pulse as instantaneous as the piston pulse.

The freely suspended disk

The next two examples considered in this section could have been treated in Sec. 7.3, for in neither case is a boundary plane present. They have been postponed to this section, however, because their solution involves techniques which have been developed in this section. The first is that of a sheet of metallic foil, in the form of a disk of radius a, somehow suspended in an unbounded medium, driven by electromagnetic forces perpendicular to its surface. Large loudspeakers, of a form roughly corresponding to this idealized model, have been constructed.

The setup can be approximated by imagining that the fluid itself is acted on by a force F per unit volume, pointed in the positive z direction, of a magnitude

$$F = \frac{F_\omega}{\pi a^2} e^{-i\omega t} \begin{cases} \delta(z) & w < a \\ 0 & w > a \end{cases}$$

where w is the distance from the center of the disk, in the xy plane, and where F_ω is the total force amplitude, driving the whole disk. Using Eq. (7.1.25), we obtain an expression for the pressure.

$$p_\omega(\mathbf{r}) = \frac{F_\omega}{\pi a^2} \int_0^{2\pi} d\phi_0 \int_0^a \left[\frac{\partial}{\partial z_0} g(w,\phi,z \mid w_0,\phi_0,z_0) \right]_{z_0=0} w_0 \, dw_0 \qquad (7.4.34)$$

To compute the far field from this source we use the approximation

$$g(r,\vartheta,\varphi \mid w_0,\phi_0,z_0) \simeq \frac{1}{4\pi r} \exp \{ik[r - w_0 \sin\vartheta \cos(\phi_0 - \phi) - z_0 \cos\vartheta]\} \qquad (7.4.35)$$

valid for $r \gg r_0$. The pressure at large distances from the free-floating disk is then

$$\begin{aligned} p_\omega(\mathbf{r}) &\simeq -ik \frac{e^{ikr}}{4\pi r} \frac{F_\omega}{\pi a^2} \cos\vartheta \int_0^a w_0 \, dw_0 \int_0^{2\pi} e^{-ikw_0 \sin\vartheta \cos(\phi_0 - \phi)} \, d\phi_0 \\ &= -ikF_\omega \frac{e^{ikr}}{4\pi r} \left[\frac{2J_1(ka \sin\vartheta)}{ka \sin\vartheta} \right] \cos\vartheta \qquad (7.4.36) \end{aligned}$$

where the factor in brackets is the same as that entering the expression for the far field from a piston in a plane. In this case, however, there is an extra directionality factor $\cos\vartheta$, which ensures antisymmetry about the xy plane.

For long wavelengths the factor in brackets, as we saw earlier, is approximately unity. Comparison with Eq. (7.1.7) shows that at low frequencies the disk radiates like a dipole source of strength $D_z = i(F_\omega/\rho ck)$. The angle distribution for the radiated intensity has an additional factor $\cos^2\vartheta$, compared with that for the piston in a plane; thus there is no radiation in the plane of the disk.

Using Eq. (7.3.15) for the near field, we can calculate the pressure on the front surface of the disk and the normal velocity of the portion of the disk a distance $w < a$ from its center.

$$p_+ = \frac{F_\omega}{2\pi a^2}\int_0^\infty J_0(\mu w)\mu\, d\mu \int_0^a J_0(\mu w_0) w_0\, dw_0 = \frac{F_\omega}{2\pi a^2} \qquad w < a$$
$$u_z = \frac{F_\omega}{2\pi a^2 \rho c k}\int_0^\infty J_0(\mu w)\mu\sqrt{k^2-\mu^2}\, d\mu \int_0^a J_0(\mu w_0) w_0\, dw_0 \tag{7.4.37}$$

where we have used Eq. (1.3.27) to compute the pressure. The pressure at the back of the disk is $p_- = -(F_\omega/2\pi a^2)$, so that the total force required to move the fluid ahead and behind the disk is $\pi a^2(p_+ - p_-) = F_\omega$, as it was assumed to be.

When ka is large, the radical in the integral for u_z is equal to k over the range of μ for which the integrand is large, so that Eq. (1.3.27) produces $u_z \simeq F_\omega/\rho c 2\pi a^2$ for $w < a$. For long wavelengths, on the other hand, the integral for u converges poorly because of the large tangential velocity near the edge of the disk. However, we can use Eqs. (7.1.12) to compute an effective normal velocity, an appropriate average over the surface of the disk. The expressions for the pressure at the disk surface, the limiting forms of the disk velocity, and the total power radiated are

$$p_+ = \frac{F_\omega}{2\pi a^2} = -p_- \qquad \langle u_z\rangle = F_\omega Y(\omega)$$
$$Y(\omega) \to \frac{1}{2\pi a^2 \rho c} \qquad ka \gg 1$$
$$\to \frac{1}{6\pi a^3 \rho\omega}\left[i(1 - \tfrac{1}{2}ik^2a^2) + \tfrac{1}{2}i(ka)^3\right] \qquad ka \ll 1 \tag{7.4.38}$$
$$\Pi = |F_\omega|^2 \operatorname{Re} Y \to \begin{cases} |F_\omega|^2 \dfrac{1}{2\pi a^2\rho c} = 2\pi a^2 \dfrac{|p_+|^2}{\rho c} & ka \gg 1 \\ |F_\omega|^2 \dfrac{k^2a^2}{12\pi a^2 \rho c} & ka \ll 1 \end{cases}$$

where Y is the radiative admittance of the disk. Its reciprocal,

$$Z_r = \frac{F_\omega}{u_z} \to \begin{cases} 2\pi a^2 \rho c & ka \gg 1 \\ 3\pi a^2 \rho c(-2ika + k^4a^4) & ka \ll 1 \end{cases}$$

has much the same form, for intermediate values of ka, as that of the impedance Z_r of Eqs. (7.4.31) for the piston in a plane baffle (there is an extra factor 2 in the present case, since the disk radiates to $z \to -\infty$ as well as to $z \to \infty$).

Radiation from an aircraft propeller

These same techniques can be used to compute the far field from an airplane propeller, whirling in the xy plane with an angular velocity ω. If there are n propeller blades, each of length a, with center at the origin, the air in the neighborhood of the disk of radius a in the xy plane is subjected to n successive pushes during each rotation of the propeller axle. Part of this push is in the $-z$ direction, but also, because of turbulence drag, there will be an accompanying force in the ϕ direction, perpendicular to the polar radius w. As with the free disk, we can assume that this driving force is concentrated in the xy plane, for $w < a$, and is greater near the propeller tips at $w = a$. A very rough approximation to the two components of the force on the medium is

$$F_z = -\left[\frac{(n+1)F}{2\pi a^{n+2}}\right] w^n e^{in(\phi-\omega t)}\,\delta(z)u(a-w)$$

$$F_\phi = \frac{(n+1)\alpha F}{2\pi a^{n+3}}\, w^{n+1} e^{in(\phi-\omega t)}\,\delta(z)u(a-w)$$

where F is approximately equal to the total backward thrust of the propeller, α is the drag-to-lift ratio (usually less than 1), and u is the step function of Eq. (1.3.21), being unity when $w < a$, zero when $w > a$.

To find the far field we insert this and the g of Eq. (7.4.35) into Eq. (7.1.25) (f_ω and $\mathfrak{T}_\omega = 0$), obtaining

$$\begin{aligned} p \simeq \frac{ik}{4\pi r}\left\{\frac{(n+1)F}{2\pi a^{n+2}}\right\} e^{ik(r-ct)}\Bigg\{\cos\vartheta \int_0^a w_0^{n+1}\,dw_0 \int_0^{2\pi} d\phi_0 e^{in\phi_0 - ikw_0 \sin\vartheta\cos(\phi_0-\varphi)} \\ + \frac{\alpha}{a}\sin\vartheta \int_0^a w_0^{n+2}\,dw_0 \int_0^{2\pi} d\phi_0 \sin(\phi_0-\varphi) e^{in\phi_0 - ikw_0\sin\vartheta\cos(\phi_0-\varphi)}\Bigg\} \\ = i^{-n-1}\frac{kF}{4\pi r}\left(\frac{\alpha n}{ka} - \cos\vartheta\right)\frac{(ka\sin\vartheta)^n}{2^{n+1}n!} \\ \times\left[\frac{2^{n+1}(n+1)!\,J_{n+1}(ka\sin\vartheta)}{(ka\sin\vartheta)^{n+1}}\right] e^{ik(r-ct)+in\varphi} \end{aligned} \tag{7.4.39}$$

where we must remember that, in this case, $k = n\omega/c$, since we are here using ω as the angular velocity of the propeller; the frequency of the radiation is $n\omega/2\pi$ since there are n blades. The quantity in brackets is roughly unity for $0 < ka\sin\vartheta < n$. Since, usually, the propeller tips are traveling somewhat slower than the speed of sound, $\omega a/c = ka/n$ is somewhat less than 1; thus the factor in brackets is nearly 1 for all values of ϑ, and the radiated intensity is, approximately,

$$I_r \simeq \frac{a^{2n}k^{2n+2}\,|F|^2}{2^{2n+6}\pi^2 r^2 (n!)^2 \rho c}\left(\frac{\alpha n}{ka} - \cos\vartheta\right)^2 \sin^{2n}\vartheta \tag{7.4.40}$$

Aside from the distorting factor in parentheses, this is the radiation from a tesseral multipole of order $n + 1$, with a strong maximum in the xy plane. Because of the distorting factor the maximum is just aft of the plane of the propeller, the peak being the more pronounced, the greater the number of blades n. The factor in parentheses comes about because of the relationship between lift and drag; the term $\alpha n/ka$ is usually somewhat less than unity, so that the intensity is noticeably greater in the backward direction (the direction of the push) than in the forward direction. Also, at the angle $\cos^{-1}(\alpha n/ka)$ to the axis in the forward direction, no sound is radiated. The whole angular pattern is further distorted when the propeller is moving with respect to the air [Eq. (11.2.34)]. In addition, the actual force is not sinusoidal, but more pulselike, with harmonic terms $e^{imn(\phi-\omega t)}$ $(m = 2, 3, \ldots)$. For these higher-frequency terms the factor in brackets in Eq. (7.4.39) imposes a preference for the axial rather than the equatorial directions. Thus the higher harmonics are louder along the axis than in the plane of the propeller.

$$\Pi \simeq \frac{a^{2n}k^{2n+2}\,|F|^2}{2^{n+4}\pi n!\,\rho c}\,\frac{1}{1\cdot 3\cdot 5\cdot \cdots \cdot (2n+3)}\left[(2n+3)\left(\frac{\alpha n}{ka}\right)^2 + 1\right] \qquad (7.4.41)$$

Comparison of the expressions for Π given in Eqs. (7.1.11) and (7.1.12) indicates that the power radiated from a multipole of order m, of strength M_m, is

$$\Pi_m = \frac{\rho c k^{2m}\,|M_m|^2}{4\pi\cdot 1\cdot 3\cdot \cdots \cdot (2m+1)}$$

Therefore the rotating n-blade propeller radiates as though it were a combination of a tesseral multipole of order n and strength M_n, minus another of order $n + 1$ and strength M_{n+1}, where

$$M_n \simeq \frac{Fa^n}{2\rho c\sqrt{2^n n!}}\,\frac{an}{ka} \qquad M_{n+1} \simeq \frac{F(a^n/k)}{2\rho c\sqrt{2^n n!}}$$

the n-pole, symmetric about the xy plane, corresponding to the effects of the propeller drag and the $(n + 1)$-pole, antisymmetric about the xy plane, caused by the propeller thrust in the negative z direction. This problem will be discussed again in Chap. 11.

The rectangular piston

Since they will be of use later, we shall work out the formulas for a flat piston of rectangular shape, of sides a and b, set in a plane rigid wall. The integrals for the far field and for the piston impedance are similar to those for the circular piston; only the boundaries of integration are now between $-\frac{1}{2}a$ and $\frac{1}{2}a$ for x_0 and between $-\frac{1}{2}b$ and $\frac{1}{2}b$ for y_0. The integral for the far

field, analogous to Eq. (7.4.29) for $\beta = 0$, is

$$p_\omega(\mathbf{r}) \simeq -ik\rho c u_\omega \frac{e^{ikr}}{2\pi r} \int_{-\frac{1}{2}a}^{\frac{1}{2}a} dx_0 \int_{-\frac{1}{2}b}^{\frac{1}{2}b} dy_0 \exp\left[-ik \sin\vartheta\,(x_0 \cos\varphi + y_0 \sin\varphi)\right]$$

$$= 2ik\rho c u_\omega ab S(\tfrac{1}{2}ka \sin\vartheta \cos\varphi) S(\tfrac{1}{2}kb \sin\vartheta \sin\varphi) \frac{e^{ikr}}{4\pi r} \qquad (7.4.42)$$

where

$$S(z) = \frac{\sin z}{z}$$

The radiated sound intensity is concentrated in a cone ahead of the piston, as is the sound from the circular piston; only now the cone is elliptic in cross section, with its wider axis in the direction of the *smaller* dimension (a or b). Function $S(z)$ has much the same behavior as $2J_1(z)/z$, the corresponding function for the circular piston, though it falls off rather less rapidly as z increases and its zeros come for different values of z.

The radiation impedance of the rectangular piston is obtained by using Eq. (7.4.7) for the Green's function, instead of Eq. (7.4.10), in Eq. (7.4.18) (with $\beta = 0$). Setting both z and z_0 zero, the pressure at the surface of the piston is

$$p_\omega(x,y) = \frac{-i}{4\pi^3} \rho c k u_\omega ab \iiint_{-\infty}^{\infty} \frac{dK_x\, dK_y\, dK_z}{K_x^2 + K_y^2 + K_z^2 - k^2} S(\tfrac{1}{2}K_x a) S(\tfrac{1}{2}K_y b) e^{-iK_x x - iK_y y}$$

$$= \frac{1}{4\pi^2} \rho c k u_\omega ab \iint_{-\infty}^{\infty} \frac{dK_x\, dK_y}{\sqrt{k^2 - K_x^2 - K_y^2}} S(\tfrac{1}{2}K_x a) S(\tfrac{1}{2}K_y b) e^{iK_x x + iK_y y}$$

and the total reaction force on the piston is

$$F_\omega = \frac{1}{4\pi^2} \rho c k u_\omega a^2 b^2 \int_0^\infty \frac{\mu\, d\mu}{k^2 - \mu^2} \int_0^{2\pi} S^2(\tfrac{1}{2}\mu a \cos\theta) S^2(\tfrac{1}{2}\mu b \sin\theta)\, d\theta \qquad (7.4.43)$$

where $\mu^2 = K_x^2 + K_y^2$. By expanding the functions S^2 in power series and integrating term by term, a power series for the integral over θ can be computed. The integral over μ can then be calculated numerically, for the appropriate values of ka and kb. Comparison of series, however, shows that a rough approximation to F_ω, valid for all values of k as long as a/b is neither very large nor very small, is

$$F_\omega \simeq \rho c u_\omega ab \frac{a^2\theta_\square(ka) - b^2\theta_\square(kb) - ia^2\chi_\square(ka) + ib^2\chi_\square(kb)}{a^2 - b^2}$$

$$\to \rho c u_\omega ab[\theta_0(ka) - i\chi_0(ka)] \qquad b \to a \qquad (7.4.44)$$

where

$$\theta_{\square}(z) = \frac{2}{z^2}\int_0^z \theta_0(y)y\,dy = \left[1 - 4\,\frac{1 - J_0(z)}{z^2}\right] \to \begin{cases} \frac{1}{16}z^2 & z \ll 1 \\ 1 & z \gg 1 \end{cases}$$

$$\chi_{\square}(z) = \frac{2}{z^2}\int_0^z \chi_0(y)y\,dy = \frac{8}{\pi z}\left[1 - \frac{\pi}{2z}\,M_0(z)\right] \to \begin{cases} \dfrac{8}{9\pi}\,z & z \ll 1 \\ \dfrac{8}{\pi z} & z \gg 1 \end{cases}$$

and $M_0(x) = (2/\pi)\displaystyle\int_0^{\pi/2} \sin(x\cos u)\,du$ is the Struve function of zero order.

