100 cps and give the position of the nodal planes and the reverberation times of each corresponding standing wave.

30 List frequencies, nodes, and reverberation times of the normal modes between $\nu = 0$ and $\nu = 100$ for the room of Prob. 29 when walls, floor, and ceiling all have an average absorption coefficient of 0.1.

31 The air in the room of Prob. 29 is started into vibration so that all the normal modes between $\nu = 98$ and $\nu = 102$ are set into motion with equal initial amplitudes. What normal modes are excited? Plot the decay curve of intensity level against time after the source is shut off at the midpoint of the room; at the midpoint of one wall; at a point 167 cm out from two walls and 250 cm up from the floor.

CHAPTER 10

COUPLING OF ACOUSTICAL SYSTEMS

10.1 SYSTEMS OF INFINITE EXTENT

In the earlier chapters of this book we studied the vibrations and wave motions of homogeneous systems of infinite, and also of finite, extent. In Chaps. 4 and 5, for example, we discussed the traveling and standing waves in uniform strings, membranes, and plates in vacuum; in Chap. 6 we took up the study of the wave motion in a homogeneous fluid of infinite extent, uncoupled to any vibrating solid system. Even in Chap. 7 we assumed that the radiated waves were produced by vibrating sources having known velocities, so that, though the waves in the medium were produced by the vibrations of the solid source, we effectively neglected to include the reaction of the medium back on the source.

In many cases this neglect is justifiable, particularly if the source is small in size compared with the wavelength and is stiff in comparison with the medium. But if the vibrating solid system is extended in size and/or if it also supports wave motion, which can be induced by the reaction of the medium, the coupling between solid and fluid systems should be included from the start in our theory. Many problems of practical interest involve the interaction between two or more wave-carrying systems, coupled together so strongly that the motion of each is noticeably affected by the other. Membranes or plates do not usually vibrate *in vacuo*; they affect and are affected by the air (or water) in contact with their surfaces; the velocity of their transverse waves is appreciably modified by the contact.

In this chapter we shall take up a few examples of this coupled motion, indicate the characteristic results of the coupling, and illustrate some of the methods by which these characteristics can be calculated. Some systems are of sufficient size so that we can observe traveling, rather than standing, waves; these will be discussed in the first section. We shall see that the coupled motion involves two (or more) different sorts of waves, each kind involving motion of both systems, but with the apportionment of energy between the systems different for the different kinds. We shall also see that the

characteristics of these different kinds of wave will depend markedly on the relationship between the wave velocities in the two (or more) separate systems.

There are two possible traveling, transverse waves on a membrane in contact with air, for example, one wave having velocity near that for the same membrane *in vacuo*, and the other with velocity near that for sound in air. Both waves involve the motion of both membrane and air. If the wave velocity of the membrane *in vacuo* is less than that of air, the former wave, but not the latter, can propagate along the membrane without attenuation. On the other hand, if the sound velocity in air is not as great as that of transverse waves on the membrane *in vacuo*, the situation is reversed; the latter kind of wave will "cling" to the membrane and propagate along it without attenuation, whereas the former kind will continually radiate energy away from the membrane into the air; the membrane wave of this kind will lose energy as it travels along, and thus will be attenuated.

Elastic, porous solid

The simplest example of coupled wave motion is that of two copenetrating media which couple together at every point in space. A porous solid, such as a piece of sound-absorbing material, for instance, couples to the air in its pores. The solid portion of the system can be considered as an interlocking mesh of elastic fibers, with a characteristic compressibility and effective mass density. *In vacuo* we could consider this mesh, for waves considerably longer than pore size, as an elastic medium of mean density ρ_s and volume compressibility κ_s (s meaning "solid" here, not isentropic). It also would support shear waves, but we are not interested in following this example further than discussing the coupling effects in the compressional waves.

The motion of the air in the pores has been discussed in Sec. 6.2; it also has an effective density ρ_a, which may not be equal to the density of air in free space, and an effective bulk modulus κ_a, which was written as $\Omega\kappa_p$ in Eq. (6.2.22). These two systems may have different displacements from equilibrium and different Eulerian velocities, defined so that $\mathbf{u}_a$ is the volume of air flowing per unit time per unit area through the pores, and $\rho_a\mathbf{u}_a$ is the fluid momentum per unit volume; $\mathbf{u}_s$ is the velocity, and $\rho_s\mathbf{u}_s$, the momentum per unit volume for the solid portion of the system.

Next we note that compressional force can be applied to the two parts of the system independently. Pressure can be applied to the air, to change its density without appreciably changing the density of the solid fibers; also, forces can be applied to the fiber network which will compress the net without appreciably affecting the interspersed air. This latter force, in the large, can be considered to have a mean compressional component p_s which will change the fiber spacing and thus the mean solid density, the gradient of which will produce acceleration of the network (there also will be shear

stresses, but these are not pertinent to our discussion). True, compression of the solid network will reduce the porosity Ω somewhat and thus will change the density of the air in the pores, but if Ω is not very small and if the compressibility of the fibers is considerably smaller than that of the air, this coupling effect will be of second order and can be neglected for small-amplitude waves.

The major part of the coupling between the two systems will come in the frictional force arising when the velocities of the solid net and the entrapped air are not equal. In Eq. (6.2.23), when $\mathbf{u}_s$ was considered to be zero, we defined this force in terms of a measurable flow resistance Φ. The generalization is obvious: the coupling force is $\Phi(\mathbf{u}_a - \mathbf{u}_s)$, acting with opposite sign on the two systems. Thus, to first order, the equations of continuity of the two portions are separate, but the equations of motion are coupled by the flow resistance.

$$\kappa_a \frac{\partial p_a}{\partial t} = -\text{div } \mathbf{u}_a \qquad \kappa_s \frac{\partial p_s}{\partial t} = -\text{div } \mathbf{u}_s$$

$$\rho_a \frac{\partial \mathbf{u}_a}{\partial t} = -\text{grad } p_a + \Phi(\mathbf{u}_s - \mathbf{u}_a) \tag{10.1.1}$$

$$\rho_s \frac{\partial \mathbf{u}_s}{\partial t} = -\text{grad } p_s + \Phi(\mathbf{u}_a - \mathbf{u}_s)$$

If no air is present ($\rho_a \to 0$, $\kappa_a \to \infty$), p_a is zero, $\mathbf{u}_a$ equals $\mathbf{u}_s$, and the solid network will transmit compressional waves with a wave velocity $c_s = \sqrt{1/\rho_s\kappa_s}$ (κ_s often is complex, because of internal friction in the fiber net). Likewise, if the density and compressibility of the fiber structure were equal to that of the air, $\mathbf{u}_s$ again would equal $\mathbf{u}_a$, and the wave velocity would equal that of air, $c_a = \sqrt{1/\rho_a\kappa_a}$. For intermediate cases the coupling will modify the wave velocity of both parts of the system.