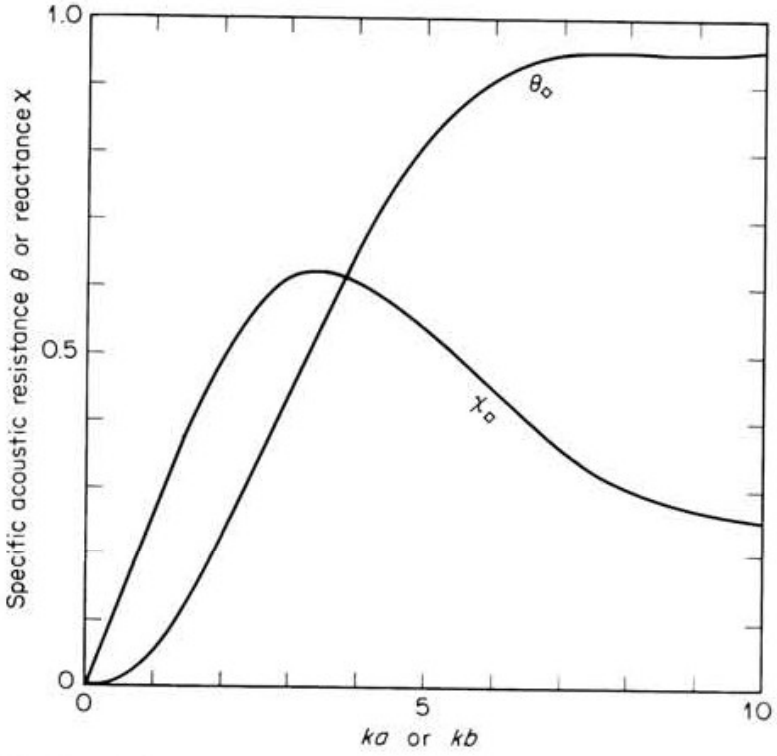


FIGURE 7.11
Impedance functions for the rectangular piston. See Eq. (7.4.44).

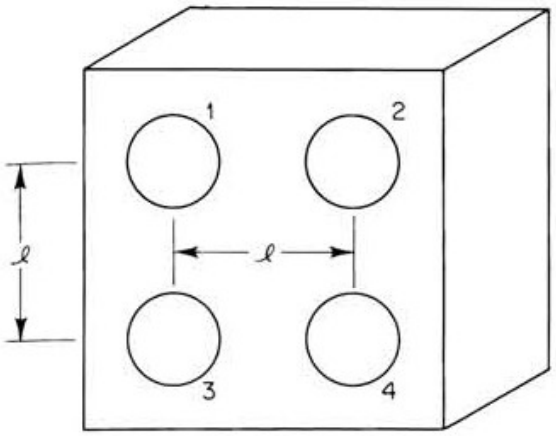
The functions θ_0, χ_0 are defined in Eqs. (7.4.31), plotted in Fig. 7.9, and given in Table IX. Functions $\theta_{\square}$ and $\chi_{\square}$ are plotted in Fig. 7.11. Approximate limiting values for the radiation impedance of the piston are thus

$$Z_r = \frac{F_\omega}{u_\omega} \to \begin{cases} \frac{1}{16}k^2(a^2 + b^2) - \dfrac{8i}{9\pi}\,k\,\dfrac{a^2 + ab + b^2}{a + b} & ka \ll 1 \\ 1 - i\,\dfrac{8}{\pi k(a + b)} & ka \gg 1 \end{cases} \tag{7.4.45}$$

for values of b/a which are neither very large nor very small.

Problems

1 An airtight box, with four loudspeakers set in the front panel as shown, is suspended in free space. Each loudspeaker has an area A and oscillates with velocity $U_0 e^{-i\omega t}$. Determine the power radiated from this system at low



frequencies when the wavelength is much larger than the dimensions of the box:

(*a*) When all speakers oscillate in phase
(*b*) When speakers 1 and 2 are in phase but 180° out of phase with 3 and 4
(*c*) When 1 and 4 are in phase and 180° out of phase with 2 and 3

2 The portion of the surface of a long cylinder of radius a which is between $-\phi_0$ and $+\phi_0$ vibrates with velocity normal to the surface, and the rest of the cylinder is rigid. Obtain a series analogous to Eqs. (7.2.19) for the average acoustic impedance over the vibrating surface.

3 A sphere and a circular disk are both driven in one-dimensional motion with a force $F_x = F_0 \exp(-i\omega t)$. The disk is driven normal to its plane. If the sphere and the disk have the same mass m and radius a, show that the velocity amplitudes at low frequencies are

$$u_s = \frac{F_0}{\omega(m + m_s)} \quad \text{and} \quad u_d = \frac{F_0}{\omega(m + m_d)}$$

where

$$m_s = \frac{2\pi}{3}\rho a^3 \quad \text{and} \quad m_d = \frac{8\pi}{3}\rho a^3$$

where ρ is the density of the surrounding fluid. For an arbitrary body the mass reaction of the fluid must in general be expressed as a tensor $\mathfrak{M}$ such that $iF_i = \omega \sum_j M_{ij} u_j$. What, then, is the ratio between the velocity amplitude and the amplitude of the driving force? What are the elements of the tensor $\mathfrak{M}$ for the sphere and for the disk?

4 Compute the impedance for an elementary dipole a distance l from a plane locally reacting wall of acoustic impedance $\rho c/\beta$ when the dipole axis is parallel to the wall. What is the impedance when the dipole axis is perpendicular to the wall?

5 A monopole is a distance h above an infinite horizontal locally reacting boundary having acoustic impedance $\rho c\zeta$. The monopole strength at frequency $\omega/2\pi$ is $S(\omega)$. Calculate the sound pressure as a function of the distance r

from the source in the horizontal plane of the source. Show that if the source height is larger than λ, the sound-pressure amplitude exhibits maxima and minima in a region given approximately by $0 < r < h^2/\lambda$ and decreases monotonically with r for sufficiently large r ($>h^2/\lambda$). If ζ is finite, show that this monotonic decrease approaches proportionality to $1/r^2$ for large r. Estimate the value of r, in terms of $|\zeta|$, above which this is approximately true.

6 Show that a point force $\mathbf{F}(t)$, applied at $\mathbf{r} = 0$ in a gas, produces a sound pressure field

$$p(\mathbf{r},t) = \frac{1}{4\pi rc}\left(\frac{\partial F}{\partial t} + \frac{cF}{r}\right)\cos\vartheta$$

where ϑ is the angle between $\mathbf{F}$ and $\mathbf{r}$ (assuming that the direction of $\mathbf{F}$ is constant). What is the result if the direction as well as the magnitude of $\mathbf{F}$ changes with time? (Parentheses indicate retarded time.)

7 A pulsating sphere of radius a has a radial surface velocity $U_0 e^{-i\omega t}$, where U_0 is independent of ϑ and φ as well as of t. Show that the acoustic radiation impedance at the surface is $\rho c\zeta_r$, where

$$\zeta_r = \frac{k^2a^2}{1 + k^2a^2} - i\frac{ka}{1 + k^2a^2}$$

where $kc = \omega$. Is this valid for any value of ka?

8 The monopole source density $s_\omega(\mathbf{r})$ of Eq. (7.1.30) has the uniform value $S_\omega/\pi a^2 l$ throughout the inside of a cylinder of radius a, with axis along the x axis and between the planes $x = \pm\frac{1}{2}l$, and is zero outside the cylinder. Change the factor $e^{-i\mathbf{k}\cdot\mathbf{r}_0}\,dv_0$ in the integral to $\exp[-ikw_0\sin\vartheta\cos(\phi - \phi_0) - ikx_0\cos\vartheta]\,w_0\,d\phi_0\,dw_0\,dx_0$, where $w_0^2 = y_0^2 + z_0^2$, and use the formulas

$$\int_0^{2\pi} e^{iq\cos\theta}\cos(m\theta)\,d\theta = 2\pi i^m J_m(q) \qquad \text{and} \qquad \int u^{m+1}J_m(u)\,du = u^{m+1}J_{m+1}(u)$$

to show that the pressure wave a distance r from the origin is

$$-ik\rho cS_\omega \frac{\sin(\frac{1}{2}kl\cos\vartheta)}{\frac{1}{2}kl\cos\vartheta}\frac{J_1(ka\sin\vartheta)}{\frac{1}{2}ka\sin\vartheta}\frac{e^{ikr-i\omega t}}{4\pi r} \qquad r \gg a, l$$

where ϑ is the angle between r and the x axis. Plot the dependence of the intensity at r on ϑ when $ka = 4$ and $kl = 3$. When kl and ka are small but not negligible, show that the far-field pattern is similar to that from a combination of a point monopole S_ω minus three quadrupoles of strengths

$$Q_{xx} = \tfrac{1}{24}l^2S_\omega \qquad Q_{yy} = Q_{zz} = \tfrac{1}{8}a^2S_\omega$$

all at the origin. *Hint*: use the expansion

$$J_m(x) = \frac{(x/2)^m}{m!}\left[1 - \frac{(x/2)^2}{m+1} + \cdots\right]$$

9 A spherical cloud of ionized gas is acted on by an oscillating electric field so that the force $\mathbf{F}$ of Eq. (7.1.24) is in the z direction and

$$F_z = \begin{cases} A(a^2 - r^2)e^{-i\omega t} & r < a \\ 0 & r > a \end{cases}$$

Show that the pressure wave produced outside the sphere is

$$p(r,\vartheta,\varphi) = \tfrac{1}{3}ia^3A\,j_2(ka)\cos\vartheta\,h_1(kr)$$

Use the integral relationship $\int z^{n+2}j_n(z)\,dz = z^{n+2}j_{n+1}(z)$.

10 The fluid inside a sphere of radius a is in violent motion so that the velocity components with frequency $\omega/2\pi$ are

$$u_x = \begin{cases} -\tfrac{1}{3}Q_\omega y^2 e^{-i\omega t} & r < a \\ 0 & r > a \end{cases} \qquad u_y = \begin{cases} Q_\omega xy e^{-i\omega t} & r < a \\ 0 & r > a \end{cases}$$

and $u_z = 0$. Use Eqs. (7.1.25) and (7.2.31) to show that the pressure wave for $r > a$, generated by this motion, has frequency $\omega/2\pi$ and amplitude

$$i\,|Q|^2\,a^4 j_3(2ka)\sin^2\vartheta\cos 2\varphi\,h_2(2kr) \qquad k = \omega/c$$

What is the pressure amplitude when $r \to \infty$? When $ka \to 0$?

11 A region of volume V contains ionized gas, which is acted on by a randomly fluctuating electromagnetic field, so that the vector $\mathbf{F}$ in Eq. (7.1.25) is not zero. The mean-square amplitudes $\langle|F_x|^2\rangle$ and $\langle|F_y|^2\rangle$ are equal, but are not equal to $\langle|F_z|^2\rangle$; the correlation length w_d and correlation time $\tau_d = 1/\omega_d$ are the same for all three components. Show that the rms intensity at r, for all frequencies, averaged over a long time interval T, is

$$I_d \to \frac{V\lambda_d}{\sqrt{32\pi}\,\rho c r^2}\,\frac{{w_d}^3}{{w_d}^2 + {\lambda_d}^2}\left(\langle|F_y|^2\rangle\cos^2\vartheta + \langle|F_x|^2\rangle\sin^2\vartheta\right)$$

for $r \to \infty$, where $\lambda_d = c/\omega_d$, ϑ is the angle between vector r and the z axis.

12 A long rigid cylinder of diameter d, placed in a transverse airstream, experiences a transverse oscillatory force because of the flow instability produced in the stream (see Chap. 11). As a result, sound is radiated (Aeolian tone) at a characteristic frequency $\omega/2\pi$. If the correlation length of the force along the cylinder is L, calculate the rms power radiated per unit length of the cylinder.

13 Suppose the specific acoustic admittance β of the surface of Eq. (7.4.21) does depend on frequency, so that the Fourier transform of the admittance term is

$$b(t) = \int_{-\infty}^{\infty} \frac{\omega\cos\vartheta}{\beta + \cos\vartheta}\,e^{-i\omega t}\,d\omega$$

By use of Prob. 23 of Chap. 1, show that the pressure wave at r, t (r very large), caused by a velocity distribution $u(x,y,t)$, is

$$\frac{i\rho}{4\pi^2 r}\int dX_0\int dY_0\int b(\tau)u\,[X_0,\ Y_0,\ T(X_0) - \tau]\,d\tau$$

where $T(X_0) = t - (r/c) + (X_0/c)\sin\vartheta$, and u, X_0, Y_0 are as defined in connection with Eq. (7.4.23).

14 The surface of Prob. 13 has acoustic impedance $R + i(K/\omega)$, so that $\beta = \rho c\omega/(\omega R + iK)$. Show that

$$b(t) = -2\pi i\left(\frac{K}{R}\right)^2\frac{(\rho c/R)\cos^2\vartheta}{[(\rho c/R) + \cos\vartheta]^3}\exp(-Wt)\,u(t)$$

where

$$W = \frac{(K/R)\cos\vartheta}{(\rho c/R) + \cos\vartheta}$$

and $u(t)$ is the unit step function of Eq. (1.3.22). If the radiating area extends over X_0 from 0 to l, over Y_0 from 0 to a (i.e., if the rz plane is parallel to two sides of the rectangular radiating area), and if the velocity of the area is a pulse $u(X,Y,t) = \Delta_0\delta(t)$, show that the pressure wave at $\mathbf{r}$, t is

$$\frac{\rho c\,\Delta_0 a}{2\pi r}\,\frac{K}{R}\,\frac{(\rho c/R)\cos\vartheta}{[(\rho c/R)+\cos\vartheta]^2}$$

$$\times\begin{cases}\exp\left[-W\left(t-\dfrac{r}{c}\right)\right]-\exp\left[-WT(l)\right] & r<ct\\ 1-\exp\left[-WT(l)\right] & r>ct>r-l\sin\vartheta\\ 0 & ct<r-l\sin\vartheta\end{cases}$$

where $T(x) = (x/c)\sin\vartheta + t - (r/c)$. Discuss the physical significance of this result; plot it as function of t for appropriate values of the constants.

15 A circular membrane of radius a, under tension, is set in an otherwise rigid plane wall. Suppose the membrane is vibrating with its (m,n)th mode [Eq. (5.2.25)] so that the u of Eq. (7.4.19) is $U_m\cos(m\phi)J_m(\pi\beta_{mn}r/a)\exp(-ik_{mn}ct)$. Show that the asymptotic form of the pressure wave a distance r from the center of the membrane is, when $r \gg a$,

$$-i\pi ka^2\rho c\beta_{mn}U_{mn}J_m(\pi\beta_{mn})\,\frac{J_m(ka\sin\vartheta)}{k^2a^2\sin^2\vartheta-\pi^2\beta_{mn}^2}\,\frac{1}{r}\exp\left[ik_{mn}(r-ct)\right]$$

where k_{mn} equals $\pi\beta_{mn}/a$ times the ratio between the wave velocity in the membrane and that in air.

16 A spherical source of radius a consists of a uniform monopole distribution of source strength $Q_0e^{-i\omega t}$ per unit volume. Show that the radiated pressure is $(-i\omega Q_0V/4\pi r)F(ka)e^{ik(r-a)-i\omega t}$, where $k = \omega/c$, $V = 4\pi a^3/3$, and

$$F(x) = \frac{6}{x^3}(x-\sin x) + \frac{3i}{x^3}(x^2-2+2\cos x)$$

17 By using Eqs. (7.4.38), determine by what factor the low-frequency output of a loudspeaker can be increased by providing it with a baffle box, in effect to convert it from a dipole to a monopole source. In making the comparison, let the amplitude of oscillation of the speaker disk be the same for the two cases. Also assume that the wavelength is larger than the dimensions of the box.

18 A small gas bubble of radius a in a liquid medium acts as a monopole source of sound when set into radial oscillation. Starting with the equilibrium conditions for the bubble, $P_i = P_0 + (2\sigma/a)$, where P_i is the pressure inside the bubble, P_0 the static pressure in the liquid, and σ the surface tension, take into account the radiative mass load on the bubble and determine the resonance frequency of the radial vibrations. Assuming that $P_i \gg P_0$, and that the compression of the gas in the bubble is adiabatic, show that, in the long-wavelength limit, when a is small compared with λ, the resonance frequency is

$$\frac{1}{2\pi}\sqrt{(3\gamma-1)\,\frac{2\sigma}{a^3\rho}}$$

where $\gamma = C_p/C_v$ for the gas in the bubble, and ρ is the density of the liquid.

19 As a result of the Alaskan earthquake of March, 1964, surface waves (Rayleigh waves) of considerable amplitude passed over the United States. These waves radiated sound into the atmosphere. At infrasonic experimental stations in Washington and in Boulder, Colo., the vertical velocity amplitude of the earth surface was found to be about $\frac{1}{2}$ cm per sec, with a period of oscillation of 25 sec. If the speed of the surface waves was five times the speed of sound in air, determine the direction of propagation and the intensity of the radiated sound wave (assumed to be a plane wave). From conservation of energy of the sound wave, determine the resulting amplitude of oscillation of the ionosphere. The gas pressure in the ionosphere is about 10^{-8} atm.

CHAPTER

8

THE SCATTERING OF SOUND

8.1 SIMPLE EXAMPLES OF SCATTERING

When a sound wave encounters an obstacle, some of the wave is deflected from its original course. It is usual to define the difference between the actual wave and the undisturbed wave, which would be present if the obstacle were not there, as the *scattered* wave. When a plane wave, for instance, strikes a body in its path, in addition to the undisturbed plane wave there is a scattered wave, spreading out from the obstacle in all directions, distorting and interfering with the plane wave. If the obstacle is very large compared with the wavelength (as it usually is for light waves and very seldom is for sound), half of this scattered wave spreads out more or less uniformly in all directions from the scatterer, and the other half is concentrated behind the obstacle in such a manner as to interfere destructively with the unchanged plane wave behind the obstacle, creating a sharp-edged shadow there. This is the case of geometrical optics; in this case the half of the scattered wave spreading out uniformly is called the *reflected* wave, and the half responsible for the shadow is called the *interfering* wave. If the obstacle is very small compared with the wavelength (as it often is for sound waves), all the scattered wave is propagated out in all directions, and there exists no sharp-edged shadow. In the intermediate cases, where the obstacle is about the same size as the wavelength, a variety of curious interference phenomena can occur.

In this chapter, since we are studying sound waves, we shall be interested in the second and third cases, where the wavelength is longer than, or at least the same size as, the obstacle. We shall not encounter or discuss sharply defined shadows. So much of the scattered wave will travel in a different direction from the plane wave that destructive interference will be unimportant, and we shall be able to separate all the scattered wave from the undisturbed plane wave. We shall be interested in the total amount of the wave that is scattered, in the distribution in angle of this wave, and in the effect of this scattered wave on the pressure at various points on the surface of the obstacle.

Scattering from a cylinder

Let us first compute the scattering, by a cylinder of radius a, of a plane wave traveling in a direction perpendicular to the cylinder's axis. If the plane wave has intensity I, the pressure wave, if the cylinder were not present, would be

$$p_p = P_0 e^{ik(x-ct)} = P_0 e^{ik(r\cos\phi - ct)} \qquad P_0 = \sqrt{\rho c I} \qquad k = \frac{2\pi}{\lambda}$$

where the direction of the plane wave has been taken along the positive x axis.

In Eq. (1.2.9) we expressed this plane wave in terms of cylindrical waves:

$$p_p = P_0 e^{ik(r\cos\phi - ct)} = P_0 \left[J_0(kr) + 2\sum_{m=1}^{\infty} i^m \cos(m\phi) J_m(kr) \right] e^{-2\pi i\nu t} \tag{8.1.1}$$

The radial velocity corresponding to this wave is

$$u_{pr} = \frac{P_0}{\rho c}\left\{ iJ_1(kr) + \sum_{m=1}^{\infty} i^{m+1}[J_{m+1}(kr) - J_{m-1}(kr)]\cos(m\phi) \right\} e^{-2\pi i\nu t}$$

When the cylinder is present with its axis at $r = 0$, the wave cannot have the form given by the above series, for the cylinder distorts the wave. There is present, in addition to the plane wave, a scattered outgoing wave of such a size and shape as to make the radial velocity of the combination zero at $r = a$, the surface of the cylinder. We shall choose the form of this outgoing wave to be the general series

$$p_s = \sum_{m=1}^{\infty} A_m \cos(m\phi)[J_m(kr) + iN_m(kr)]e^{-2\pi i\nu t}$$

$$u_{sr} = \frac{1}{\rho c}\left\{ iA_0[J_1(kr) + iN_1(kr)] + \frac{i}{2}\sum_{m=1}^{\infty} A_m \cos(m\phi)[J_{m+1}(kr) - J_{m-1}(kr) + iN_{m+1}(kr) - iN_{m-1}(kr)] \right\} e^{-2\pi i\nu}$$

The combination $J + iN$ has been chosen because it ensures that all the scattered wave is outgoing.

Our first task is to find the values of the coefficients A which make the combination $u_{pr} + u_{sr}$ equal zero at $r = a$. Equating u_{sr} to $-u_{pr}$ at $r = a$ term by term, we obtain

$$A_m = -\epsilon_m P_0 i^{m+1} e^{-i\gamma_m} \sin\gamma_m \qquad P_0 = \sqrt{\rho c I}$$

$$\tan\gamma_0 = -\frac{J_1(ka)}{N_1(ka)} \qquad \tan\gamma_m = \frac{J_{m-1}(ka) - J_{m+1}(ka)}{N_{m+1}(ka) - N_{m-1}(ka)} \tag{8.1.2}$$

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for all values of m larger than unity. These phase angles γ_m have already been defined in Eqs. (7.3.7), in connection with the

UNIVERSIDAD NACIONAL

radiation of sound from a cylinder. Values of some of them are given in Table V. The behavior of these phase angles completely determines the behavior of the scattered wave. It is interesting to notice the close connection between the waves scattered by a cylinder and the waves radiated by the same cylinder when it is vibrating. The quantities needed to compute one are also needed to compute the other.

The pressure and radial velocity of the scattered wave at large distances from the cylinder are

$$p_s \simeq -\sqrt{\frac{2\rho c I a}{\pi r}}\,\psi_s(\phi)e^{ik(r-ct)} \qquad u_s \simeq \frac{p_s}{\rho c}$$

$$\psi_s(\phi) = \frac{1}{\sqrt{ka}} \sum_{m=0}^{\infty} \epsilon_m \sin(\gamma_m) e^{-i\gamma_m} \cos(m\phi)$$

The intensity of the scattered part, at the point $(r,\phi)(kr \gg 1)$, is therefore

$$\Upsilon_s \simeq \frac{Ia}{\pi r}\,|\psi_s(\phi)|^2 \tag{8.1.3}$$

$$|\psi_s|^2 = \frac{1}{ka} \sum_{m,n=0}^{\infty} \epsilon_m \epsilon_n \sin\gamma_m \sin\gamma_n \cos(\gamma_m - \gamma_n) \cos(m\phi) \cos(n\phi)$$

where $\epsilon_0 = 1$, $\epsilon_m = 2 \quad (m > 0)$. This intensity is plotted as a function of ϕ on a polar plot in Fig. 8.1, for different values of $\eta = 2\pi va/c = 2\pi a/\lambda$.

It is interesting to notice the change in directionality of the scattered

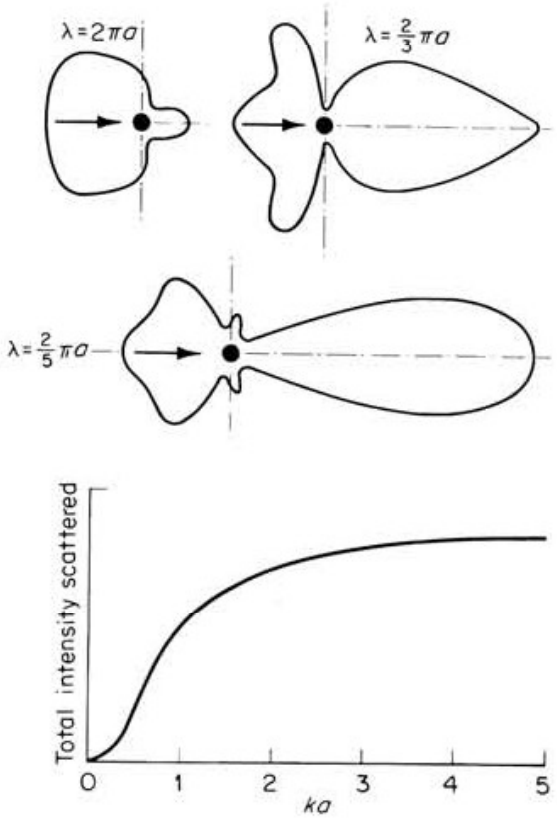


FIGURE 8.1
The scattering of sound waves from a rigid cylinder of radius a. Polar diagrams show the distribution in angle of the intensity of the scattered wave; lower graph shows the dependence of the scattered intensity on $\eta = 2\pi a/\lambda$.

wave as the wavelength is changed. For very long wavelengths (η small) but little is scattered, and this is scattered almost uniformly in all the backward directions. As the frequency is increased, the distribution in angle becomes more and more complicated, diffraction peaks appearing and moving forward, until for very short wavelengths (much shorter than those shown in Fig. 8.1), one-half of the scattered wave is concentrated straight forward (the interfering beam), and the other half is spread more or less uniformly over all the other directions, giving a polar plot which is a cardioid, interrupted by a sharp, very high peak in the forward direction, as shown in Eq. (8.1.4) below.

For very long wavelengths only the two cylindrical waves corresponding to $m = 0$ and $m = 1$ are important in the scattered wave. As shown in Eq. (7.3.7),

$$\gamma_0 \simeq -\gamma_1 \simeq \frac{\pi k^2 a^2}{4} \gg -\gamma_2, -\gamma_3, - \ldots \qquad \text{when } ka \to 0$$

The first-order approximation for the scattered intensity at long wavelengths is therefore

$$I_s \simeq \frac{\pi \omega^3 a^4}{8c^3 r} I(1 - 2\cos\phi)^2 \qquad \omega a \ll c$$

Short-wavelength limit

For wavelengths very small compared with the circumference of the cylinder, the relatively simple approximation of "geometrical acoustics" is valid, with the "scattered" wave dividing into two parts, the "reflected" and the "shadow-forming" waves. However, the process of demonstrating that the series of Eqs. (8.1.3) really does behave in a simple manner involves mathematical manipulation of considerable intricacy.

In the first place, the series of Eqs. (8.1.3), which is in a form useful for calculating scattering at longer wavelengths, is not particularly suitable for showing in detail the interference between part of the scattered wave and the primary wave to form the shadow. In optics, one differentiates between *Fraunhofer diffraction*, where intensities are measured at distances so large that the angle subtended by the diffracting object is small compared with the ratio $(\lambda/2\pi a) = 1/ka$, and *Fresnel diffraction*, for distances large compared with the wavelength but not extremely large compared with $2\pi a$. The Fresnel-diffraction formulas show the shadow with its related diffraction bands, but at the great distances involved in the Fraunhofer-diffraction formulas, the shadow has become blurred again. Series (8.1.3) is for distances corresponding to Fraunhofer diffraction, so that what can be demonstrated is the separation of the scattered wave into a reflected wave and a shadow-forming wave, but not the details of the interference with the incident wave, which are characteristic of the Fresnel formulas.

In the second place, the simplicity of the formulas for very short wavelengths only appears as an average; the scattered intensity varies rapidly with angle in a complicated sort of way, and only the average intensity per degree (or per minute) varies smoothly. This rapid fluctuation is seldom measured, however, for any small change of frequency or of position of the cylinder will blur it out, leaving only the average intensity. Consequently, our calculations should separate the rapid fluctuations of intensity from the average behavior of the reflected wave.

When all the necessary manipulations are made, the expression for the scattered intensity at short wavelengths is

$$\frac{I_s}{I} \simeq \frac{a}{r}\sin\left(\frac{\phi}{2}\right) + \frac{1}{\pi kr}\cot^2\frac{\phi}{2}\sin^2(ka\sin\phi) + (\text{rapidly fluctuating terms}) \qquad kr \gg ka \gg 1 \quad (8.1.4)$$

The first term of this expression constitutes the reflected intensity, which, for a cylinder, reflects more in the backward direction ($\phi \simeq \pi$) than in a forward direction ($\phi \simeq 0$). The second term is the shadow-forming beam, concentrated in the forward direction within an angle $\pi/ka = \lambda/2a$ which is smaller the smaller λ is compared with a. The third term contains rapidly fluctuating quantities that average to zero, and so may be neglected.

Total scattered power

The total power scattered by the cylinder per unit length is obtained by multiplying I_s by r and integrating over ϕ from 0 to 2π. The cross terms in the sum of Eqs. (8.1.3) disappear, because of the integral properties of the characteristic functions $\cos(m\phi)$, leaving the result

$$\Pi_s = 4aI\frac{1}{ka}\sum_{m=0}^{\infty}\epsilon_m\sin^2(\gamma_m) \simeq \begin{cases}\dfrac{6\pi^5a^4}{\lambda^3}I & \lambda \gg 2\pi a \\ 4aI & \lambda \ll 2\pi a\end{cases} \quad (8.1.5)$$

The limiting value for total *scattered* power for very short wavelengths is the power contained in a beam *twice* as wide as the cylinder ($4a$). This is due to the fact, discussed above, that the scattered wave includes *both* the reflected *and* the shadow-forming waves, the first and second terms of Eq. (8.1.4). The integral of r times the first term is the total *reflected* power, which is just $2aI$; and the integral of r times the second term is also $2aI$, showing that the shadow-forming wave has enough power just to cancel the primary wave behind the cylinder.

The quantity $\Pi_s/4aI$ is plotted in Fig. 8.1 as a function of $ka = 2\pi a/\lambda$. Notice that in spite of the various peculiarities of the distribution in angle of the intensity, the total scattered intensity turns out to be a fairly smooth function of ka. We also should note that Π_s is not usually measured experimentally, because of the difficulty of separating primary from scattered

waves at small angles of scattering. What is usually measured is more nearly the quantity

$$\Pi_{\text{exp}} \simeq 2\int_{\Delta}^{\pi-\Delta} I_s r\, d\phi$$

where Δ is a small angle, if the experimental conditions are good, but never zero. It turns out that Π_{exp} is very nearly equal to Π_s for the longer wavelengths; but as the wavelength is made smaller, the shadow-forming beam [the second term in Eq. (8.1.4)] is less and less included in the integral, until for very short wavelengths Π_{exp} is equal to $\frac{1}{2}\Pi_s$. The transition from Π_s to $\frac{1}{2}\Pi_s$ comes at values of ka near $\pi/2\Delta$, at wavelengths λ near $4a\Delta$.

The force on the cylinder

Returning now to the expression for the total pressure due to both the undisturbed plane wave and the scattered wave, we find, after some involved juggling of terms [the use, for instance, of the last of Eqs. (7.3.2)], that the total pressure at the surface of the cylinder at an angle ϕ from the x axis is

$$p_a = (p_p + p_s)_{r=a} = \frac{4P_0}{\pi ka} e^{-2\pi i\nu t} \sum_{m=0}^{\infty} \frac{\cos(m\phi)}{E_m} e^{i[-\gamma_m + (\pi m/2)]} \qquad (8.1.6)$$

where the quantities E_m are defined in Eq. (7.3.7). This expression is proportional to the expression in Eq. (7.3.8), giving the pressure at some distance from the cylinder due to a vibrating line element on the cylinder, when we make the necessary change from ϕ to $\pi - \phi$ (since now $\phi = 0$ is in the direction *opposite* to the source). This is an example of the *principle of reciprocity*. The pressure at a point A due to a source at point B is equal to the pressure at B due to a source at A, everything else being equal. Therefore the polar curves of Fig. 7.5 show the distribution of intensity about a cylinder having a line source, and they also show the distribution of the square of the pressure on the surface of the cylinder due to a line source at very large distances away from the cylinder (distances so large that the wave has become a plane wave by the time the wave strikes the cylinder).

When η is very small ($\lambda \gg a$), the expression for the pressure at $r = a$ reduces to

$$p \xrightarrow[ka\to 0]{} P_0(1 + 2ika\cos\phi)e^{-2\pi i\nu t}$$

which approaches in value the pressure $P_0 e^{-2\pi i\nu t}$ of the plane wave alone as η goes to zero.

The net force on the cylinder per unit length is in the direction of the plane wave and is

$$F = a\int_0^{2\pi} p\cos\phi\, d\phi = 4aP_0 \frac{1}{kaE_1} e^{-i\omega t - i\gamma_1 + i\pi/2}$$

$$\simeq \begin{cases} i\omega \dfrac{4\pi^2 a^2}{c} P_0 e^{-i\omega t} & \omega \ll \dfrac{c}{a} \\ \sqrt{4a\lambda} P_0 e^{i(\omega/c)(a-ct)+i\pi/4} & \lambda \ll 2\pi a \end{cases} \qquad (8.1.7)$$

This force lags behind the pressure which the plane wave would have at $r = 0$ if the cylinder were not there by an angle $-\gamma_1 + (\pi/2)$.