To demonstrate this we ask for the plane-wave solutions, setting

$$p_a = P_a e^{ikx - i\omega t} \qquad u_a = \frac{\omega\kappa_a}{k} p_a = \frac{\omega}{\rho_a c_a{}^2 k} p_a$$

$$p_s = P_s e^{ikx - i\omega t} \qquad u_s = \frac{\omega\kappa_s}{k} p_s = \frac{\omega}{\rho_s c_s{}^2 k} p_s$$

and solving for k as a function of ω, the κ's, and ρ's.

$$i\omega\Phi\kappa_s P_s = (\omega^2\alpha_a{}^2 - k^2)P_a \qquad \text{and} \qquad i\omega\Phi\kappa_a P_a = (\omega^2\alpha_s{}^2 - k^2)P_s$$

where

$$\alpha_s{}^2 = \kappa_s\rho_s + i\frac{\kappa_s\Phi}{\omega} \qquad \text{and} \qquad \alpha_a{}^2 = \kappa_a\rho_a + i\frac{\kappa_a\Phi}{\omega}$$

or

$$k^2 = \tfrac{1}{2}\omega^2(\alpha_s{}^2 + \alpha_a{}^2) \pm \tfrac{1}{2}[\omega^4(\alpha_s{}^2 - \alpha_a{}^2)^2 - 4\omega^2\kappa_a\kappa_s\Phi^2]^{\frac{1}{2}} \tag{10.1.2}$$

When $|c_a^2 - c_s^2| \gg 4\Phi^2/\omega^2\rho_a\rho_s$, the approximation

$$k^2 \simeq \left(\frac{\omega}{c_s}\right)^2 + i\Phi\frac{\omega\rho_s}{c_s^2} + \frac{\Phi^2}{\rho_s\rho_a}\frac{1}{c_a^2 - c_s^2}$$

or

$$\simeq \left(\frac{\omega}{c_a}\right)^2 + i\Phi\frac{\omega\rho_a}{c_a^2} + \frac{\Phi^2}{\rho_s\rho_a}\frac{1}{c_s^2 - c_a^2}$$

is valid, indicating two possible waves, with two velocities,

$$\frac{\omega}{k} \simeq c_a\left(1 + \frac{\Phi^2}{2\omega^2\rho_a\rho_s}\frac{c_a^2}{c_a^2 - c_s^2}\right) \quad \text{or} \quad c_s\left(1 + \frac{\Phi^2}{2\omega^2\rho_a\rho_s}\frac{c_s^2}{c_s^2 - c_a^2}\right)$$

The result is reminiscent of that of Eqs. (3.1.3) for a pair of simple coupled oscillators. The velocity of the faster wave is increased by the coupling; that of the slower wave is diminished. Inserting the result into the equation relating the magnitudes P_a and P_s shows that the wave with velocity near c_a has most of its energy in air motion; that with velocity near c_s carries most of its energy in motion of the solid network.

This particular example will not be carried further; all that was intended was to point out the parallel with the coupled oscillators, to demonstrate that there will be as many different kinds of waves as there are different systems interacting, and that the particular wave which has its velocity near that of one of the component systems will entrust its energy chiefly to that component. In the more interesting and more complicated examples taken up in the rest of this section, these general characteristics will appear, though somewhat modified by the complications of the energy flow, which can be concentrated close to a component system, traveling only in one direction, or else can radiate outward to infinity.

Flexible string

As an example of somewhat greater practical interest, we take up next the effect of the surrounding medium on a flexible string of diameter $2a$, of infinite length, under tension T. According to the discussion of Chap. 4, the velocity of transverse waves in such a string, if coupling to the surrounding medium is ignored, is $c_s = \sqrt{T/\pi a^2\rho_s}$, where ρ_s is the density of the string material, usually much larger than ρ, the density of the surrounding medium. Thus we have two dimensionless parameters, $\alpha = c/c_s$, the ratio of acoustic wave velocity in the medium to the string-wave velocity, and $\mu = \rho/\rho_s$, the ratio of density of the medium to that of the string material. There also are two wavenumbers, $k = \omega/c$ for the medium, and K for the string, which equals ω/c_s *in vacuo* but which differs from this in the presence of the medium.

We first look for possible free vibrations of the coupled system, to see whether there can be a simple-harmonic wave propagating along the string, for example. If such a wave were possible, the dependence of all parts of

the system on z, the distance along the string, and on time t, would be via the factor $\exp(iKz - i\omega t)$. Presumably, this is possible only when there is a particular relationship between frequency $\omega/2\pi = kc/2\pi$ and the string wavenumber K, which must be determined. The ratio (ω/K) is then the phase velocity of the wave along the string.

Suppose the displacement $y(z,t)$ of the string, from its equilibrium position along the z axis, is in the yz plane. If the wave is to have the characteristics we have assumed, the displacement should be

$$\eta = A \exp(iKz - i\omega t)$$

and the velocity, normal to the string surface ($r = a$) at the surface, is

$$u_r = -i\omega A \cos\varphi \exp(iKz - i\omega t)$$

where r, φ are the polar coordinates in the xy plane, φ being the angle between r and the yz plane of displacement of the string.

The pressure wave in the medium outside the string, produced by the string's motion, would then be

$$p(r,\varphi,z,t) = P_0 \cos\varphi \, H_1^{(1)}(qr) \exp(iKz - i\omega t)$$

where $H_1^{(1)}(u) = J_1(u) + iN_1(u) \to -i\sqrt{\pi/2iu}\, e^{iu}$ $(u \to \infty)$ is the Hankel function for outgoing waves for $m = 1$, and $q = \sqrt{k^2 - K^2}$ when $k = \omega/c > K$ and $= i\sqrt{K^2 - k^2}$ when $K > k$. If $K > k$, the argument of the Hankel function is imaginary, and the wave attenuates exponentially; no acoustic energy radiates to $r \to \infty$.

The radial velocity of the medium at $r = a$ is

$$u_r = \frac{1}{ik\rho c}\left(\frac{\partial p}{\partial r}\right)_{r=a} = -i\frac{P_0}{\rho c}\frac{q}{k}\cos\varphi \, H_1'(qa) \exp(iKz - i\omega t)$$

where

$$H_1'(u) = \frac{\partial H_1^{(1)}(u)}{\partial u} = \tfrac{1}{2}H_0(u) - \tfrac{1}{2}H_2(u) = H_0(u) - \frac{1}{u}H_1(u)$$

omitting the superscripts for convenience. This must equal the radial velocity of the string surface, which provides a relationship between P_0 and A, the amplitude of motion of the string,

$$P_0 = \frac{k^2\rho c^2}{qH_1'(qa)}A \tag{10.1.3}$$

The wave motion in the medium reacts back on the string. The net force in the y direction, per unit length of string, is

$$F(z,t) = \int_0^{2\pi} d\varphi \, a\cos\varphi \, p(a,\varphi,z,t) = \pi a P_0 H_1(qa) \exp(iKz - i\omega t)$$

$$= \frac{\pi a k^2 \rho c^2}{q}\frac{H_1(qa)}{H_1'(qa)} A \exp(iKz - i\omega t) \tag{10.1.4}$$

The equation of motion of the string is therefore

$$\pi a^2 \rho_s \frac{\partial^2 \eta}{\partial t^2} = T \frac{\partial^2 \eta}{\partial z^2} + F \qquad T = \frac{\pi a^2 \rho_s}{c_s^2}$$

or

$$(k^2\alpha^2 - K^2)A = \frac{k^2\alpha^2\mu}{qa} \frac{H_1(qa)}{H_1'(qa)} A \tag{10.1.5}$$

where $\alpha = c/c_s$ and $\mu = \rho/\rho_s$. This is the equation which relates K and k. Usually, the wavelength of sound in the medium is much longer than the string diameter $2a$. When this is the case we can use the first few terms in the series expansion for the Hankel function.

$$H_1(u) \simeq -\frac{2i}{\pi u} + \tfrac{1}{2}u\left[1 - \frac{i}{\pi} + \frac{2i}{\pi} \ln (Cu)\right] \qquad C = 0.8905$$

$$\frac{H_1(u)}{H_1'(u)} \simeq -u[1 - u^2 \ln (Cu) + \tfrac{1}{2}\pi i u^2] \qquad \text{if } u \text{ is real}$$

If u is positive-imaginary, the imaginary term in brackets in the last equation vanishes, and the logarithm is of the magnitude of u. Since $u = qa$ and $u^2 = k^2a^2 - K^2a^2$, we obtain an approximate relation between K, the wavenumber in the z direction, and k, the wavenumber in the medium, ω/c, determined by the frequency.

$$K \simeq \begin{cases} k\alpha\{1 + \mu - (1 - \alpha^2 - \mu\alpha^2)k^2a^2\mu[\ln (Cka\sqrt{1 - \alpha^2 - \mu\alpha^2}) - \tfrac{1}{2}\pi i]\}^{\frac{1}{2}} \\ \qquad\qquad \text{when } (1 + \mu)\alpha^2 < 1 \\ k\alpha\{1 + \mu + (\alpha^2 + \mu\alpha^2 - 1)k^2a^2\mu \ln (Cka\sqrt{\alpha^2 + \mu\alpha^2 - 1})\}^{\frac{1}{2}} \\ \qquad\qquad \text{when } (1 + \mu)\alpha^2 > 1 \end{cases} \tag{10.1.6}$$