The quantity $|F/P_0|$ is the net force on the cylinder per unit length per unit pressure of the plane wave. This quantity, divided by $2\pi a$, is plotted in Fig. 8.2 as a function of $ka = \eta$. We note that for small frequencies the force is proportional to the frequency (that is, to η), but that when η becomes larger than unity (i.e., when the wavelength becomes smaller than the circumference of the cylinder), the linear relation breaks down and the force diminishes with increasing frequency thenceforth. This result is of interest

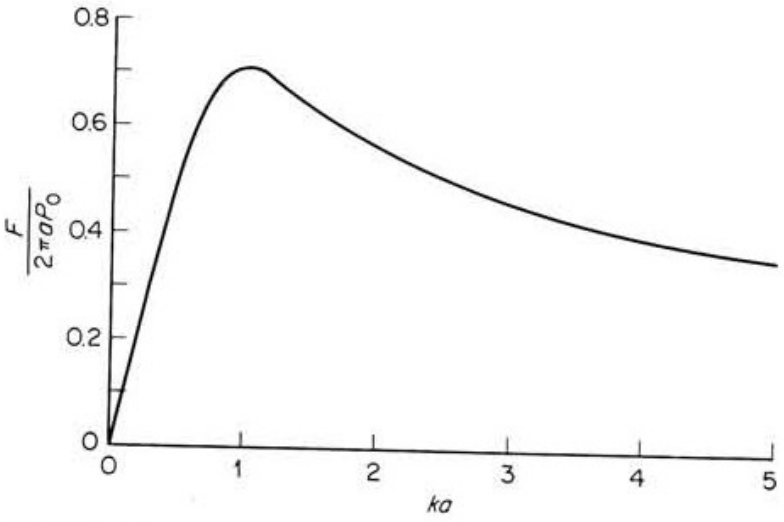


FIGURE 8.2
Amplitude of sideward force per unit length F on a cylinder of radius a due to the passage of a plane wave of pressure amplitude P_0, plotted as a function of $2\pi a/\lambda$.

in connection with the so-called velocity-ribbon microphone, which consists of a light metal strip more or less open to the atmosphere, pushed to and fro by the sound wave. The ribbon is in a transverse magnetic field, so that the motion induces an emf along the ribbon, which actuates an amplifier. The net force on the strip is, of course, not exactly the same function of η as that given in Eq. (8.1.7) for the cylinder, but the behavior will be the same, in general. The force on the strip will be proportional to the frequency for small frequencies, but this linear dependence will break down when the wavelength becomes smaller than twice the width of the strip.

There is an approximate method of finding the net force on the cylinder which gives the correct result for wavelengths longer than the circumference of the cylinder. If the pressure in the plane wave is $P_0 e^{ik(x-ct)}$, the pressure at the surface of the cylinder due to the plane wave is $P_0 e^{i\eta \cos\phi - 2\pi i \nu t}$. If η is small, this can be expanded into $p_p \simeq P_0(1 + i\eta \cos\phi)e^{-2\pi i \nu t}$. This is the pressure due to the plane wave; there is also a scattered wave, enough of a wave to make the radial velocity at the surface come out to zero. This

scattered wave contributes a term $i\eta P_0 \cos\phi e^{-2\pi i\nu t}$ to this approximation to the pressure, so that the net pressure is that given in Eq. (8.1.6), and the net force is the limiting value given in (8.1.7). We thus see that even for very long waves the distortion of the plane wave due to the presence of the cylinder contributes a factor 2 to the net force on the cylinder.

Scattering from an inhomogeneity

The two-dimensional example just concluded has illustrated a number of general characteristics of scattered sound; they can, however, be more generally displayed in terms of the Green's function. A sound wave is scattered, not only by a solid object, but also by a region in which the acoustic properties of the medium, such as its density or compressibility, differ from their values in the rest of the medium. Turbulent air scatters, as well as generates, sound. A rough patch on a plane wall scatters, as well as reflects, sound. Fog particles in air, and bubbles in water, scatter sound.

When an object or region scatters sound, some of the energy carried by the incident wave is dispersed. The energy lost to the incident wave may be absorbed by the scatterer, or it may simply be deflected from its original course. The amount of energy lost per second by an incident plane wave, divided by the incident wave's intensity (energy per second per unit area), is called the *cross section* Σ of the object or region. It is the cross-sectional area of that portion of the incident wave which is removed by the scatterer. If the energy is absorbed by the object, the ratio is called the absorption cross section; if the sound is simply deflected, so it goes to infinity in a different direction, it is called the scattering cross section; the sum is the total cross section. In any case, the incident plane wave is reduced in intensity because of the loss. If there are, on the average, N equivalent scatterers per unit volume of the medium, then, as the plane wave progresses, each unit volume removes a fraction $N\Sigma$ of the wave's intensity. In other words, the intensity is attenuated by the fraction $N\Sigma$ per unit length; thus the intensity of a plane wave traversing a scatterer-filled medium is $I_0 e^{-N\Sigma x}$, where x is the distance of penetration.

If the scattering object is a region in which the medium differs in acoustic properties from the rest of the medium, the wave solution can be written in terms of the volume integral of Eq. (7.1.17). For example, suppose, in the region R, which can be inscribed in a sphere of radius a at the origin, the density of the fluid is ρ_e and its compressibility is $\kappa_e = 1/\rho_e c_e^2$ (so that the wave velocity in R is c_e), whereas these quantities have values ρ and $\kappa = 1/\rho c^2$ throughout the rest of the medium.

To obtain the appropriate equations of motion from the general equations (6.2.6) and (6.2.11) or (6.4.1) to (6.4.22), in the present case, we must distinguish between the variations of pressure and density caused by the sound wave and the variations of density and compressibility inherent in the inhomogeneous nature of the medium. If these inhomogeneities are caused

by "foreign bodies" in a fluid of density ρ and compressibility κ, then, in the absence of a sound wave, the density, as a function of $\mathbf{r}$ and t, will be $\rho + \delta_e(\mathbf{r},t)$, where δ_e is zero except inside one of the "foreign bodies." These bodies move as wholes. They may be fog droplets in air or bubbles in water, and in the meaning of Eq. (6.1.9), the total time derivative of $\rho + \delta_e$ is zero.

$$0 = \frac{d}{dt}(\rho + \delta_e) = \frac{\partial \delta_e}{\partial t} + \mathbf{u} \cdot \text{grad } \delta_e \qquad \rho \text{ is constant}$$

Now we add a sound wave, with its velocity $\mathbf{u}$, its pressure p, and its additional density change δ, related by the usual first-order equation $\delta = (\rho + \delta_e)\kappa_e p$, where κ_e is equal to the constant value κ in the homogeneous part of the fluid but is different from κ in each inhomogeneous region. The equation of continuity for density (6.2.6) thus becomes

$$\frac{\partial}{\partial t}(\rho + \delta_e + \delta) \equiv \frac{\partial}{\partial t}(\delta_e + \delta) = -\text{div } [(\rho + \delta_e + \delta)\mathbf{u}]$$
$$\simeq -(\rho + \delta_e) \text{ div } \mathbf{u} - \mathbf{u} \cdot \text{grad } \delta_e$$

where we have neglected the two terms involving both δ and $\mathbf{u}$ as being much smaller than the others. Applying the previous two equations, we eventually obtain

$$\rho_e \kappa_e \frac{\partial p}{\partial t} = -\rho_e \text{ div } \mathbf{u} \qquad \text{or} \qquad \text{div } \mathbf{u} = -\kappa_e \frac{\partial p}{\partial t} \tag{8.1.8}$$

where $\rho_e = \rho + \delta_e$ is the fluid density, equal to the constant value ρ outside the foreign bodies; κ_e likewise is κ outside, in the homogeneous region.

The argument regarding the force equation (6.2.11) in the absence of turbulence is rather similar, and if we neglect viscosity and heat conduction, the result is

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_e} \text{grad } p \tag{8.1.9}$$

The combination of these two equations, eliminating $\mathbf{u}$, is

$$\text{div}\left(\frac{1}{\rho_e} \text{grad } p\right) - \kappa_e \frac{\partial^2 p}{\partial t^2} = 0 \tag{8.1.10}$$

If turbulence is present, so that the fluid velocity is the sum of $\mathbf{U}$, the turbulent velocity, and $\mathbf{u}$, the small acoustic velocity, then the right-hand side of the equation, instead of being zero, is equal to $(1/\rho)\nabla \cdot \mathfrak{J} \cdot \nabla$, where $\mathfrak{J}$ is the kinetic tensor of Eq. (6.2.10). The velocity components entering the elements of this tensor are of two kinds: the components of the acoustic velocity $\mathbf{u}$ of Eq. (8.1.8), which are usually quite small compared with c, the velocity of sound, and the components of the turbulent velocity $\mathbf{U}$, which may not be small compared with c. Thus in the region where $\mathbf{U}$ differs from zero, the

kinetic term to insert on the right in Eq. (8.1.10) is

$$\frac{1}{\rho}\nabla\cdot\mathfrak{J}\cdot\nabla \simeq \frac{1}{\rho}\left[\frac{\partial^2}{\partial x^2}U_x{}^2 + 2\frac{\partial^2}{\partial x\,\partial y}U_xU_y + \cdots\right] + \frac{2}{\rho}\left[\frac{\partial^2}{\partial x^2}R_{xx} + \frac{\partial^2}{\partial x\,\partial y}(R_{xy}+R_{yx}) + \cdots\right]$$

$$R_{xx} = \rho u_x U_x \qquad R_{xy} = \tfrac{1}{2}\rho(u_xU_y + u_yU_x) \qquad \cdots$$

where we have omitted the very small terms u_xu_y, etc.

The U_iU_j terms in $\mathfrak{J}$ are independent of the acoustic wave; if they vary with time, they represent sources of sound, of the sort discussed in Chap. 7. We are interested, at present, in the scattering of sound, coming from some other source, by the inhomogeneities in R, not by the generation of sound within R. Therefore we drop the terms U_iU_j from our discussion, assuming their effects have been computed by the methods of Chap. 7. The portion of the $\nabla\cdot\mathfrak{J}\cdot\nabla$ term which is of importance here is that arising from the cross products, which may be written $(2/\rho)\boldsymbol{\nabla}\cdot\mathfrak{R}\cdot\boldsymbol{\nabla}$, with $\mathfrak{R}$ having elements R_{xy}, etc., as already defined. See Sec. 11.4 for further discussion.

Thus the equation for the acoustic pressure, either inside the region R or outside, where ρ_e and κ_e have the constant values ρ and κ, can be written in the form

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}\gamma_\kappa(\mathbf{r},t) + \operatorname{div}\left[\gamma_\rho(\mathbf{r},t)\operatorname{grad} p\right] - 2\boldsymbol{\nabla}\cdot\mathfrak{R}\cdot\boldsymbol{\nabla} \quad (8.1.11)$$

where

$$\gamma_\kappa = \begin{cases}\dfrac{\kappa_e-\kappa}{\kappa} & \text{inside } R\\ 0 & \text{outside } R\end{cases} \qquad \gamma_\rho = \begin{cases}\dfrac{\rho_e-\rho}{\rho_e} & \text{inside } R\\ 0 & \text{outside } R\end{cases}$$

and where $c^2 = 1/\rho\kappa$. We have added the terms $(1/\rho)\nabla^2 p - \rho\kappa(\partial^2 p/\partial t^2)$ on both sides of Eq. (8.1.10), and then multiplied by ρ to obtain it in a form appropriate for changing into an integral equation. We have omitted the source terms involving U_iU_j for the reasons given two paragraphs above.

Equation (8.1.11) is homogeneous in the unknown quantity p, though we have put the terms representing the effect of R on the right, as though they were some sort of source term. In a sense they are source terms, for they represent sources of scattered sound, produced by the interaction between the sound wave p and the nonuniformities in R; but they do not represent any new energy being introduced into the sound field, as would terms on the right-hand side which were independent of p, discussed in Chap. 7.

If the irregularities in R are stationary in time, as they would be if R were a "foreign particle" in the fluid, such as a fog droplet in the air or a bubble in water (in these cases $\mathfrak{R}$ is usually zero), we can profitably consider

the case where the acoustic motion has a single frequency $\omega/2\pi$ and use the time-independent equation for the pressure amplitude,

$$\nabla^2 p_\omega + k^2 p_\omega = -k^2 \gamma_\kappa(\mathbf{r}) p_\omega - \operatorname{div}\left[\gamma_\rho(\mathbf{r}) \operatorname{grad} p_\omega\right] \qquad \text{where } k = \frac{\omega}{c} \tag{8.1.12}$$

From this equation we can obtain steady-state single-frequency solutions and transient solutions, if needed, by the usual Fourier transform procedures.

The integral equation

As with radiation problems, we shall find it much more useful to deal with an integral equation for the pressure wave, rather than with a differential equation plus boundary conditions. To obtain the integral equation we return to Eq. (7.1.17), giving p_ω in terms of a volume integral of $f_\omega G_\omega$ plus a surface integral of p_ω and G_ω. For the present case, with an unbounded medium, the Green's function is $g_\omega = e^{ikr}/4\pi r$. We cannot neglect the surface integral over the sphere at infinity in the present case, for we are now feeding energy into the medium, from infinity, via the incident wave. In fact, it turns out that the surface integral of Eq. (7.1.17), over the sphere at infinity, just produces $p_i(\mathbf{r})$, the incident wave, which comes from infinity and is scattered by region R.

Thus the integral equation for the scattering of an incident wave $p_i(\mathbf{r})$ by a single region R of different ρ and κ is

$$\begin{aligned} p_\omega(\mathbf{r}) &= p_i(\mathbf{r}) + \iiint_R \{k^2 \gamma_\kappa(\mathbf{r}_0) p_\omega(\mathbf{r}_0) - \operatorname{div}\left[\gamma_0(\mathbf{r}_0) \nabla_0 p_\omega(\mathbf{r}_0)\right]\} g_\omega(\mathbf{r} \mid \mathbf{r}_0)\, dv_0 \\ &= p_i(\mathbf{r}) + \iiint_R (k^2 \gamma_\kappa p_\omega g_\omega + \gamma_\rho \nabla_0 p_\omega \cdot \nabla_0 g_\omega)\, dv_0 \end{aligned} \tag{8.1.13}$$

where

$$g_\omega(\mathbf{r} \mid \mathbf{r}_0) = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}_0|} \exp\left(ik\,|\mathbf{r} - \mathbf{r}_0|\right)$$

To obtain the second form from the first we have integrated by parts, as we did with Eq. (7.1.25). In fact, our discussion of this integral equation can go very much as it did for Eq. (7.1.25). The integral here represents the scattered wave produced by the interaction between the incident wave and the irregularities of ρ and κ inside R. The first term comes from the fact that R has not the same compressibility as the surrounding medium, which results in a monopole source distribution of source density $(ik/\rho c)\gamma_\kappa(\mathbf{r}) p_\omega(\mathbf{r})$. The second term gives rise to dipole scattered wavelets from each dv_0; because of the difference in density, region R does not move in response to the force grad p_ω with the same velocity as does the surrounding medium; this produces a dipole source density $(1/ik\rho c)\gamma_\rho(\mathbf{r})$ grad $p_\omega(\mathbf{r})$. Note that these sources depend on p_ω, the actual pressure field inside R, not on the incident pressure p_i.

This is an inhomogeneous integral equation; instead of the volume term being the inhomogeneous term, as it was in the case of radiation, the incident wave $p_i(\mathbf{r})$ (a specified function) is the inhomogeneous term. We have transformed a differential equation, which is homogeneous because it represents the fact that no energy is being added mechanically in the finite part of space, into an integral equation, which is inhomogeneous because energy is being added from infinity, via the incident wave p_i.

The scattered wave

At large distances from the scattering region R ($r \gg a$, the radius of the sphere enclosing R), the Green's function takes on the simple form

$$g_\omega(\mathbf{r} \mid \mathbf{r}_0) \to \frac{1}{4\pi r} \exp(ikr - i\mathbf{k}_s \cdot \mathbf{r}_0)$$

where vector $\mathbf{k}_s$ has magnitude k, and is pointed in the direction of the radius vector $\mathbf{r}$ from the origin, inside R, to the point of observation. If, for example, the incident wave were a plane wave of amplitude A and wavenumber $k = \omega/c$, the exact form for the asymptotic behavior of the pressure wave would be

$$p_\omega(\mathbf{r}) \to Ae^{i\mathbf{k}_i\cdot\mathbf{r}} + A\left(\frac{e^{ikr}}{r}\right)\Phi_e(\mathbf{k}_s) \tag{8.1.14}$$

$$\Phi_s(\mathbf{k}_s) = \frac{k^2}{4\pi A}\iiint_R \left[\gamma_\kappa(\mathbf{r}_0)p_\omega(\mathbf{r}_0) - i\gamma_\rho(\mathbf{r}_0)\frac{\mathbf{a}_r}{k}\cdot\nabla_0 p_\omega\right]\exp(-i\mathbf{k}_s\cdot\mathbf{r}_0)\,dv_0$$

where $\mathbf{a}_r = \mathbf{k}_s/k$ is the unit vector in the direction of the observer. The second term for p_ω is, of course, a scattered wave, radiating outward from the region R, with an amplitude depending on vector $\mathbf{k}_s$, which points in the direction of the observer. Thus, if we knew the exact form of the sound field p_ω in R, we could compute the exact form of the scattered wave at large distances.