When the string is heavy and the tension is not great enough to make $c_s = \sqrt{T/\pi a^2 \rho_s}$ larger than c, quantity $\alpha^2(1 + \mu)$ is larger than unity. In this case the wavelength $2\pi/K$ of transverse waves on the string is smaller than the wavelength $2\pi/k$ of waves in the medium, and an outgoing wave, carrying energy from the string to $r \to \infty$, cannot be fitted onto the wave in the string. The radial wavenumber component $q \simeq ik\sqrt{\alpha^2(\mu + 1) - 1}$ is pure-imaginary, and the function $H_1(qr)$ is a real exponential, attenuating to zero as $r \to \infty$. None of the energy possessed by the string or by the motion of the fluid close to the string can be radiated away, except along the string to $z \to \infty$. The phase velocity of this wave in the z direction is [see discussion of Eq. (9.1.17)]

$$c_\varphi = \frac{kc}{K} \simeq c_s\{1 - \tfrac{1}{2}\mu[1 + (\alpha^2 + \mu\alpha^2 - 1)k^2a^2 \ln (0.8905ka\sqrt{\alpha^2 + \mu\alpha^2 - 1})]\}$$
$$\text{when } \alpha^2(1 + \mu) > 1 \tag{10.1.7}$$

a speed somewhat less than c_s, and thus definitely less than c. The effect is as though the string were loaded with the additional mass of some of the neighboring medium. Being a mass load, however, no energy is lost and K is real, implying no attenuation of the wave as it traverses the string.

The situation is quite different if the string is light enough and the tension great enough so that $1/\alpha^2 \equiv (c_s/c)^2 > 1 + \mu$. In this case q is real, energy can radiate away to the medium, and the wavenumber K has an imaginary part, representing attenuation in the z direction. To the first approximation, in the small quantity μ, the attenuation constant and the phase velocity for the string wave are

$$\begin{aligned} &\operatorname{Im} K \simeq ka(1-\alpha^2)\frac{\pi k^2a^2\mu}{4} \qquad \mu \ll 1;\ \alpha^2(1+\mu) < 1 \\ &c_\varphi = \frac{kc}{\operatorname{Re} K} \simeq c_s\{1 - \tfrac{1}{2}\mu[1 - k^2a^2(1-\alpha^2)\ln(Cka\sqrt{1-\alpha^2})]\} \end{aligned} \tag{10.1.8}$$

The transverse wave in the string is then

$$\eta = A\exp(iKz - i\omega t) = Ae^{-z\operatorname{Im}(K)}\exp[i\operatorname{Re} K(z - c_\varphi t)]$$

The wave in the medium is a conical one, propagating in a direction at an angle Ψ, such that

$$\cos\Psi = \frac{\operatorname{Re} K}{k} \simeq \alpha\sqrt{1+\mu}\,[1 - \tfrac{1}{2}(1-\alpha^2)k^2a^2\mu\ln(Cka\sqrt{1-\alpha^2})]$$

$$\sin\Psi \simeq \sqrt{1-\alpha^2}(\mu+1)[1 + \tfrac{1}{2}k^2a^2\alpha^2\mu\ln(Cka\sqrt{1-\alpha^2})] \qquad \mu \ll 1$$

The wave, and its asymptotic form, is

$$\begin{aligned} p &= P_0\cos\varphi\, H_1(qr)\exp(iKz - i\omega t) \\ &\xrightarrow[r\to\infty]{} -iP_0\cos\varphi\sqrt{\frac{2}{\pi iqr}}\exp[ik(r\sin\Psi + z\cos\Psi - ct) \\ &\qquad - \tfrac{1}{4}k\alpha(1-\alpha^2)\pi k^2a^2\mu z_0] \end{aligned} \tag{10.1.9}$$

where $q \simeq k\sin\Psi$, and P_0 is related to A by Eq. (10.1.3). Quantity $z_0 = z - r\cot\Psi$ is the point on the string where the conical wave reaching point r, φ, z started from, as shown in Fig. 10.1. When $\alpha^2(\mu+1) < 1$, the radial

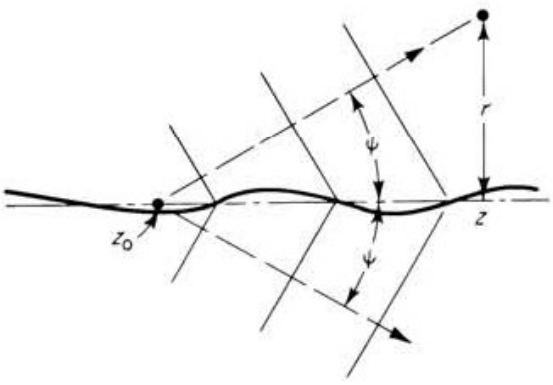


FIGURE 10.1
Conical radiation from traveling wave on a string.

wavenumber q has a negative-imaginary part, and K has a positive-imaginary part. Therefore the wave attenuates as we increase z, holding r constant, but increases exponentially as we increase r, holding z constant, since the wave comes from farther to the left on the string (z_0 greater negative), and thus from a portion of the string which has not attenuated so much.

Incidentally, we can verify Eq. (7.3.6) for the radiation impedance per unit length of the string, at least for very long string wavelengths. We insert the approximate formula for the Hankel function in Eq. (10.1.4) and substitute $U/-i\omega$ for A, where U is the velocity amplitude of the string. The ratio of F to U is the desired impedance

$$Z_{\text{rad}} \simeq -i\omega(\pi a^2\rho)[1 - a^2(k^2 - K^2)\ln(Ca\sqrt{k^2 - K^2})] + \tfrac{1}{2}\pi^2\rho c a^4 k(k^2 - K^2) \tag{10.1.10}$$

(if $K > k$, omit the resistive term and use $K^2 - k^2$ in the second reactive term). This is equivalent to Eq. (7.3.6) if we neglect the logarithmic term and set $K \to 0$. Formula (10.1.10) shows, however, that the radiation resistance is strongly dependent on the wavelength $2\pi/K$ of the wave on the string, vanishing completely if this wavelength is shorter than the wavelength $2\pi/k$ of the sound in the medium. Under the next heading we shall show that the radiation resistance of a string of finite length differs even more from that given by Eq. (7.3.6), unless c_s is very much greater than c.

Arbitrary waveshape

The wave motion portrayed by Eqs. (10.1.8) is valid for the simple-harmonic wave in a string of infinite extent. Because of the attenuation of the wave for $\alpha^2(1 + \mu) < 1$, it implies an infinite amplitude at $z \to -\infty$. A more realistic situation would be that of a string of infinite extent, with only a finite portion of the string allowed to move transversally. We continue to neglect the effect of the outer boundaries of the acoustic medium, but we couple this infinite medium to a string of finite length, fastened rigidly at both ends, for example. We could consider the string as being a very narrow cylinder of finite length, but this would introduce difficulties at each end; it would be easier to continue to consider the string as having infinite extent but held motionless over all but a finite portion. The added motionless part makes it easier to fit boundary conditions, but does not appreciably affect the wave motion if the string radius is small compared with the wavelength.