Function $\Phi_s(\mathbf{k}_s)$ is called the *angle-distribution factor* for the scattered wave, or sometimes the *scattered amplitude*. Its square $|\Phi_s|^2$, divided by $\rho c r^2$, is the intensity of the scattered wave at the observation point $\mathbf{r}$ ($r \gg a$). Ordinarily, the scattered wave must be measured a considerable distance from the scatterer in order to separate the scattered wave from the incident wave; so the function Φ_s is usually the goal of our calculations. An important point is that Φ_s is the three-dimensional Fourier transform of the function

$$\begin{aligned}\varphi_s(\mathbf{r}) &= \frac{2\pi^2k^2}{A}\left[\gamma_\kappa(\mathbf{r})p_\omega(\mathbf{r}) - i\gamma_\rho(\mathbf{r})(\mathbf{a}_r/k)\cdot\nabla p_\omega(\mathbf{r})\right] \\ &= \iiint \Phi_s(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{r}}\,dv_K \\ \Phi_s(\mathbf{K}) &= \frac{1}{8\pi^3}\iiint \varphi_s(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\,dv_r \\ dv_K &= dK_x\,dK_y\,dK_z \qquad dv_r = dx\,dy\,dz\end{aligned} \tag{8.1.15}$$

for the special value $\mathbf{K} = \mathbf{k}_s$. This property of the scattered amplitude will be of considerable utility in our analyses.

A few other general properties of Φ_s, resulting from the Fourier transform relationship, merit exposition here. For example, we can generalize the convolution integral of Prob. 23 of Chap. 1 to obtain an alternative form for Φ_s. If we define the spatial Fourier transforms of p_ω and γ as

$$P(\mathbf{K}) = \frac{1}{8\pi^3}\iiint p_\omega(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\,dv_r \qquad \Gamma(\mathbf{K}) = \frac{1}{8\pi^3}\iiint \gamma(\mathbf{r})e^{-i\mathbf{K}\cdot\mathbf{r}}\,dv_r$$

and if we remember that the Fourier transform of grad p_ω is $i\mathbf{K}P(\mathbf{K})$, then the angle-distribution factor can be written

$$\Phi_s(\mathbf{k}_s) = \frac{2\pi^2k^2}{A}\iiint P(\mathbf{K})\left[\Gamma_\kappa(\mathbf{k}_s - \mathbf{K}) + \frac{\mathbf{K}\cdot\mathbf{a}_r}{k}\,\Gamma_\rho(\mathbf{k}_s - \mathbf{K})\right] dv_K \quad (8.1.16)$$

the convolution in this case being in wavenumber space rather than in distance space. If, for example, in the region R, p_ω resembles a traveling wave $A\exp(i\mathbf{k}_i\cdot\mathbf{r})$, its Fourier transform will have a sharp peak at $\mathbf{K} = \mathbf{k}_i$, resembling the transform of $A\exp(i\mathbf{k}_i\cdot\mathbf{r})$, which is $A\,\delta(\mathbf{K} - \mathbf{k}_i)$. In this case $\Phi_s(\mathbf{k}_s)$ would be approximately equal to

$$(2\pi^2k^2/A)[\Gamma_\kappa(\mathbf{k}_s - \mathbf{k}_i) - \Gamma_\rho(\mathbf{k}_s - \mathbf{k}_i)\cos\vartheta]$$

where ϑ is the angle between $\mathbf{k}_s$ and $\mathbf{k}_i$. The dependence on ϑ comes explicitly on the cosine factor, but also implicitly, because the vector $\mathbf{k}_s - \mathbf{k}_i$ depends, both in magnitude and direction, on ϑ, the angle of scattering.

The intensity of the scattered wave, at large distances from R, is given by a double integral which can be transformed into the Fourier transform of an autocorrelation function.

$$\begin{aligned} I_s &= \frac{k^4}{16\pi^2\rho cr^2}\iiint_R f_\omega(\mathbf{r}_0,\mathbf{a}_r)e^{-i\mathbf{k}_s\cdot\mathbf{r}_0}\,dv_0\iiint_R f_\omega^*(\mathbf{r}_1,\mathbf{a}_r)e^{i\mathbf{k}_s\cdot\mathbf{r}_1}\,dv_1 \\ &= \frac{k^4V}{16\pi^2\rho cr^2}\iiint_R e^{-i\mathbf{k}_s\cdot\mathbf{l}}\Upsilon_\omega(\mathbf{l})\,dv_l \end{aligned} \quad (8.1.17)$$

where

$$f_\omega(\mathbf{r},\mathbf{a}_r) = \gamma_\kappa(\mathbf{r})p_\omega(\mathbf{r}) - i\gamma_\rho(\mathbf{r})\frac{\mathbf{a}_r}{k}\cdot\nabla p_\omega(\mathbf{r})$$

$$\Upsilon_\omega(\mathbf{l}) = \frac{1}{V}\iiint_R f_\omega(\mathbf{r},\mathbf{a}_r)f_\omega^*(\mathbf{r}+\mathbf{l},\mathbf{a}_r)\,dv_r \qquad \mathbf{k}_s = k\mathbf{a}_r$$

V being the volume of the scattering region R, and $\mathbf{l} = \mathbf{r}_0 - \mathbf{r}_1$. We notice that Υ_ω is zero when $l > 2a$, twice the radius of the sphere enclosing R. Thus, when $ka \ll 1$, the scattered intensity depends only on the volume integral of Υ_ω, and its directionality depends only on the directionality of f_ω.

The Born approximation

We have seen that, once the exact form of the pressure distribution p_ω is determined inside the scattering region R, then all the properties of the sound field outside R can be computed, including the asymptotic behavior of the scattered wave. The integral equation (8.1.13) can occasionally be solved exactly; a few examples will be given in the next section. It may be solved approximately by variational methods; an example will be given in the problems. Or it may be solved by successive approximations. Suppose we take, for our first approximation, the field produced when the unknown p_ω in the integral of Eq. (8.1.13) is replaced by the known function p_i, the incident wave.

$$\begin{aligned} p_\omega(\mathbf{r}) &\simeq p_i(\mathbf{r}) + p_1(\mathbf{r}) \\ p_1(\mathbf{r}) &= \iiint_R [k^2\gamma_\kappa(\mathbf{r}_0)p_i(\mathbf{r}_0)g_\omega(\mathbf{r} \mid \mathbf{r}_0) + \gamma_\rho(\mathbf{r}_0)\,\nabla_0 p_i \cdot \nabla_0 g_\omega]\,dv_0 \end{aligned} \tag{8.1.18}$$

If $p_1(\mathbf{r})$ turns out to be considerably smaller than $p_i(\mathbf{r})$ throughout R, then this approximation, called the *Born approximation* [Eq. (4.5.16)], will give us a satisfactory representation of the behavior of the scattered sound. If this is not the case, we might add higher terms in the approximation series.

$$\begin{aligned} p_\omega(\mathbf{r}) &= p_i(\mathbf{r}) + p_1(\mathbf{r}) + p_2(\mathbf{r}) + p_3(\mathbf{r}) + \cdots \\ p_{n+1}(\mathbf{r}) &= \iiint_R [k^2\gamma_\kappa(\mathbf{r}_0)p_n(\mathbf{r}_0)g_\omega(\mathbf{r} \mid \mathbf{r}_0) + \gamma_\rho(\mathbf{r}_0)\,\nabla_0 p_n \cdot \nabla_0 g_\omega(\mathbf{r} \mid \mathbf{r}_0)]\,dv_0 \end{aligned} \tag{8.1.19}$$

For the sorts of functions γ we encounter in acoustics, this series converges, but it usually converges slowly unless the γ's are quite small (in which case p_1 is sufficient). Thus, when the Born approximation is inadequate, we turn to variational methods, or else to exact solutions, when these are possible.

When the γ's are small and the Born approximation is valid, the calculation of the scattered wave is fairly simple. For example, when the incident wave is a plane wave, $p_i(\mathbf{r}) = A \exp(i\mathbf{k}_i \cdot \mathbf{r})$, the angle-distribution function is nearly equal to

$$\begin{aligned} \Phi_b(\boldsymbol{\mu}) &= \frac{k^2}{4\pi} \iiint_R [\gamma_\kappa(\mathbf{r}_0) + \gamma_\rho(\mathbf{r}_0)\cos\vartheta] \exp(i\boldsymbol{\mu} \cdot \mathbf{r}_0)\,dv_0 \\ &= 2\pi^2 k^2[\Gamma_\kappa(\boldsymbol{\mu}) + \Gamma_\rho(\boldsymbol{\mu})\cos\vartheta] \qquad \boldsymbol{\mu} = \mathbf{k}_s - \mathbf{k}_i \end{aligned} \tag{8.1.20}$$

where $\cos\vartheta = \mathbf{k}_i \cdot \mathbf{k}_s/k^2$ is the cosine of the angle between $\mathbf{k}_s$ and $\mathbf{k}_i$ (i.e., the angle of scattering between the direction of the incident plane wave and the radius vector $\mathbf{r}$ to the observer), and the Γ's are the spatial Fourier transforms of the γ's, as defined in Eq. (8.1.16).

If R is a sphere of radius a and both the γ's depend only on the radius r_0, then the angle coordinates of dv_0 can be integrated over and

$$\Phi_b(\boldsymbol{\mu}) = \frac{k^2}{\mu}\int_0^a [\gamma_\kappa(r_0) + \gamma_\rho(r_0)\cos\vartheta]\sin(\mu r_0) r_0\, dr_0$$
$$\to \tfrac{1}{3}k^2a^3\left(\left\langle\frac{\kappa_e - \kappa}{\kappa}\right\rangle + \left\langle\frac{\rho_e - \rho}{\rho_e}\right\rangle\cos\vartheta\right) \qquad \mu a \ll 1 \quad (8.1.21)$$

where $\mu = |\mathbf{k}_s - \mathbf{k}_i| = 2k\sin(\vartheta/2)$. Thus, at long wavelengths, the dipole wave, proportional to the mean value $\langle\gamma_\rho\rangle$ over R, has the same order of dependence on ka as the monopole wave, which is proportional to the mean value $\langle\gamma_\kappa\rangle$ over R. Other properties of the Born approximation to the scattered wave will be taken up in the problems.

Scattering from turbulence

Our second example of the use of the Born approximation is the case where the γ's and $\mathfrak{R}$ are small, but vary in a random manner with time and position within R. If the rms values are small, the approximate solution should be satisfactory. This time we shall not analyze the wave first into simple-harmonic components, but shall use the time-dependent Green's function of Eq. (7.1.18), with a source function

$$f(\mathbf{r},t) \simeq -\frac{1}{c^2}\gamma_\kappa(\mathbf{r},t)\frac{\partial^2 p_i}{\partial t^2} - \operatorname{div}[\gamma_\rho(\mathbf{r},t)\operatorname{grad} p_i(\mathbf{r},t)] + 2\boldsymbol{\nabla}\cdot\mathfrak{R}\cdot\boldsymbol{\nabla} \quad (8.1.22)$$

where the components of $\mathbf{u}$ in the elements of $\mathfrak{R}$ are set equal to the components of $\mathbf{u}_i$, related to p_i via the equation $-(\partial\mathbf{u}_i/\partial t) = (1/\rho)\operatorname{grad} p_i$.

Again we specialize by assuming that the incident wave is a plane wave, $p_i = A\exp(i\mathbf{k}_i\cdot\mathbf{r} - ik_ict)$, and since we are interested in the asymptotic form of the scattered wave, we use the asymptotic form of the Green's function,

$$g(\mathbf{r},t\,|\,\mathbf{r}_0,t_0) \to \frac{1}{4\pi r}\,\delta\left[t_0 - t + \frac{r}{c} - \frac{1}{c}(\mathbf{r}_0\cdot\mathbf{a}_r)\right]$$

where $\mathbf{a}_r$ is a unit vector pointed along $\mathbf{r}$. Thus $\mathbf{r}_0\cdot\mathbf{a}_r$ equals r_0 times the cosine of the angle between $\mathbf{r}_0$ and $\mathbf{r}$, and $k_i\mathbf{r}_0\cdot\mathbf{a}_r$ equals $\mathbf{k}_s\cdot\mathbf{r}_0$, where $\mathbf{k}_s = k_i\mathbf{a}_r$ as previously. Dealing first with the γ_κ term, we can write the integral for the asymptotic form of this part of the scattered wave, eliminating the delta function by integrating over t_0.

$$p_{s\kappa}(\mathbf{r},t) \to \frac{Ak_i^2}{4\pi r}\int dt_0\iiint dv_0\,\gamma_\kappa(\mathbf{r}_0,t_0)\,\delta\left(t - t_0 - \frac{r}{c} + \frac{1}{c}\mathbf{r}_0\cdot\mathbf{a}_r\right)$$
$$\times\exp[i\mathbf{k}_i\cdot\mathbf{r}_0 - ik_ict_0] \quad (8.1.23)$$
$$= \frac{Ak_i^2}{4\pi r}e^{ik_ir}\iiint dv_0\,\gamma_\kappa\left(\mathbf{r}_0, t - \frac{r}{c} + \frac{1}{c}\mathbf{r}_0\cdot\mathbf{a}_r\right)$$
$$\times\exp[i(\mathbf{k}_i - \mathbf{k}_s)\cdot\mathbf{r}_0 - ik_ict]$$

To demonstrate that this approximate formula for the scattered wave does indeed display a frequency spread, we now carry out a frequency analysis by taking the Fourier transform of $p_{s\kappa}$ for the frequency $\omega/2\pi$ (not necessarily equal to $k_i c/2\pi$).

$$P_{s\kappa}(\mathbf{r},\omega) \to \frac{Ak_i^2}{4\pi r} e^{ik_i r} \iiint dv_0 \int dt \gamma_\kappa \left(\mathbf{r}_0, t - \frac{r}{c} + \frac{1}{c}\mathbf{r}_0 \cdot \mathbf{a}_r\right) \times \exp\left[i(\mathbf{k}_i - \mathbf{k}_s)\cdot \mathbf{r}_0 + i(\omega - k_i c)t\right]$$

This is the four-dimensional Fourier transform of the fluctuating function $\gamma_\kappa(\mathbf{r},t)$.

$$\Gamma_\kappa(\mathbf{K},\omega_\gamma) = (2\pi)^{-4} \int dt_1 \iiint dv_1 \, \gamma_\kappa(\mathbf{r}_1,t_1) \exp\left(-i\mathbf{K}\cdot\mathbf{r}_1 + i\omega_\gamma t_1\right)$$

for a specific value of $\mathbf{K}$ and ω_κ, multiplied by a phase factor, which arises because the proper integration variable is not t but $t_1 = t - (r/c) + (1/c)\mathbf{r}_0 \cdot \mathbf{a}_r$. Carrying out the integrals in turn, we obtain

$$P_{s\kappa}(\mathbf{r},\omega) \to \frac{Ak_i^2}{8\pi^3 r} e^{ik_i r} \int dt_1 \iiint dv_0 \, \gamma_\kappa(\mathbf{r}_0,t_1) \times \exp\left[i(\mathbf{k}_i - \mathbf{k}_s)\cdot\mathbf{r}_0 + i(\omega - k_i c)\left(t_1 + \frac{r}{c} - \frac{1}{c}\mathbf{r}_0\cdot\mathbf{a}_r\right)\right]$$

$$= \frac{2\pi^2 A}{r} k_i^2 e^{i\omega r/c} \Gamma_\kappa\left(\frac{\omega}{c}\mathbf{a}_r - \mathbf{k}_i, \omega - k_i c\right) \qquad (8.1.24)$$

Because of the random motion within R, the transform of γ is not a delta function $\delta(\omega - k_i c)$ exactly at the frequency $k_i c/2\pi$ of the incident wave, but has a spread in frequency roughly equal to the reciprocal of the correlation time interval $2\pi\tau_c = 2\pi/\omega_c$. Thus the scattered wave can have frequencies $\omega/2\pi$ and wavenumber ω/c which differ from the incident values $k_i c/2\pi$ and k_i. The transform Γ_κ also has a broader peak at $(\omega/c)\mathbf{a}_r \simeq k_i$, which produces a dependence on the angle between $\mathbf{a}_r$ and $\mathbf{k}_i$ in the scattered wave.

The other parts of the scattered wave may be computed by similar procedures, though the calculation is further complicated because of the divergence operators, which give difficulty when integrated with a delta function. The safe way is to integrate by parts, obtaining derivatives of the delta function, such as $\delta'(x)$, which has the property that $\int f(x)\delta'(x-a)\,dx = f'(a)$, where the prime indicates differentiation with respect to the argument. One then can integrate over t_0 and eventually obtain for the ω component of the complete scattered wave

$$p_s(\mathbf{r},\omega) \to \frac{2\pi^2 A}{r} e^{i\omega r/c} \left\{k_i^2 \Gamma_\kappa(\boldsymbol{\mu}_s,\Delta_s) - \cos\vartheta \left[k_i \frac{\omega}{c} \Gamma_\rho(\boldsymbol{\mu}_s,\Delta_s) - 2\left(\frac{\omega}{c}\right)^2 \Omega_r(\boldsymbol{\mu}_s,\Delta_s)\right]\right\} \qquad (8.1.25)$$

where

$$\boldsymbol{\mu}_s = \frac{\omega}{c}\,\mathbf{a}_r - \mathbf{k}_i \qquad \Delta_s = \omega - k_i c$$

and ϑ is the angle between $\mathbf{a}_r$ and $\mathbf{k}_i$ (the scattering angle). Functions Γ_ρ and Ω_r are the Fourier transforms, respectively, of γ_ρ and of $1/c$ times the component U_r, along $\mathbf{r}$, of the turbulent velocity $\mathbf{U}$. The details of these calculations will be taken up in the problems.