Thus our next task is to survey the relationship between the transverse motion of a string, only a portion of which moves, to the coupled wave motion in the medium. We assume that the string displacement $\eta(z)e^{-i\omega t}$ is entirely in the yz plane and is simple-harmonic with frequency $\omega/2\pi$; $\eta(z)$ is nonzero over only a finite range of z. Adapting Eq. (7.1.17), using a

Green's function with zero radial gradient at $r = a$, we have

$$p_\omega(r) = -a\iint \frac{\partial p}{\partial r_0} G_\omega(r,\varphi,z \mid a,\varphi_0,z_0)\, d\varphi_0\, dz_0$$

where we have omitted the common factor $e^{-i\omega t}$. For motion in the yz plane this becomes

$$P_\omega(r) = -\rho a\omega^2 \int_0^{2\pi} \cos\varphi_0\, d\varphi_0 \int_{-\infty}^{\infty} \eta(z_0) G_\omega(r,\varphi,z \mid a,\varphi_0,z_0)\, dz_0$$

The appropriate Green's function is

$$G_\omega = \frac{i}{8\pi} \sum_{m=0}^{\infty} \epsilon_m \cos m(\varphi - \varphi_0) \int_{-\infty}^{\infty} e^{iK(z-z_0)}\, dK \times H_m(qr)\left[J_m(qr_0) - \frac{J'_m(qa)}{H'_m(qa)} H_m(qr_0)\right] \quad (10.1.11)$$

when $r_0 < r$, where $q^2 = k^2 - K^2$ and $k = \omega/c$. Inserting this in the integral for p and using the formula

$$H'_m(u)J_m(u) - J'_m(u)H_m(u) = \frac{2i}{\pi u}$$

derived from the last of Eqs. (7.3.2), we have for the radiated wave and the reaction force on the string,

$$p_\omega = \frac{\rho\omega^2}{2\pi} \cos\varphi \int_{-\infty}^{\infty} e^{iKz} \frac{H_1(qr)}{H'_1(qa)} \frac{dK}{q} \int_{-\infty}^{\infty} e^{-iKz_0} \eta(z_0)\, dz_0$$

$$\to -i\rho\omega^2 \cos\varphi \sqrt{\frac{2}{\pi i r}} \int_{-\infty}^{\infty} e^{iKz+iqr} \left[\frac{Y(K)}{q^{\frac{3}{2}} H'_1(qa)}\right] dK \qquad kr \to \infty \quad (10.1.12)$$

$$F_\omega(z) = -\pi\rho c^2 k^2 a^2 \int_{-\infty}^{\infty} e^{iKz} \left[\frac{H_1(qa)}{qaH'_1(qa)}\right] Y(K)\, dK$$

where $Y(K)$ is the Fourier transform of the string displacement

$$Y(K) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(z_0) e^{-iKz_0}\, dz_0 \qquad \eta(z) = \int_{-\infty}^{\infty} Y(K) e^{iKz}\, dK$$

If η were a traveling wave $Ae^{i\varkappa z}$, then $Y(K)$ would be the delta function $A\delta(K - \varkappa)$, and F would have the form of Eq. (10.1.4), with $\varkappa$ instead of K. If, however, η were some function which goes to zero rapidly enough so that the integral of $|\eta|^2$ over z from $-\infty$ to $+\infty$ is finite, then, according to Eq. (1.3.17), $Y(K)$ goes to zero, as $|K| \to \infty$, rapidly enough so that the integral of $|Y|^2$, over the whole range of K, is finite; in fact, so that

$$\int_{-\infty}^{\infty} |\eta(z)|^2\, dz = 2\pi \int_{-\infty}^{\infty} |Y(K)|^2\, dK$$

The spread of $Y(K)$ in K depends on the degree of "jaggedness" of η. If η is smoothly varying so that components of wavelength shorter than $\lambda_{\min}$ are unimportant, then $Y(K)$ will be negligible for values of $|K|$ larger than $2\pi/\lambda_{\min}$. In particular, if there is no "kink" or rapid change in slope of η with a radius of curvature smaller than about $20a$ (where a is the radius of the string), then Y is effectively zero for $|K|$ larger than about $1/3a$.

When such is the case, the quantity in brackets in the integral for $F_\omega(z)$ in Eqs. (10.1.12) can be approximated by the first term in its series expansion.

$$\begin{aligned}\frac{H_1(qa)}{qaH_1'(qa)} &\simeq -1 + a^2(k^2 - K^2)[\ln (Ca\sqrt{k^2 - K^2}) - \tfrac{1}{2}\pi i] \qquad |K| < k \\ &\simeq -1 - a^2(K^2 - k^2) \ln (Ca\sqrt{K^2 - k^2}) \qquad |K| > k\end{aligned} \tag{10.1.13}$$

(where $C = 0.8905$) over the range of K, for which $Y(K)$ is nonnegligible. If the string radius is very small, this approximation is valid even for string shapes with a fair amount of "jaggedness." For example, if η is zero for $|z| > \frac{1}{2}l$, as it would be if we were representing the motion of a string of finite length, held rigid at $\pm\frac{1}{2}l$, as long as $l \gg a$, the "kinks" at the two supports will not be perfectly sharp in actual practice, but will be curves instead of cusps, with a curvature radius of about $20a$ or greater. If this is the case, the integral for the reactive force can be greatly simplified.

$$F_\omega(z) \simeq \pi\rho c^2 k^2 a^2 \int_{-\infty}^{\infty} e^{iKz} Y(K)[1 + a^2(K^2 - k^2) \ln (Ca\sqrt{K^2 - k^2})]\, dK \tag{10.1.14}$$

where the quantity in brackets is real and positive at $K \to \infty$, and the contour of integration (C in Fig. 10.2) goes just above the real axis of K from $-\infty$ to 0 and just below the real axis from 0 to $+\infty$ in order that p_ω be an outward-traveling wave.

The first term in the brackets can be integrated immediately, for it is the inverse transform of Y. Thus, as was indicated in Eq. (10.1.10), the largest part of the reaction force is $\pi a^2 \rho \omega^2 \eta(z)$, an inertial reaction, proportional to $-\omega^2$ times the displacement η, the added mass load being equal to an amount of fluid medium equal to the volume of the string [this corresponds to the μ term in braces in Eqs. (10.1.6)]. The logarithmic term is more difficult; in fact, it can be worked out only when we know the formula for $Y(K)$ (or for η). It is important, however, for it is responsible for the resistive part of the reaction, the part producing energy loss. We of course do not know the exact form of η (in fact, we are in the process of determining its form).

But suppose we assume that this part of the reaction does not change the shape of the vibrating string very much, that its shape is roughly the same

as if it were vibrating in a vacuum, between supports at $\pm\frac{1}{2}l$, that is, that

$$\eta(z) \simeq \begin{cases} 0 & z < -\frac{1}{2}l \\ A\cos\dfrac{\pi z}{l} & -\frac{1}{2}l < z < \frac{1}{2}l \\ 0 & z > \frac{1}{2}l \end{cases}$$

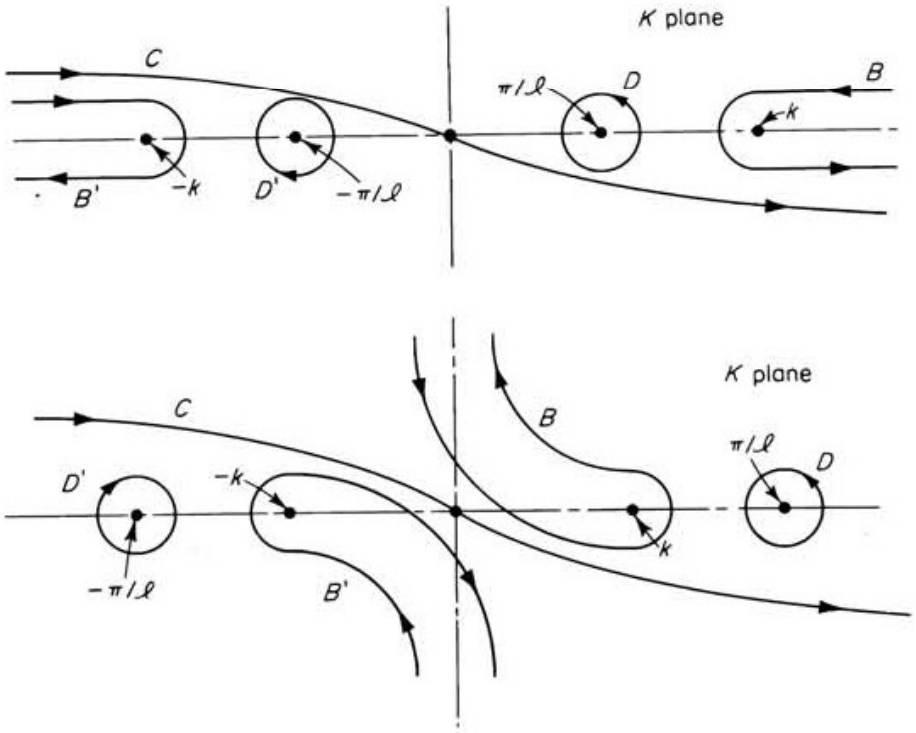


FIGURE 10.2
Contours of integration for the reaction integral.