Because of the random motion, the phases of Γ_κ, Γ_ρ, and Ω_r are random functions of their arguments. Since, usually, γ_κ, γ_ρ, and U_r/c vary independently, when we square p_s to obtain the intensity of the scattered wave, the cross terms average out, and the intensity involves just the sum of the squares of Γ_κ, Γ_ρ, and Ω_r separately. Again we go into the details in regard to Γ_κ here, leaving the rest of the calculation for a problem. Thus the spectrum level of the sound scattered by γ_κ, analyzed over the time interval at T, is

$$I_\kappa(\mathbf{r}) \to \frac{4\pi^4 |A|^2 k_i^4}{\rho c r^2} \left| \Gamma_\kappa\left(\frac{\omega}{c}\,\mathbf{a}_r - \mathbf{k}_i,\ \omega - k_i c\right)\right|^2 \tag{8.1.26}$$

where $|\Gamma_\kappa(\boldsymbol{\mu}_s,\Delta_s)|^2$ is the spatial-temporal spectrum density, for wavenumber vector $\boldsymbol{\mu}_s$ and frequency difference $\Delta_s/2\pi$, of the inhomogeneity factor $\gamma_\kappa(\mathbf{r},t)$ in R. If the fluctuations in R are isotropic, $|\Gamma_\kappa|^2$ will depend on the magnitude of $\boldsymbol{\mu}_s$, not its direction, and if its fluctuations have equal contributions for frequencies up to $\omega_c/2\pi$ and from wavenumbers of magnitudes up to $1/l_c$ (i.e., if the correlation time is $1/\omega_c$ and the correlation length is l_c), then $|\Gamma_\kappa(\mathbf{K},\omega_\gamma)|^2$ will be maximum at $\boldsymbol{\mu}_s = 0$ and $\Delta_s = 0$ and will drop to zero when $|\boldsymbol{\mu}_s|$ is larger than $1/l_c$, when $|\Delta_s|$ is larger than ω_c.

Therefore Eq. (8.1.26) shows that the greatest amount of this scattered intensity, measured at $\mathbf{r}$, is for a frequency equal to $k_i c/2\pi = \omega_i/2\pi$ and for a direction $\mathbf{a}_r$ for which $(\omega/c)\mathbf{a}_r = \mathbf{k}_i$; it drops to zero for frequencies such that $|\omega - \omega_i|/2\pi$ is greater than $\omega_c/2\pi$ and when $|(\omega/c)\mathbf{a}_r - \mathbf{k}_i|$ is greater than $1/l_c$. To make these points a little clearer, we work out the case for which the autocorrelation function for γ_κ has the simple form

$$\Upsilon_\kappa(\mathbf{l},\tau) \equiv \frac{1}{VT}\int_{-T}^{0} dt \iiint_R \gamma_\kappa(\mathbf{r},t)\gamma(\mathbf{r}+\mathbf{l},\, t+\tau)\, dv_r$$

$$= \langle \gamma_\kappa{}^2 \rangle \exp\left[-\frac{1}{2}\left(\frac{l}{l_c}\right)^2 - \tfrac{1}{2}\omega_c{}^2\tau^2 \right]$$

where $\langle \gamma_\kappa{}^2 \rangle$ is the mean square of γ_κ, averaged over region R and over time interval T. Generalization of Eq. (2.3.13) indicates that an autocorrelation function of this form will have a spectrum density

$$|\Gamma_\kappa(\mathbf{u},\Delta)|^2 = \frac{VT_c{}^3 l}{(2\pi)^6 \omega_c} \langle \gamma_\kappa{}^2 \rangle \exp\left[-\tfrac{1}{2}\mu^2 l_c{}^2 - \frac{1}{2}\left(\frac{\Delta}{\omega_c}\right)^2 \right]$$

Therefore the intensity of the sound scattered at frequency $\omega/2\pi$ from a

plane wave, of amplitude A and frequency $\omega_i/2\pi$, going in the direction $\mathbf{k}_i$, by the region R with irregularity γ_κ characterized above, is

$$I_\kappa(\mathbf{r}) \simeq \frac{|A|^2 k_i^4 V T l_c^3}{16\pi^2 \rho c \omega_c r^2} \langle \gamma_\kappa^2 \rangle \exp\left[-\tfrac{1}{2} l_c^2 \left| \frac{\omega}{c} \mathbf{a}_r - \mathbf{k}_i \right|^2 - \frac{(\omega - k_i c)^2}{2\omega_c^2} \right] \quad (8.1.27)$$

which exhibits the dependence on frequency and on the angle between the direction $\mathbf{k}_i$ of the incident wave and the direction $\mathbf{a}_r$ to the observer.

Finally, Eq. (1.3.17) shows that the total intensity at $\mathbf{r}$, averaged over a time interval T, is the same as the integral of $I_\kappa(\mathbf{r})$ over ω.

$$I_{\kappa t}(\vartheta) \equiv \frac{1}{T}\int I_t(t)\, dt = \frac{2\pi}{T}\int I_\kappa\, d\omega \simeq \frac{|A|^2 V k_i^4}{\sqrt{32\pi}\, \rho c r^2} \frac{l_c^3}{\sqrt{1 + k_c^2 l_c^2}} \langle \gamma_\kappa^2 \rangle$$
$$\times \exp\left[\frac{-k_i^2 l_c^2}{2(k_c^2 l_c^2 + 1)} (4 \sin^2 \tfrac{1}{2}\vartheta + k_c^2 l_c^2 \sin^2 \vartheta) \right] \quad (8.1.28)$$

where $k_c = \omega_c/c$, and ϑ is the angle of scattering, between $\mathbf{a}_r$ and $\mathbf{k}_i$. Integration of this, times $2\pi r^2 \sin\vartheta\, d\vartheta$ over ϑ from 0 to π, would equal the total power scattered by the inhomogeneous term γ_λ. Again we see that the scattered intensity falls off as the scattering angle increases. For high frequencies (k_i large) it falls off rapidly, particularly if $k_c l_c = l_c/c\tau_c$ is large. When the correlation length l_c is small compared with c times the correlation time τ_c, and the scattered peak in the k_i direction is broader and at lower frequencies ($k_i l_c \ll 1$), the scattered intensity is nearly isotropic.

More calculation of the same sort indicates that the total spectrum density of sound scattered by random fluctuations in compressibility, density, and turbulent velocity is

$$I_\omega(r) \to \frac{|A|^2 V T l_c^3}{16\pi^2 \rho c \omega_c r^2} \left[k_i^4 \langle \gamma_\kappa^2 \rangle + k_i^2 \left(\frac{\omega}{c}\right)^2 \langle \gamma_\rho^2 \rangle \cos^2\vartheta \right.$$
$$\left. + 4\left(\frac{\omega}{c}\right)^4 \langle M_t^2 \rangle \cos^2\vartheta \right] \exp\left[-\tfrac{1}{2}\mu_s^2 l_c^2 - \frac{1}{2}\left(\frac{\Delta_s}{\omega_c}\right)^2 \right] \quad (8.1.29)$$

where γ_κ and γ_ρ are defined in Eq. (8.1.11), and $M_t = U_r/c$, so that $\langle M_t^2 \rangle$ is the mean-square Mach number of the turbulent motion, independent of ϑ, if the turbulence is isotropic [see Eq. (11.4.15) for further discussion]. The quantities μ_s and Δ_s are given in Eq. (8.1.25). The total scattered intensity, for all wavelengths, is then

$$I_s(\mathbf{r}) \to \frac{|A|^2 V}{\sqrt{32\pi}\, \rho c l_c r^2} \frac{\alpha^4}{\sqrt{1 + \alpha^2}} \left\{ \eta^4 \langle \gamma_\kappa^2 \rangle + \eta^2 \langle \gamma_\rho^2 \rangle \frac{1 + \eta^2(1 + \alpha^2 \cos\vartheta)^2}{1 + \alpha^2} \right.$$
$$\left. + 12\langle M_t^2 \rangle \frac{1 + 2\eta^2(1 + \alpha^2\cos\vartheta)^2 + \tfrac{1}{3}\eta^4(1 + \alpha^2\cos\vartheta)^4}{(1 + \alpha^2)^2} \right\}$$
$$\times \exp\left[\frac{-\eta^2\alpha^2}{2(1 + \alpha^2)} (4 \sin^2 \tfrac{1}{2}\vartheta + \alpha^2 \sin^2\vartheta) \right] \quad (8.1.30)$$

where $\eta = k_i/k_c = \omega_i/\omega_c$, and $\alpha = k_c l_c$. We have assumed that although γ_κ, γ_ρ, and U_r/c fluctuate independently, their correlation lengths and times are roughly equal.

Ordinarily, in a turbulent region, the fluctuations of κ and of ρ are proportional to M_t^2, so that if the turbulent Mach number is small, the first

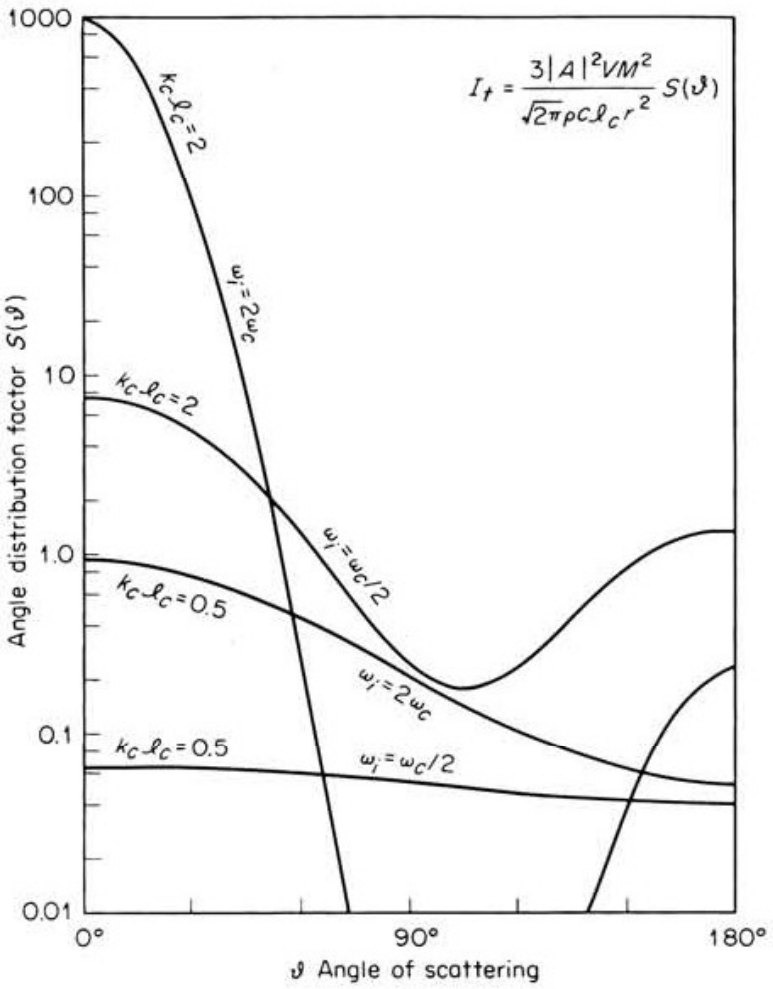


FIGURE 8.3
Angle distribution of total intensity scattered by isotropic turbulence of Mach number M.

two terms are small compared with the last one in the braces. The angle distribution for this last term, assuming γ_κ and γ_ρ are negligible, is plotted in Fig. 8.3, to display the very large variety in the directionality of the scattering from a turbulent region.

8.2 SCATTERING BY SPHERES

To obtain more accurate formulas for sound scattering we must specialize our calculations for objects of relatively simple shape. In the preceding section we worked out the scattering from circular cylinders; in this section

we take up the other simple shape, the sphere. This case is of considerable practical importance; many scattering objects are more or less spherical. Scattering from a few other shapes can be computed exactly, from disks and ellipsoids, for example. But these computations are much more complex and introduce no new principles; so we omit their discussion.

If the sphere is composed of material of quite different acoustical properties from those of the medium, it will turn out to be advantageous to fit boundary conditions at the surface of the sphere, rather than to use the volume integral of Eq. (8.1.13). First let us go through the simplest case, that of a rigid, motionless sphere of radius a, centered at the origin.

The rigid sphere

The analysis follows the same lines as that for the cylinder. The expression for a plane wave traveling to the right along the polar axis is

$$p_p = Ae^{ik(r\cos\vartheta - ct)} = A\sum_{m=0}^{\infty}(2m+1)i^m P_m(\cos\vartheta)j_m(kr)e^{-2\pi ivt} \quad (8.2.1)$$

where $A = \sqrt{\rho_0 cI}$, and the factors P_m and j_m are defined in Eqs. (7.2.4) and (7.2.11), respectively. The expression for the wave scattered from a sphere of radius a whose center is the polar origin is

$$p_s = -A\sum_{m=0}^{\infty}(2m+1)i^{m+1}e^{-i\delta_m}\sin\delta_m P_m(\cos\vartheta) \times [j_m(kr) + in_m(kr)]e^{-2\pi ivt} \quad (8.2.2)$$

where the angles δ_m have been defined in Eqs. (7.2.15) in connection with the radiation from a sphere. The values of some of them are given in Table VIII.

The intensity of the scattered wave and the total power scattered are

$$\begin{aligned} I_s &\simeq \frac{a^2 I}{r^2}\frac{1}{k^2a^2}\sum_{m,n=0}^{\infty}(2m+1)(2n+1) \\ &\qquad \times \sin\delta_m \sin\delta_n \cos(\delta_m - \delta_n)P_m(\cos\vartheta)P_n(\cos\vartheta) \\ &\simeq \begin{cases} \dfrac{16\pi^4\nu^4a^6I}{9c^4r^2}(1-3\cos\vartheta)^2 & ka \ll 1 \\ I\left[\dfrac{a^2}{4r^2} + \dfrac{a^2}{4r^2}\cot^2\left(\dfrac{\vartheta}{2}\right)J_1^2(ka\sin\vartheta)\right] & ka \gg 1 \end{cases} \end{aligned} \quad (8.2.3)$$

$$\Pi_s = 2\pi a^2 I\frac{2}{k^2a^2}\sum_{m=0}^{\infty}(2m+1)\sin^2\delta_m \simeq \begin{cases} \dfrac{256\pi^5a^6}{9\lambda^4}I & \lambda \gg 2\pi a \\ 2\pi a^2 I & \lambda \ll 2\pi a \end{cases}$$

The discussion concerning the short-wavelength limit for the scattering from a sphere is similar to that preceding and following Eq. (8.1.4) for the

cylindrical case. The total power scattered is that contained in an area of primary beam equal to twice the cross section πa^2 of the sphere. Half of this is reflected equally in all directions from the sphere (the first term in the last expression for I); and the other half is concentrated into a narrow beam which tends to interfere with the primary beam and cause the shadow

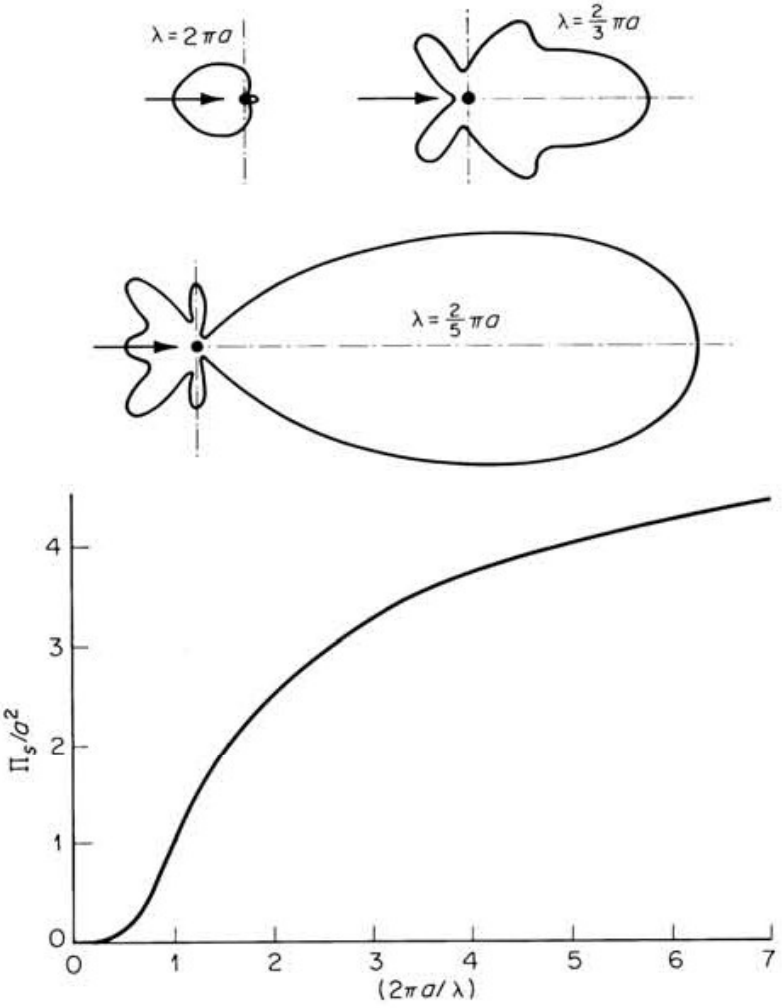


FIGURE 8.4
Distribution-in-angle of intensity scattered from a sphere of radius a and total power scattered Π, per unit incident intensity.