The Fourier transform for this function is

$$Y(K) = -\frac{A}{2l}\frac{e^{iKl/2} + e^{-iKl/2}}{K^2 - (\pi/l)^2} \tag{10.1.15}$$

In this case the transverse force the medium exerts on the string, per unit length of string, in the range $-\frac{1}{2}l < z < \frac{1}{2}l$ (it does not matter what force is exerted on the rest of the string; it is being held motionless), becomes

$$F_\omega(z) \simeq \pi a^2\rho\omega^2 A\cos\frac{\pi z}{l} - \frac{\pi a^2\rho\omega^2 A a^2}{2l}\int_C \frac{K^2 - k^2}{K^2 - (\pi/l)^2} \times (e^{iKl/2} + e^{-iKl/2}) \ln (aC\sqrt{K^2 - k^2})e^{iKz}\, dK$$

where the contour of integration is the one marked C in Fig. 10.2.

For the exponential $e^{iK(\frac{1}{2}l+z)}$, if $z > -\frac{1}{2}l$, the contour must be closed by a semicircle around the upper half of the K plane, which can be shrunk down to the contour D around the pole of the integrand at π/l, plus the infinite loop B around the branch point at $K = k$. For the exponential $e^{-iK(\frac{1}{2}l-z)}$, if $z < \frac{1}{2}l$, the closing semicircle must be around the lower half-plane; it can be reduced to the circle D' around the pole at $-(\pi/l)$ plus the loop B' around the branch point $-k$. The shape of the loops B, B' depends on whether k is larger or smaller than π/l, in other words, whether the wavelength $\lambda_s = 2\pi/\alpha k$ of the wave in the string (if vibrating *in vacuo*) is smaller or larger than $\lambda = 2\pi/k$, the wavelength of sound in the medium, i.e., whether the string wave velocity $c_s = c/\alpha$ is smaller or larger than the medium velocity c. If $c_s > c$ $(\alpha < 1)$, then $\lambda_s = 2l$ will be greater than $\lambda = 2\pi/k$ and $\pi/k < l$; in this case the contours will be those shown in the upper half of Fig. 10.2. Conversely, if c_s is smaller than c, then k is smaller than π/l, and the contours to use are shown in the lower half of Fig. 10.2.

In either case, Eq. (1.2.15) shows that the integrals D and D', around the poles, when $|z| < \frac{1}{2}l$, are

$$D + D' = -\frac{\pi^2 i\rho\omega^2 A a^2}{l}\left(\frac{ie^{i\pi z/l}}{2\pi/l} + \frac{ie^{-i\pi z/l}}{2\pi/l}\right)\left[\left(\frac{\pi}{l}\right)^2 - k^2\right]\ln\left[aC\sqrt{\left(\frac{\pi}{l}\right)^2 - k^2}\right]$$

If $z > \frac{1}{2}l$, the contour D' does not exist, and the $e^{-i\pi z/l}$ term is missing; if $z < -\frac{1}{2}l$, the $e^{i\pi z/l}$ term vanishes. If $k > \pi/l$, the logarithmic factor has an imaginary part, $-(i\pi/2)$.

This part of the reaction force is proportional to $\cos(\pi z/l)$, that is, proportional to η (in the range $|z| < l/2$). But the integrals B and B' result in a function of z which is not proportional to $\cos(\pi z/l)$. The difference in value of the logarithmic factor between the incoming and outgoing parts of B is $i\pi$; all other factors are unchanged as the path goes around $K = k$. Thus, if $k > \pi/l$,

$$B = -\frac{i\pi^2 a^2\rho\omega^2 A a^2}{2l}\int_k^\infty \frac{K^2 - k^2}{K^2 - (\pi/l)^2}\, e^{iK(\frac{1}{2}l+z)}\, dK$$

and

$$B' = -\frac{i\pi^2 a^2\rho\omega^2 A a^2}{2l}\int_k^\infty \frac{K^2 - k^2}{K^2 - (\pi/l)^2}\, e^{iK(\frac{1}{2}l-z)}\, dK$$

These integrals can be made to converge more rapidly by changing the path of integration to one from k to $+i\infty$. It can be seen that $B + B'$ is not proportional to $\cos(\pi z/l)$, and thus that the total expression for F is not proportional to η. Therefore $A\cos(\pi z/l)$ is not an exact solution for the shape of a string vibrating between fixed supports, when in contact with an acoustic medium.

Variational calculation of frequency

Since an exact solution in closed form is not possible, we set up a variational procedure to find a close approximation, at least for the natural frequency of vibration of the string. We are considering a string of infinite extent, held fixed along the z axis except for the portion between $-\frac{1}{2}l$ and $+\frac{1}{2}l$, which can move transversally in the yz plane, with a displacement $\eta(z)e^{-i\omega t}$. The motion is subject to the string tension $T = \pi a^2 \rho_s c_s^2 = \pi a^2 \rho c^2 (1/\mu\alpha^2)$ $(\mu = \rho/\rho_s,\ \alpha = c/c_s)$ and to the reaction force $F_\omega(z)e^{-i\omega t}$ of the surrounding acoustic medium. We wish to determine the shape of the string when it vibrates under the influence of these forces, and more important, we wish to compute the natural frequency $\omega_n = k_n/c$ of this motion. The real part of $\omega_n/2\pi$ is the frequency of free vibration; the imaginary part of ω_n (which must be negative) is the damping constant, related to the loss of energy by radiation. What we have seen in the foregoing discussion is that the damping constant depends strongly on α, the ratio of wave velocities in medium and string.

We first set down the equation of motion for the free portion of the string. Rewriting the equation for F somewhat, to assist in the subsequent manipulations, we have

$$-\pi a^2 \rho_s \omega^2 \eta = T \frac{\partial^2 \eta}{\partial z^2} + F_\omega(z) \qquad -\tfrac{1}{2}l < z < \tfrac{1}{2}l$$

or

$$\omega^2 \eta(z) = -\left(\frac{c}{a}\right)^2 \left[\frac{\partial^2 \eta}{\partial z^2} + \frac{\mu\omega^2}{2\pi} \int_{-\frac{1}{2}l}^{\frac{1}{2}l} G_\omega(z \mid z_0)\eta(z_0)\, dz_0\right] \qquad (10.1.16)$$

where

$$G_\omega = \int_{-\infty}^{\infty} e^{iK(z-z_0)} \left[\frac{H_1(qa)}{qaH_1'(qa)}\right] dK$$

We multiply this equation by $\eta(z)$ and integrate over z from $-\frac{1}{2}l$ to $+\frac{1}{2}l$, obtaining

$$\omega^2 = \frac{1}{\int \eta^2\, dz} \left[\left(\frac{c}{a}\right)^2 \int \left(\frac{\partial \eta}{\partial z}\right)^2 dz + \frac{\mu\omega^2}{2\pi} \int \eta(z)\, dz \int G_\omega(z \mid z_0)\eta(z_0)\, dz_0\right] \qquad (10.1.17)$$

with all integrations from $-\frac{1}{2}l$ to $+\frac{1}{2}l$. We have integrated the first term by parts, utilizing the fact that $\eta = 0$ at the end points of the integration.