(the second term in the last expression). If the experimentally measured "total scattered power" includes everything from $\vartheta = \Delta$ to $\vartheta = \pi - \Delta$, Π_{exp} will equal Π_s for wavelengths longer than $4a/\Delta$ and will approach $\frac{1}{2}\Pi_s$ for wavelengths much shorter than $4a/\Delta$.

Figure 8.4 shows polar curves of the scattered intensity as a function of the angle of scattering ϑ, for different values of $\eta = ka$, and a curve Π_s as function of η. As with the cylinder, the directionality of the scattered wave increases as the frequency increases.

The force on the sphere

The total pressure at a point on the sphere an angle ϑ from the polar axis (note that the point $\vartheta = 0$ is the point farthest away from the source of the sound) turns out to be

$$p_a = Ae^{-i\omega t}\left(\frac{1}{ka}\right)^2 \sum_{m=0}^{\infty} \frac{2m+1}{B_m} P_m(\cos\vartheta) e^{-i(\delta_m - \frac{1}{2}\pi m)} \qquad (8.2.4)$$
$$\simeq (1 + \tfrac{3}{2}ika\cos\vartheta)Ae^{-i\omega t} \qquad ka \ll 1$$

As with the cylinder, this expression is proportional to that for the pressure, at large distances, due to a simple source set in the sphere. The curves of Fig. 7.1 therefore show the dependence of p_a^2 on ϑ.

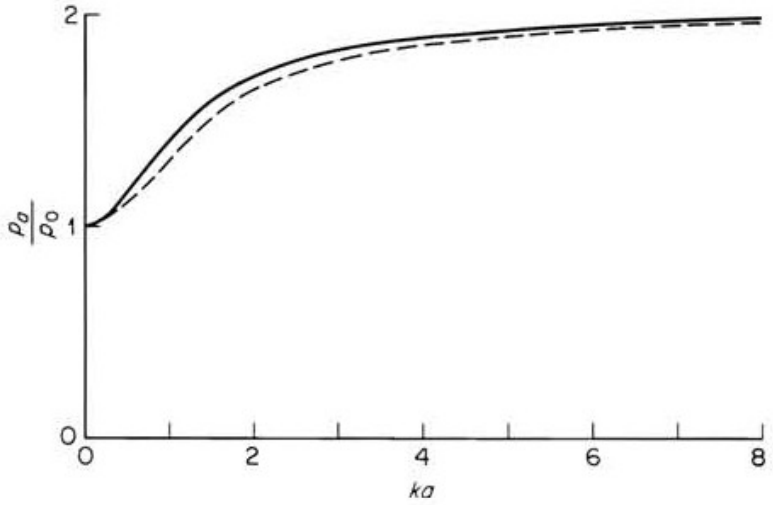


FIGURE 8.5
Ratio of pressure amplitude at a point on a sphere to pressure amplitude of the plane wave striking the sphere, plotted as function of $\eta = 2\pi a/\lambda$. Solid line is the pressure at a point facing the incident wave; dotted line is the average pressure for a circular area around this point with an angular radius of 30°.

The amplitude of the pressure at the point nearest the source of sound ($\vartheta = \pi$), for a plane wave of unit pressure amplitude, is plotted in Fig. 8.5 as a function of η. We notice that for wavelengths long compared with the circumference of the sphere, the pressure at $\vartheta = \pi$ equals the pressure of the plane wave, but that for shorter wavelengths the distortion of the wave due to the presence of the sphere makes p_a differ from A. This general fact will be true for obstacles of other than spherical shape, even though the dependence of p_a on η will be somewhat different from that given in Eq. (8.2.4). Therefore a microphone measures the pressure of the wave striking it only as long as its circumference is smaller than the wavelength of the wave. For smaller wavelengths a correction must be made for the distortion due to the presence of the microphone.

By using Eqs. (7.2.5) we can find the average value of the pressure on that portion of the surface of the sphere contained between the angles $\vartheta = \pi$ and $\vartheta = \pi - \vartheta_0$:

$$\begin{aligned} p_{\text{avg}} &= \frac{1}{1 - \cos\vartheta_0} \int_{\pi-\vartheta_0}^{\pi} p_a \sin\vartheta \, d\vartheta \\ &= Ae^{-i\omega t} \sum_{m=0}^{\infty} \frac{i^m e^{-i\delta m}}{k^2 a^2 B_m} \frac{P_{m-1}(\cos\vartheta_0) - P_{m+1}(\cos\vartheta_0)}{P_{-1}(\cos\vartheta_0) - P_1(\cos\vartheta_0)} \qquad (8.2.5) \\ &\simeq Ae^{-i\omega t}[1 + \tfrac{3}{4}ika(1 + \cos\vartheta_0)] \qquad ka \ll 1 \end{aligned}$$

where $P_{-1}(\cos\vartheta_0) = 1$. The dotted line in Fig. 8.5 gives the values of $p_{\text{avg}}/A_0 e^{-i\omega t}$ as function of $\eta = ka$ for the case $\vartheta_0 = 30°$.

Nonrigid sphere

To compute the scattering from a sphere which is not perfectly rigid, we shall find it convenient to use the full machinery of the Green's function integral. The fluid medium fills all space outside the surface of the sphere of radius a centered at the origin. In this region the volume integral of Eq. (7.1.17) is zero, and the surface integral is over the sphere at infinity and that at $r = a$. The integral over the sphere at infinity produces the incident wave p_i, as mentioned in connection with Eq. (8.1.13). Thus Eq. (7.1.17) becomes

$$p_\omega(r,\vartheta,\varphi) = p_i + a^2 \int_0^{2\pi} d\varphi_0 \int_0^{\pi} \sin\vartheta_0 \, d\vartheta_0 \Big[p_\omega(a,\vartheta_0,\varphi_0) \frac{\partial}{\partial a} g_\omega - g_\omega(r,\vartheta,\varphi \mid a,\vartheta_0,\varphi_0) \frac{\partial}{\partial a} p_\omega(a,\vartheta_0,\varphi_0) \Big] \qquad (8.2.6)$$

where the incident wave p_i is usually the plane wave Ae^{ikz} and where we have replaced $\partial/\partial n_0$ in the integral by $-(\partial/\partial a)$. The appropriate series for the Green's function is given in Eq. (7.2.31). To make the integral equation specific, we need to determine the ratio between p_ω and $\partial p_\omega/\partial r$ at the surface of the sphere $r = a$. This ratio is, of course, fixed by the acoustic properties of the material inside $r = a$ [which is not a part of the region in which Eq. (8.2.6) holds]. If the material is such that its surface at $r = a$ is one of local reaction, then $u_r = -(\beta/\rho c)p$ or $\partial p/\partial r = -ik\beta p$ just outside $r = a$, where $\rho c/\beta$ is the point impedance, and β is the specific admittance of the surface.

On the other hand, if the material of the scattering sphere allows sound waves to propagate through it, the surface at $r = a$ will be one of extended reaction, and we shall have to determine the ratio between u_r and p at $r = a$ by investigating the wave motion inside $r = a$. Suppose the material of the sphere has density ρ_e and compressibility κ_e, so that wave motion propagates

with velocity $c_e = 1/\sqrt{\rho_e \kappa_e}$ in it. In this case the acoustic pressure inside $r = a$ can be represented by a series of terms of the form $Y_{mn}^{\sigma}(\vartheta,\varphi) j_m(k_e r)$, where $k_e = \omega/c_e$; Y_{mn}^{σ} and j_m are the spherical harmonics and spherical Bessel functions discussed in Sec. 7.2. We must use j_m, not n_n, because this wave cannot go to infinity at $r = 0$.

For each spherical harmonic of order m, the relation between u_r and p just inside $r = a$ is then

$$u_r = \frac{1}{i\omega\rho_e}\left(\frac{\partial p}{\partial r}\right)_{r=a} = \frac{1}{i\rho_e c_e} j_m'(k_e a) = -\frac{\beta_m}{\rho c} j_m(k_e a) \qquad (8.2.7)$$

where

$$\beta_m = i\frac{\rho c}{\rho_e c_e}\left[\frac{j_m' k_e a}{j_m(k_e a)}\right]$$

and where j_m' represents the derivative of j_m with respect to its argument. We have expressed the ratio between the portions of u_r and p, which belong to the spherical harmonic Y_{mn}^{σ}, in terms of an effective admittance β_m, by analogy with the locally reacting case. Here, however, β_m is a function of the order m of the spherical harmonic, whereas for the locally reacting case β is independent of m.

In either case we shall have to expand the solution $p_\omega(r,\vartheta,\varphi)$ outside $r = a$ into a spherical harmonic series

$$p_\omega(r,\vartheta,\varphi) = \sum_{mn\sigma} Y_{mn}^{\sigma}(\vartheta,\varphi) q_{mn}^{\sigma}(kr) \qquad k = \frac{\omega}{c} \qquad (8.2.8)$$

where q_{mn}^{σ} is a combination of $j_m(kr)$ and $h_m(kr)$, which fits the boundary conditions at $r = a$. In fact, the boundary condition is that, just outside $r = a$,

$$u_r = \frac{1}{i\omega\rho}\left(\frac{\partial q}{\partial r}\right)_{r=a} = \frac{1}{i\sigma\rho c} q_{mn}^{\sigma\prime}(ka) = -\frac{\beta_m}{\rho c} q_{mn}^{\sigma}(ka)$$

by definition of β_m. Quantity $q_{mn}^{\sigma\prime}(\zeta) = dq_{mn}^{\sigma}(\zeta)/d\zeta$. However, the ratio between u_r and p at $r = a$, required by the acoustic properties of the material inside $r = a$, is given by combining the equations of the foregoing paragraphs. We see that

$$q_{mn}^{\sigma\prime}(ka) = -i\beta_m q_{mn}^{\sigma}(ka) \qquad \text{where } \beta_m = i\frac{\rho c}{\rho_e c_e}\left[\frac{j_m'(k_s a)}{j_m(k_e a)}\right] \qquad (8.2.9)$$

for a surface of extended reaction at $r = a$. For a surface of local reaction, of point impedance $\rho c/\beta$, the relation between q_{mn}' and q_{mn} is the same, but $\beta_m = \beta$, independent of m, instead of being given by the second equation. This fixes the relationship between the first and second terms in the brackets in Eq. (8.2.6), for either kind of surface reaction.

Solving the integral equation

We can now carry out the surface integral of Eq. (8.2.6), using the expansion (7.2.31) for the Green's function, the expansion (8.2.8) for p_ω, and Eq. (8.2.9). Because of the integral relationships between the spherical harmonics, given in Eqs. (7.2.7), the double summation reduces to a single sum, and the integral simplifies considerably. For example, the first term in the surface integral turns out to be

$$\frac{ik^2a^2}{4\pi}\int_0^{2\pi} d\varphi_0 \int_0^{\pi} \sin\vartheta_0\, d\vartheta_0 \sum_{st\tau} q_{st}{}^{\tau'}(ka)\, Y_{st}{}^{\tau}(\vartheta_0,\varphi_0)$$
$$\times \sum_{mn\sigma} (2m+1)\frac{(m-n)!}{(m+n)!}\, Y_{mn}{}^{\sigma}(\vartheta_0,\varphi_0)\, Y_{mn}{}^{\sigma}(\vartheta,\varphi) j'_m(ka) h_m(kr)$$
$$= ik^2a^2 \sum_{mn\sigma} Y_{mn}{}^{\sigma}(\vartheta,\varphi) h_m(kr) q_{mn}{}^{\sigma}(ka) j'_m(ka)$$

where we have chosen the form of the Green's function appropriate for $r \geqslant a$.

Thus Eq. (8.2.6) becomes

$$p_\omega(r,\vartheta,\varphi) = p_i(r,\vartheta,\varphi) + ik^2a^2 \sum_{mn\sigma} [j'_m(ka) + i\beta_m j_m(ka)] q_{mn}{}^{\sigma}(ka)\, Y_{mn}{}^{\sigma}(\vartheta,\varphi) h_m(kr) \quad (8.2.10)$$

which will enable us to evaluate the pressure anywhere outside the sphere, as soon as we have computed the values of the constants $q_{mn}(ka)$. The first term on the right-hand side is the incident wave; the second term, made up of outgoing Hankel functions, is the scattered wave. We compute the values of $q_{mn}(ka)$ by requiring that Eq. (8.2.10) hold on the boundary $r = a$. The equation

$$p_i(a,\vartheta,\varphi) = \sum_{mn\sigma} q_{mn}{}^{\sigma}(ka)\, Y_{mn}{}^{\sigma}(\vartheta,\varphi)\{1 - ik^2a^2 h_m(ka)[j'_m(ka) + i\beta_m j_m(ka)]\}$$

simply states that the total wave, given by Eq. (8.2.8), minus the scattered wave, at $r = a$, equals the incident wave.

One further simplification can be made in this equation. The last of Eqs. (7.2.12) states that $j'_m(ka)h_m(ka) = j_m(ka)h'_m(ka) - (i/k^2a^2)$. Therefore the equation determining the coefficients $q_{mn}(ka)$, the equivalent of the integral equation (8.2.6), becomes

$$p_i(a,\vartheta,\varphi) = -ik^2a^2 \sum_{m,n,\sigma} q_{mn}{}^{\sigma}(ka)\, Y_{mn}{}^{\sigma}(\vartheta,\varphi) j_m(ka)[h'_m(ka) + i\beta_m h_m(ka)] \quad (8.2.11)$$

The incident wave also may be expanded in terms of spherical harmonics.

$$p_i(r,\vartheta,\varphi) = \sum_{m,n,\sigma} A_{mn}{}^{\sigma}\, Y_{mn}{}^{\sigma}(\vartheta,\varphi) j_m(kr)$$

Since the incident wave is the complete solution *in the absence* of the scattering sphere, it must be finite at $r = 0$, and thus only the Bessel functions $j_m(kr)$ can be used. If the incident wave is a plane wave of amplitude $A = \sqrt{\rho c I}$, intensity I, and wavelength $\lambda = 2\pi/k$, traveling in the direction of the spherical axis (z axis), the form is

$$p_i = A\sum_{m=0}^{\infty}(2m+1)i^m P_m(\cos\vartheta)j_m(kr) \qquad \text{or} \qquad A_{mn} = A(2m+1)i^m\delta_{\sigma 1}\delta_{n0} \tag{8.2.12}$$

since $Y_{mo}^{1}(\vartheta,\varphi) = P_m(\cos\vartheta)$.

Therefore the coefficients of expansion of p_ω on the surface $r = a$ have values

$$q_{mn}{}^{\sigma}(ka) = i\frac{A_{mn}{}^{\sigma}}{k^2a^2}\,[h_m'(ka) + i\beta_m h_m(ka)]^{-1}$$

$$= \frac{(2m+1)i^{m+1}A}{h_m'(ka) + i\beta_m h_m(ka)}\,\frac{\delta_{\sigma 1}\delta_{n0}}{k^2a^2} \qquad \text{(for a plane wave)} \tag{8.2.13}$$

For the rest of this chapter we shall concentrate exclusively on the case of an incident plane wave, for which the second expansion holds for $q_{mn}{}^{\sigma}(ka)$. The more general case will be taken up later.