The expression on the right of this equation is a variational expression; if we vary the shape of η, the coefficient of $\delta\eta$ inside the integral is zero if η is a solution of Eq. (10.1.16); so for shapes near this, the variation of the expression is second order. Therefore, if we insert a trial function $\chi(\gamma)$ instead of η in the expression of Eq. (10.1.17) and adjust γ so the expression's derivative with respect to γ is zero, the difference between χ and the correct η will be minimal, and the expression's value will differ from the correct ω^2 to the second order of this difference. We know, of course, that there are an infinite number of exact solutions and a corresponding infinity of allowed

values of ω, corresponding to the fundamental and the higher harmonics of the vibrating string. The lowest allowed value of ω^2 is that for the fundamental, so, as we vary χ, the absolute minimum of the variational expression is the square of the ω for the fundamental mode. We shall see later how to determine the higher harmonics. Note that the variational expression is independent of the magnitude of χ, since η^2 (or χ^2) enters both numerator and denominator. Since Eq. (10.1.16) is linear and homogeneous in η, a constant times a solution is a solution.

If the coupled medium is light enough ($\mu = \rho/\rho_s \ll 1$) so that the radiative reaction is small compared with the effect of string tension, we should expect that η does not differ much from the shape of the same string *in vacuo*. For the fundamental mode, therefore, we can insert $\cos(\pi z/l)$ for χ in the variational expression and expect that the resulting value would differ very little from the correct value of ω^2 for the fundamental. Thus a second-order approximation to the fundamental eigenvalue is

$$\omega^2 \simeq \frac{2}{l}\left\{\left(\frac{\pi c}{\alpha l}\right)^2 \frac{l}{2} + 2\pi\mu\omega^2 \int_{-\infty}^{\infty} |Y(K)|^2 \left[\frac{H_1(qa)}{qaH_1'(qa)}\right] dK\right\}$$

where $\alpha = c/c_s$; $q^2 = k^2 - K^2$; and $Y(K)$ is the Fourier transform of $\cos(\pi z/l)$, given in Eq. (10.1.15), without the factor A.

If $l \gg a$, we can assume, as before, that Y^2 is negligible in the range of integration, where approximation (10.1.13) is no longer valid, and the simpler form can be used for the quantity in brackets in the integral. Then we can use Eq. (1.3.17) for the first term of the approximate form, and the variational expression becomes

$$\omega^2 \simeq \left(\frac{\pi c_s}{l}\right)^2 - \mu\omega^2 - \frac{\pi\mu\omega^2 a^2}{l^3}\int_{-\infty}^{\infty} (K^2 - k^2) \times \left[\frac{e^{iKl/2} + e^{-iKl/2}}{K^2 - (\pi/l)^2}\right]^2 \ln\left(aC\sqrt{K^2 - k^2}\right) dK \quad (10.1.18)$$

the contour being the one marked C in Fig. 10.2, as before. Also as before, this can be broken into the set $B + D$ for the terms $1 + e^{iKl}$ in the denominator, and the set $B' + D'$ for the terms $1 + e^{-iKl}$. The integrals D and D' around the second-order poles are evaluated by means of Eq. (1.2.13), and we have

$$\omega^2 \simeq \left(\frac{\pi c_s}{l}\right)^2 - \mu\omega^2 - \mu\omega^2 a^2\left[\left(\frac{\pi}{l}\right)^2 - k^2\right] \ln\left[aC\sqrt{\left(\frac{\pi}{l}\right)^2 - k^2}\right] + B + B'$$

for $k < \pi/l$. If $k > \pi/l$, the logarithmic factor becomes

$$\ln\left[aC\sqrt{k^2 - \left(\frac{\pi}{l}\right)^2}\right] - \tfrac{1}{2}i\pi$$

These first three terms for the eigenvalue are the same as those in Eq. (10.1.6) for a traveling wave of infinite extent. If k is smaller than π/l (i.e., if $c + \mu c > c_s$), these first three terms are real; a wave of infinite extent, if it moves slower than the speed of sound in the medium (times $1 + \mu$), loses no energy by radiation to $r \to \infty$, as was noted earlier. For the finite string, however, some energy is lost even when $c(1 + \mu) > c_s$, because of the sudden change of slope of η at $z = \pm\frac{1}{2}l$. This contribution to the imaginary part of ω^2 comes from the contours B and B' of Fig. 10.2, which arise because of the form of $Y(K)$, that is, because $\eta(z)$ is zero for $|z| > \frac{1}{2}l$. The form of these integrals differs, depending on whether k is greater or less than π/l.

When $k < \pi/l$ ($l < \frac{1}{2}\lambda$ or $c + \mu c > c_s$), the contours in the lower part of Fig. 10.2 are to be used. The difference in the logarithmic factor, between the incoming and outgoing part of B, is $i\pi$; that for B' is $-i\pi$. The resulting integrals, for those parts going from $K = 0$ to $K = \pm i\infty$, cancel each other; the parts from 0 to $\pm k$ add, so that

$$\begin{aligned} B + B' &\simeq -\frac{2i\pi^2\mu\omega^2a^2}{l^3}\int_0^k \frac{k^2 - K^2}{[(\pi/l)^2 - K^2]^2}(1 + e^{iKl})\,dK \\ &= -\frac{2i\mu\omega^2k^3a^2}{\pi^2}\int_0^{\pi/2} \frac{\sin^3\theta(1 + e^{ikl\cos\theta})}{[1 - (kl/\pi)^2\cos^2\theta]^2}\,d\theta \\ &\to -\frac{2i\mu\omega^2k^3a^2}{\pi^2}\left(\frac{4}{3} + \frac{i}{4}kl\right) \qquad 2l \gg \lambda \end{aligned}$$

Thus, when the wave velocity in the string is very much smaller than that in the medium, the formula for the resonance frequency of the fundamental mode, correct to first order in the small quantity $\mu\pi^2a^2/l^2$, is

$$\omega_1 \simeq \frac{\pi c\gamma_s}{l}\left\{1 - \frac{\mu/2}{1 + \mu}\left(\frac{\pi a}{l}\right)^2\left[(1 - \gamma_s^2)\ln\left(\frac{a\pi C}{l}\sqrt{1 - \gamma_s^2}\right) + \frac{8i}{3\pi}\gamma_s^3\right]\right\} \tag{10.1.19}$$

when $c_s \ll c$, where $\gamma_s^2 = [c_s^2/c^2(1 + \mu)] = [1/\alpha^2(1 + \mu)]$, and $C = 0.8905$.

The last term in brackets is the only imaginary part of ω_1; when $c_s < c$ the sole reason for loss of energy is the finite extent of the string displacement. This result differs radically from that of the simple Eq. (7.3.6); energy loss from a string of finite extent is quite different from energy loss from a string of infinite extent, particularly when c_s is less than c. Of course, with the strings used in musical instruments, the coupling with the air is not primarily direct, but is transmitted through the string supports, which are not rigid, to a sounding board, which is a much more efficient acoustic radiator than is the string itself. Consequently, Eq. (10.1.19) is not applicable for the calculation of the damping constant of a violin or piano string, for example; it applies only to the case of the string with truly rigid supports.

In the other limit, when $c_s \gg c$ $(kl \gg \pi)$, the contours B and B' are those shown in the upper half of Fig. 10.2, with the integrations carried from k to $\pm i\infty$. The approximate result is

$$\omega_1 \simeq \frac{\pi c\gamma_s}{l}\left\{1 + \frac{\mu/2}{1+\mu}\left(\frac{\pi a}{l}\right)^2(\gamma_s{}^2 - 1)\ln\left(\frac{\pi aC}{l}\sqrt{\gamma_s{}^2 - 1}\right)\right.$$
$$\left. - \frac{i\mu}{1+\mu}\left(\frac{\pi a}{l}\right)^2\left[\frac{1+\gamma_s{}^2}{\pi}\coth^{-1}(\gamma_s) - \frac{\gamma_s}{\pi} + \frac{\pi}{2}(\gamma_s{}^2 - 1)\right]\right\} \qquad c_s \gg c \tag{10.1.20}$$

which has a larger damping constant than does the case for $c_s \ll c$. For the intermediate range, when c_s is roughly equal to c, the integrals B and B' must be computed more accurately.

Calculations for the higher modes can be carried out by using $\chi = \cos(\pi nz/l)$ for n odd or $\chi = \sin(\pi nz/l)$ for n even in Eq. (10.1.17). But these are subject to increasing uncertainties, as the following discussion will indicate. We can write Eq. (10.1.16) as $\omega^2\eta = \mathfrak{G}_\omega\eta$, where $\mathfrak{G}_\omega$ is the differential and integral operator written on the right of Eq. (10.1.16). When the exact solution η_n for the nth harmonic is used, the quantity $\mathfrak{G}_\omega\eta_n$ will equal $\omega_n{}^2\eta_n$. The eigenvalues $\omega_n{}^2$ are complex because of the energy loss to the medium, but their real parts are all positive and can be arranged in an ordered sequence so that $\mathrm{Re}\ \omega_{n+1}^2 > \mathrm{Re}\ \omega_n{}^2$, with $\omega_1{}^2$ the eigenvalue for the fundamental. Furthermore, it is not difficult to show that the η_n's are mutually orthogonal; i.e.,

$$\int_{-l/2}^{l/2}\eta_n(z)\eta_m(z)\,dz = \begin{cases} 0 & n \neq m \\ N_n & n = m \end{cases}$$

where the complex quantity N_m is the normalizing constant for η_m.

Choosing an arbitrary form for η is equivalent to choosing an arbitrary expansion $\eta = \sum A_n\eta_n$ in terms of the eigenfunctions. Inserting this into Eq. (10.1.17) shows that the variational equation is equivalent to

$$\frac{\sum A_n{}^2N_n\omega_n{}^2}{\sum A_n{}^2N_n} = W - iV$$

where $V \ll W$ in most cases. Of course, we do not know the η_n's or ω_n's (otherwise we should not be using the variational procedure), but this representation of the variational expression is useful because it tells us more about the limitations of the method. For example, since $\mathrm{Re}\ \omega_1{}^2$, though positive, is smaller than the real part of any other $\omega_n{}^2$, we can see that W can never be smaller than $\mathrm{Re}\ \omega_1{}^2$. Thus, as we vary the A_n's by varying the shape of χ, the absolute minimal value of W is $\mathrm{Re}\ \omega_1{}^2$, which only occurs when $\chi = \eta_1$ ($A_n = 0$, for all n except $n = 1$).

To find $\omega_2{}^2$ by means of the variational procedure, we must choose a series for χ which has $A_1 = 0$; in other words, we must arrange that the trial

function, though variable in form, always is orthogonal to η_1 (i.e., that $\int \chi\eta_1\, dz = 0$). When $A_1 = 0$, the argument used for ω_1^2 works for ω_2^2; the minimal value of W for all shapes orthogonal to η_1 is Re ω_2^2, which occurs only when $\eta = \eta_2$. And so on. But we do not know η_1 exactly; so we cannot find the family of trial functions exactly orthogonal to η_1. If we make the trial functions for η_2 orthogonal to the best approximation we can find for η_1, these trial functions may have a small component of η_1 in them (A_1 is not exactly zero), and thus we cannot guarantee that the minimal value of W for these functions is Re ω_2^2; it may be somewhat smaller than this. And if we then go on to ω_3^2 by using a set of trial functions orthogonal to the approximations for η_1 and η_2, we are compounding our errors.

Thus the variational procedure used here (sometimes called the Rayleigh-Ritz variational procedure) is most accurate in determining the fundamental mode, and rapidly loses utility when it is used to compute higher modes. In contrast, the procedure used in Eq. (9.4.19), though it involves more computation, loses efficacy more slowly as n is increased. This will be illustrated in a problem.

In the case of the string, we have illustrated the point, mentioned at the beginning of this section, that the behavior of the wave representing coupled motion of string and medium depends markedly on the ratio α of wave velocities in the two systems, when uncoupled. The other point mentioned earlier, that there are always two kinds of waves, with two velocities, has not been emphasized in this example. The wave of Eq. (10.1.9) is obviously related to that of the string *in vacuo*. There are, of course, waves in the air, going with velocity c, which move the string slightly; these would be the other set of waves. In the case of the string, however, the coupling is so small that the characteristics of this second set are not appreciably affected by the presence of the string (at least in regard to wave velocity); we can calculate their space distribution by use of Sec. 8.1.

If the interacting systems involve greater coupling, such as a membrane plus surrounding air, the second class of waves may have its velocity also modified by the coupling. In fact, as we shall see, for some wave-velocity ratios, waves in the medium may be "trapped" by the plate; so they travel parallel to the plate surface, the majority of the energy being carried by the medium near the plate. This can be demonstrated by solving the problem of the coupled motion of a thin plate with the medium surrounding it.

Free motion of an infinite plate

We consider a thin plate of thickness $2h$ and density ρ_p, with modulus of elasticity Q and Poisson ratio s (usually equal to $\frac{1}{3}$). The analogous, but simpler, example of a membrane in contact with a medium will be given as a problem. According to Eq. (5.3.9), the equation of motion for the plate's transverse displacement, when driven by a simple-harmonic force $F_\omega(x,y)e^{-i\omega t}$,

is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\eta-\gamma^4\eta=\frac{\gamma^4}{2kh\rho_p c\omega}F_\omega \tag{10.1.21}$$

where $\gamma^4=3\omega^2\rho_p(1-s^2)/Qh^2$ and $k=\omega/c$ as before, ρ and c being the density and sound velocity, respectively, of the medium. As an example of magnitudes, for a sheet of steel 2 mm thick, with $\rho_p=8$ and $Q=2\times10^{12}$ dynes per sq cm, parameter γ is approximately equal to $\sqrt{\nu/4{,}400}$ (ν being the frequency), and the ratio of the wave velocity of the plate ω/γ to the velocity of sound c in the air is $(c_p/c)\simeq\sqrt{\nu/10^4}$. The plate, of course, is a dispersive medium for transverse waves.

First let us ask what sort of free sinusoidal waves can travel along the plate in contact with the medium. We assume that the transverse displacement of the plate and its transverse velocity are

$$\eta=A\exp(iKx-i\omega t)\qquad u_z=-i\omega\eta$$

where we have placed the xy plane at the mid-plane of the plate when in equilibrium. The pressure waves above and below the plate must have the same dependence on x; because of symmetry the two are equal and opposite in sign at the plate surface.

$$p_+=P\exp(iKx+izq-i\omega t)\qquad p_-=-P\exp(iKx-izq-i\omega t)$$

$$u_z=\pm\frac{q}{\rho ck}p\qquad q=\sqrt{k^2-K^2}\qquad\text{or}\qquad i\sqrt{K^2-k^2}$$

where we have assumed that h is so small compared with λ that we can assume that the boundary conditions are at $z=0$. The reaction force of these two waves on the plate is $F_\omega=(p_--p_+)_{z=0}$. Therefore there are two equations relating A and P, one equating the normal velocity of plate and neighboring air, the other with $-2P\exp(iKx-i\omega t)$ for F_ω in Eq. (10.1.21). The combination relates the amplitude A of plate motion to amplitude P of acoustic wave and also fixes the relationship between plate wavenumber and frequency $kc/2\pi$ at which there is free vibration.

$$-i\omega A=\left(\frac{q}{\rho ck}\right)P\qquad K^4-\gamma^4=i\frac{\rho\gamma^4}{\rho_p qh}=i\frac{\mu\gamma^4}{q} \tag{10.1.22}$$

where $\mu=\rho/\rho_p h$.