Absorption and scattering

Inserting the second form of Eq. (8.2.13) into Eq. (8.2.10), we obtain, for the total sound field outside the sphere, for an incident plane wave,

$$p_\omega(r,\vartheta) = Ae^{ikr\cos\vartheta} - A\sum_{m=0}^{\infty}(2m+1)i^m P_m(\cos\vartheta) \times \frac{j_m'(ka) + i\beta_m j_m(ka)}{h_m'(ka) + i\beta_m h_m(ka)}\,h_m(kr)$$

$$= A\sum_m (2m+1)i^m P_m(\cos\vartheta)[j_m(kr) - \tfrac{1}{2}(1+R_m)h_m(kr)]$$

$$\to \frac{iA}{2kr}\sum_m (2m+1)P_m(\cos\vartheta)[R_m e^{ikr} + (-1)^m e^{-ikr}] \qquad r\to\infty \tag{8.2.14}$$

In the last expression we have used the asymptotic formulas for j_m and h_m given in Eqs. (7.2.9) and (7.2.12). The last two formulas have been simplified by defining a *reflection coefficient* R_m.

$$1 + R_m = 2\frac{j_m'(ka) + i\beta_m j_m(ka)}{h_m'(ka) + i\beta_m h_m(ka)} \qquad R_m = \frac{h_m'^{*}(ka) + i\beta_m h_m^{*}(ka)}{h_m'(ka) + i\beta_m h_m(ka)}$$

with $h_m'^{*}(ka) = j_m'(ka) - in_m'(ka)$, $h_m^{*} = j_m - in_m$ being the complex conjugates of the Hankel functions $h_m' = j_m' + in_m'$ and $h_m = j_m + in_m$, respectively. These results deserve detailed discussion.

The asymptotic expression indicates that the pressure wave outside the sphere can be expressed as a series of waves traveling radially outward from the origin (characterized by the factor e^{ikr}) plus another series traveling radially inward (with the factor e^{-ikr}). The waves traveling inward are unaffected by the nature of the spherical surface at $r = a$; they have been determined solely by the shape of the incident plane wave. The outward-bound waves, on the other hand, have amplitudes proportional to the factors R_m, which depend on the surface admittances β_m. We may call this factor, the ratio between the mth outgoing and incoming waves, the *reflection coefficient* for the mth partial wave. If β_m is purely imaginary (reactive), R_m is the ratio between the quantity $h'_m + i\beta_m h_m$ and its complex conjugate, so that $|R_m| = 1$. In this case each wave is reflected from the surface $r = a$ with no change in amplitude, but a change in phase; no energy is lost at the sphere's surface.

On the other hand, if β_m has a real part, $|R_m|$ is less than unity. Less energy is reflected than is incident on the sphere; some of the energy is absorbed. Finally, if the material inside the sphere has the same compressibility and density as the medium outside, so that $\beta_m = i[j'_m(ka)/j_m(ka)]$, then $1 + R_m = 0$, $R_m = -1$ for all m, there is no scattering or absorption, and the incident plane wave is undistorted.

The asymptotic form of the scattered wave is

$$p_s(r,\vartheta,\varphi) = -\tfrac{1}{2}A \sum_m (2m+1) i^m (1 + R_m) P_m(\cos\vartheta) h_m(kr)$$
$$\to A \frac{e^{ikr}}{r} \Phi(\vartheta) \qquad \Phi(\vartheta) = \frac{i}{2k} \sum_m (2m+1)(1 + R_m) P_m(\cos\vartheta) \tag{8.2.15}$$

Factor Φ, the angle-distribution factor for the scattered wave, is equal to $1/A$ times the factor Φ_s of Eq. (8.1.14). The total power scattered, divided by the incident intensity $I = |A|^2/\rho c$, is equal to the *scattering cross section* [see discussion preceding Eq. (8.1.8)].

$$\Sigma_s = 2\pi \int_0^\pi |\Phi(\vartheta)|^2 \sin\vartheta \, d\vartheta = \frac{4\pi}{k^2} \sum (2m+1) \left| \frac{j'_m(ka) + i\beta_m j_m(ka)}{h'_m(ka) + i\beta_m h_m(ka)} \right|^2$$
$$= \frac{\pi}{k^2} \sum_m (2m+1) |1 + R_m|^2 \to \frac{4\pi}{k^2} \sum_m (2m+1) \sin^2 \delta_m \qquad \text{when } \beta \to 0 \tag{8.2.16}$$

[see Eqs. (8.2.3)]. If β has a real part, energy is absorbed by the sphere as well as scattered to infinity. To calculate the power absorbed, we compute the energy flux, $-\mathrm{Re}\,(pu_r^*)$ at $r = a$, into the sphere. Equations (8.2.8) and (8.2.13) enable us to calculate the pressure at the surface, whereas Eq. (8.2.9) shows that

$$(u_r)_{r=a} = -\frac{A}{\rho c k^2 a^2} \sum_m \frac{(2m+1) i^m \beta_m}{h'_m(ka) + i\beta_m h_m(ka)} P_m(\cos\vartheta)$$

If the admittance β_m has a real part ξ_m and imaginary part σ_m, so that $\beta_m = \xi_m - i\sigma_m$, then the *absorption cross section* Σ_a is the power absorbed by the sphere, divided by I_i.

$$\Sigma_a = \frac{k^4a^4}{4\pi a^2}\sum_m \frac{(2m+1)\xi_m}{[h_m'(ka) + i\beta_m h_m(ka)]^2} = \frac{\pi}{k^2}\sum_m (2m+1)(1 - |R_m|^2) \quad (8.2.17)$$

Quite a bit of algebra is required to show that the two series are equal.

Having written out the two expressions for the power withdrawn from the incident beam, we can write down the sum, called the *total cross section.*

$$\Sigma = \Sigma_s + \Sigma_a = \frac{2\pi}{k^2}\sum_m (2m+1)(1 + \mathrm{Re}\, R_m) - \frac{4\pi}{k}\,\mathrm{Im}\,[\Phi(0)] \quad (8.2.18)$$

since the imaginary part of the angle-distribution factor Φ, for $\vartheta = 0$, is proportional to the sum of the real parts of $(2m+1)(1+R_m)$. We note that the forward-directed scattering function Φ is proportional to the *total* cross section. Since Im $[\Phi(0)]$ forms the shadow, it is not surprising that it measures the total power abstracted from the incident wave, whether absorbed or scattered.

Long wavelengths

To show how these formulas can be used and to demonstrate the range of validity of the Born approximation [Eq. (8.1.20)] for the case of a spherical scatterer, we shall work out the limiting forms for the angle-distribution function Φ and for the cross section Σ when ka is small (i.e., when λ is larger than $2\pi a$). Insertion of Eqs. (7.2.9) and (7.2.12) into Eqs. (8.2.9), (8.2.13), and (8.2.15) produces approximate formulas for angle-distribution function, cross sections, and pressure and velocity distributions at the surface of the sphere, to the lowest order in the small quantity ka.

$$\begin{aligned}
\Phi(\vartheta) &\simeq \tfrac{1}{3}k^2a^3\left(\frac{\kappa_e - \kappa}{\kappa} + \frac{3\rho_e - 3\rho}{2\rho_e + \rho}\cos\vartheta\right) \\
p_\omega(a,\vartheta) &\simeq A\left(1 + ika\,\frac{3\rho_e}{2\rho_e+\rho}\cos\vartheta\right) \qquad p_s(r,\vartheta) \to A\,\frac{e^{ikr}}{r}\,\Phi(\vartheta) \\
u_r(a,\vartheta) &\simeq \frac{A}{\rho c}\left(\tfrac{1}{3}ika\,\frac{\kappa_e}{\kappa} + \frac{3\rho}{2\rho_e+\rho}\cos\vartheta\right) \qquad (8.2.19) \\
\Sigma_s &\simeq \tfrac{4}{9}\pi a^2(ka)^4\left(\left|\frac{\kappa_e - \kappa}{\kappa}\right|^2 + \frac{1}{3}\left|\frac{3\rho_e - 3\rho}{2\rho_e+\rho}\right|^2\right) \\
\Sigma_a &\simeq \tfrac{4}{3}\pi a^2(ka)\,\mathrm{Im}\left(\frac{\kappa_e - \kappa}{\kappa} + \frac{3\rho_e - 3\rho}{2\rho_e+\rho}\right)
\end{aligned}$$

These formulas also deserve extended discussion. For a heavy, incompressible sphere ($\kappa_e \ll \kappa$, $\rho_e \gg \rho$) the angle distribution of the scattered

intensity has the factor $(-1 + \frac{3}{2}\cos\vartheta)^2$, with 25 times as much scattered backward ($\vartheta = \pi$) as forward ($\vartheta = 0$) and with a node of zero intensity at $\vartheta = \cos^{-1}\frac{2}{3} \simeq 48°$. The radial velocity of the surface of the sphere in this case is zero, of course, and the pressure at the surface is $A(1 + \frac{3}{2}ika\cos\vartheta)$, whereas the incident pressure at $r = a$ is $A(1 + ika\cos\vartheta)$. This indicates the error made by the Born approximation, which uses the incident pressure. The modified approximation of Eq. (8.1.21) checks with the present exact solutions when $\rho_e - \rho$ is small compared with ρ_e. But for a very heavy, incompressible sphere the modified Born approximation would predict an angle distribution of $(-1 + \cos\vartheta)^2$, with nothing scattered forward, instead of the $(-1 + \frac{3}{2}\cos\vartheta)^2$ of the exact solution. The unmodified Born approximation, obtained by substituting p_i for p_ω in the integral of Eq. (8.1.14), would be much less satisfactory, for the effect of ρ_e would be the same as the effect of κ_e, and there would be *no* cosine term, of second order in ka, in the resulting $\Phi(\vartheta)$.

For a light, compressible sphere ($\kappa_e > \kappa$, $\rho_e \ll \rho$) the angle factor for the scattered intensity is $[(\kappa_e/\kappa) - 3\cos\vartheta]^2$, which again has a predominance in the backward direction. The pressure at the surface of the sphere is A, with negligible dependence on ϑ, but the radial velocity at $r = a$ is large, being $(A/\rho c)[ika(\kappa_e/\kappa) + 3\cos\vartheta]$; the sphere moves back and forth along the z axis with three times the velocity of the surrounding fluid; so it is not surprising that the scattering cross section is large. This is the case with air bubbles in water. The absorption cross section differs from zero only if the compressibility or density of the material of the sphere, or both, are complex, corresponding to frictional effects.

Porous sphere

In this connection it is interesting to work out the scattering from a sphere of porous material, of porosity Ω and flow resistance Φ_e, as discussed in connection with Eq. (6.2.23). We first assume that the material of the pore walls is stationary, the air flowing through the pores as described in Sec. 6.2. From Eq. (6.2.25) we see that the effective compressibility and density of the fluid in the pores, for sound of frequency $\omega/2\pi$, are

$$\kappa_e = \kappa_p\Omega \qquad \rho_e = \rho_p + i\frac{\Phi_e}{\omega}$$

To this approximation the compressibility has no imaginary part, though inclusion of thermal transfer would introduce a small imaginary term.

For many porous acoustic materials in air, $\Phi_e/\rho_p\omega$ ranges from 1 to 10 for frequencies of 1,000 cps and is proportionally larger for lower frequencies; so the imaginary part of ρ_e is often larger than its real part. When this is true we can write

$$\frac{3\rho_e - 3\rho}{2\rho_e + \rho} \simeq \frac{3}{2} + \frac{3}{4}i\frac{\rho ck}{\Phi_e}$$

If, in addition, $\kappa_p\Omega$ is not very different from κ, the expressions of Eq. (8.2.19) become

$$\Phi(\vartheta) \simeq \tfrac{1}{2}k^2a^3\left[1 + \tfrac{1}{2}i\frac{\rho ck}{\Phi_e}\right]\cos\vartheta$$

$$p_\omega(a,\vartheta) \simeq A\left[1 + ika\left(\tfrac{3}{2} - \tfrac{3}{4}i\frac{\rho ck}{\Phi_e}\right)\cos\vartheta\right]$$

$$u_r(a,\vartheta) \simeq \frac{A}{\rho c}\left[\tfrac{1}{3}ika - \tfrac{3}{2}i\frac{\rho ck}{\Phi_e}\cos\vartheta\right] \tag{8.2.20}$$

$$\Sigma_s \simeq \tfrac{1}{3}\pi a^2(ka)^4\left[1 + \frac{1}{4}\left(\frac{\rho ck}{\Phi_e}\right)^2\right]$$

$$\Sigma_a \simeq \pi a^2(ka)\frac{\rho ck}{\Phi_e}$$

Note that the angle-distribution function $\Phi(\vartheta)$ is not to be confused with flow resistance Φ_e.

When the effective compressibility is nearly equal to that of air, the scattered wave is dipolelike; the air flows back and forth through the pores, the flow resistance ensuring that the dipole is out of phase with the driving pressure. The absorption cross section is larger than the scattering cross section if ka is small.

When the framework of the pore walls is not held at rest, yet another correction must be added because of the motion of the solid material. If the pore-wall material is effectively incompressible, the radially symmetric term ($m = 0$) in the series for the pressure inside the sphere is unaffected by this motion, and the term $(\kappa_e - \kappa)/\kappa$ in Eqs. (8.2.19) is unchanged. However, the $m = 1$ term, representing linear motion along the z axis, is changed because the solid part of the sphere as a whole will move back and forth (as well as the air in the pores) unless the pore-wall material has infinite mass. Suppose this material has density ρ_m, large but not infinite. Then the mass of a unit volume of the sphere, exclusive of the fluid contained in the pores, is $\rho_m(1 - \Omega)$, where $1 - \Omega$ is the fraction of the volume occupied by the solid material. This is usually larger than ρ_e.

When this is the case, for the $m = 1$ term (proportional to $\cos\vartheta$), the impedance to the fluid flow is equivalent to a parallel circuit, the resistance Φ_e of the porous material being shunted by the reactance $-i\omega\rho_m(1 - \Omega)$ of the solid material, which is forced into motion by the drag of the air moving through the pores. In other words, the Φ_e of Eqs. (8.2.20), for the $m = 1$ term only, must be replaced by

$$\Phi_p = \left[\frac{1}{\Phi_e} + \frac{i}{\omega\rho_m(1 - \Omega)}\right]^{-1}$$

and the factor $\rho ck/\Phi_e$ in the first three of Eqs. (8.2.20) must be replaced by

$$\frac{\rho ck}{\Phi_e} + i\,\frac{\rho}{\rho_m(1-\Omega)}$$

where the susceptance term is usually 1/2 to 1/10 of the magnitude of the conductance term. The spherically symmetric ($m = 0$) components are unaffected, to this approximation, the velocity being out of phase with the pressure at $r = a$. The axial ($m = 1$) components, on the other hand, are changed by the motion of the solid material, both in regard to phase relations and to magnitude ratios.

The fourth and fifth of Eqs. (8.2.20) then become

$$\begin{aligned}\Sigma_s &= \tfrac{1}{3}(\pi a^2)(ka)^4\left\{1 + \frac{1}{4}\left(\frac{\rho ck}{\Phi_e}\right)^2 + \frac{1}{4}\left[\frac{\rho}{\rho_m(1-\Omega)}\right]^2\right\} \\ \Sigma_a &= \pi a^2(ka)\,\frac{\rho ck}{\Phi_e}\end{aligned} \tag{8.2.21}$$

indicating that the absorption cross section is not affected by the motion of the solid material (to this approximation) and the scattering cross section is not changed much when $\rho/\rho_m(1-\Omega)$ is smaller than $\rho ck/\Phi_e$.

For larger spheres or shorter wavelengths these approximate formulas are not valid, and the exact formulas of Eqs. (8.2.15) to (8.2.17) must be used.

Viscosity and thermal conduction

Finally, we can combine these equations with Eqs. (6.4.17) and (6.4.25) to find the effect of viscosity and thermal conduction on the scattered wave. The incident-plus-scattered wave is, of course, a propagational one; in air the attenuation effects can be neglected, except at the surface of the scatterer. Thus the incident plane wave p_i is still given by Eq. (8.2.1), with $k = \omega/c$, and the wave scattered by the sphere can still be written

$$p_s = A\sum_m B_m P_m(\cos\vartheta)h_m(kr) \to A\,\frac{e^{ikr}}{ikr}\sum_m i^m B_m P_m(\cos\vartheta)$$

where $B_m = -\frac{1}{2}(2m+1)i^m(1+R_m)$, R_m being the reflection coefficient of Eq. (8.2.14). In addition, in the boundary layer just outside the sphere, there are thermal and shear waves, of an amplitude sufficient to satisfy the boundary conditions on the temperature and tangential velocity at the surface. These also modify the pressure and radial-velocity conditions somewhat, so that coefficients B_m, and thus the reflection coefficients R_m, have not the values they would have if these small effects were neglected.

Usually, these effects can be neglected, particularly if the sphere is fairly large or is "soft." They are proportionally more important for small scatterers and for a cloud of small spheres (such as fog or rain); the effect can be quite noticeable. For a sphere of radius a, large compared with the