First let us find the various roots of the equation for K, regardless of whether the plate wave goes to infinity at $x=\pm\infty$, but insisting that the radiated wave is outgoing and noninfinite at $z=\pm\infty$ (i.e., that only the sheet of $\sqrt{K^2-k^2}$ with positive-real or positive-imaginary part be used). There are four such roots, two real and two imaginary, when $k<\gamma$ (when the plate velocity is smaller than that in air). When both μ/γ and k/γ are

quite small, approximate expressions for K are

$$\begin{aligned} K = \pm K_+ \qquad & K_+ \simeq \gamma\left[1 + \frac{\mu}{4(\gamma^2 - k^2)^{\frac{1}{2}}}\right] \\ & q_+ \simeq i\sqrt{\gamma^2 - k^2}\left[1 + \frac{\mu\gamma^2}{4(\gamma^2 - k^2)^{\frac{3}{2}}}\right] \\ K = \pm iK_- \qquad & K_- \simeq \gamma\left[1 + \frac{i\mu}{4(\gamma^2 + k^2)^{\frac{1}{2}}}\right] \\ & q_- \simeq \sqrt{\gamma^2 + k^2}\left[1 + \frac{i\mu\gamma^2}{4(\gamma^2 + k^2)^{\frac{3}{2}}}\right] \qquad \mu = \frac{\rho}{\rho_p h} \end{aligned} \tag{10.1.23}$$

For the steel plate 2 mm thick, μ/γ is roughly equal to $\sqrt{1/100\nu}$ when the plate is in air, and is $\sqrt{10{,}000/\nu}$ when it is in water. The other roots, near $K = \pm k$, are not allowed, since they correspond to $\sqrt{K^2 - k^2}$ being negative-imaginary, when $k < \gamma$, which is not allowed. When μ is not small compared with γ, these formulas are not valid, but the roots still are two real and two nearly pure-imaginary.

The first pair of roots corresponds to wave propagation along the plate with no attenuation; the corresponding wave in air attenuates with increasing $|z|$, so that no energy is radiated away to $|z| \to \infty$. The second pair, being imaginary, cannot correspond to free-wave motion in an infinite plate; they are used only to fit boundary conditions in a finite or semi-infinite plate. We should note, however, that the corresponding q is nearly real, corresponding to energy loss to $z \to \pm\infty$; the presence of a boundary induces loss of energy at the edge even for the case $k < \gamma$. The wave velocity for the roots $\pm K_+$ is close to ω/γ, the wave velocity for the plate in vacuum, when $\mu < \gamma$. Most of the energy is in the plate.

At higher frequencies, when $k > \gamma$, all six roots are allowed. Formulas for small values of μ/γ are

$$\begin{aligned} K = \pm K_+ \qquad & K_+ \simeq \gamma\left[1 + \frac{i\mu}{4(k^2 - \gamma^2)^{\frac{1}{2}}}\right] \\ & q_+ \simeq \sqrt{k^2 - \gamma^2}\left[1 - \frac{i\mu\gamma^2}{4(k^2 - \gamma^2)^{\frac{3}{2}}}\right] \\ K = \pm iK_- \qquad & K_- \simeq \gamma\left[1 + \frac{i\mu}{4(k^2 + \gamma^2)^{\frac{1}{2}}}\right] \\ & q_- \simeq \sqrt{k^2 + \gamma^2}\left[1 + \frac{i\mu\gamma^2}{4(k^2 + \gamma^2)^{\frac{3}{2}}}\right] \\ K = \pm K_s \qquad & K_s \simeq k\left[1 + \frac{1}{2}\left(\frac{\mu}{k}\right)^2\left(\frac{\gamma^4}{k^4 - \gamma^4}\right)^2\right] \qquad q_s \simeq i\mu\frac{\gamma^4}{k^4 - \gamma^4} \end{aligned} \tag{10.1.24}$$

The first root again corresponds to the free wave of the plate in vacuum. This time ($c_p > c$), some of its energy can radiate away to $|z| \to \infty$. The pressure wave above the plate is, approximately,

$$p_+ \simeq -i\rho c^2 \frac{kA}{\sqrt{1-(\gamma/k)^2}} \exp\left[ik(x\cos\Psi + z\sin\Psi) - \frac{\rho\cot\Psi}{4\rho_p h}x_0 - i\omega t\right] \tag{10.1.25}$$

where $\cos\Psi = \gamma/k$, and $x_0 = x - z\cot\Psi$. This is a plane wave, moving in a direction at angle Ψ to the x axis, with velocity $\omega/k = c$, with amplitude proportional to the plate amplitude at the point x_0, the point at which the wave which arrives at (x,z) was generated. The geometry is similar to that of Fig. 10.1, with (z,r) changed to (x,z) and with the present wave a plane wave, instead of the conical wave produced by the string. Since high-frequency wave motion can radiate to $|z| \to \infty$, the plate wave attenuates with x, and the acoustic wave attenuates with x_0.

In contrast to this wave is the *surface wave*, corresponding to the third pair of roots (only allowed when $k > \gamma$). This wave has most of its energy carried in the medium; since q is imaginary and small, the ratio of P to A is large, and the disturbance in the medium attenuates slowly as $|z|$ increases. This wave has velocity slightly less than the acoustic wave velocity in the medium. The plate moves just enough to cause the sound wave to "cling" to the plate, with its wavefronts normal to the plate and with wave amplitude largest near the plate, diminishing exponentially as $|z|$ increases. This sort of wave, produced by plate-medium coupling when $c_p > c$, has no attenuation in the x direction; thus no energy is lost to $|z| \to \infty$. Such a wave does not exist for a plate in vacuum, nor does it exist in a medium without a plate; it requires the interaction of both.

Linear driving force

If the plate is subject to a transverse driving force $F(K)e^{iKx-i\omega t}$ per unit area of plate, in addition to the radiation reaction of the surrounding medium, the steady-state waves in the plate and medium can be obtained from Eqs. (10.1.21) and (10.1.22).

$$\eta = A(K)e^{iKx-i\omega t} \qquad \left(K^4 - \gamma^4 - \frac{\mu\gamma^4}{\sqrt{K^2-k^2}}\right)A(K) = \frac{\gamma^4}{2h\rho_p\omega^2}F(K)$$

$$p = \pm P(K)\exp\left(iKx \mp z\sqrt{K^2-k^2} - i\omega t\right) \qquad P(K) = -\frac{\rho\omega^2}{\sqrt{K^2-k^2}}A(K) \tag{10.1.26}$$

where the upper signs in the expression for p are used for $z > 0$, the lower signs for $z < 0$. For the p wave to have the right asymptotic behavior, the square root must be in the first quadrant of the complex plane; we take it to be real and positive as K approaches $+\infty$.

If the driving force is simple-harmonic but not a sinusoidal function of x, the force shape can be built up as a Fourier integral.

$$f(x) = \int_{-\infty}^{\infty} F(K)e^{iKx}\,dK \qquad F(K) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-iKx}\,dx$$

The steady-state driven motion is then $\eta = a(x)e^{-i\omega t}$, and $p = \pm p(x,\pm z)e^{-i\omega t}$, where

$$a(x) = \int_{-\infty}^{\infty} A(K)e^{iKx}\,dK = \frac{\gamma^4}{2h\rho_p\omega^2}\int_C \frac{F(K)\exp(i\kappa x)\sqrt{\kappa^2 - k^2}}{(K^4 - \gamma^4)\sqrt{K^2 - k^2} - \mu\gamma^4}\,dK$$

$$p(x,z) = \tfrac{1}{2}\mu\gamma^4\int_C \frac{F(K)\exp(iKx - z\sqrt{K^2 - k^2})}{(K^4 - \gamma^4)\sqrt{K^2 - k^2} - \mu\gamma^4}\,dK \qquad z > 0 \tag{10.1.27}$$

where the contour of integration is the one marked C in Fig. 10.3, and $\mu = \rho/\rho_p h$. When x is positive this contour must be completed by a semicircle in the upper half of the K plane. It then surrounds the roots K_+ and iK_- of Eq. (10.1.23) if $k < \gamma$, or the roots K_+, iK_-, and K_s of Eq. (10.1.24) if $k > \gamma$. It also must avoid the branch cut, which starts from $K = k$ and goes to infinity in the first quadrant.

If the driving force $f(X)$ is a line force $F_0\delta(x)$ along the line $x = 0$, the transform is simply $F(K) = F_0/2\pi$, having no additional singular points in the K plane. In this case the integrals reduce to circuits around the two or three poles of the integrand, plus a U-shaped integral around the branch cut, as shown in Fig. 10.3. If $x < 0$, the circuit is completed around the lower half-plane, and the circuits are similar ones around the other poles, with a U around $-k$ going to $-i\infty$. To compute the integrals around the poles we must calculate the residues (see page 15) at each, by setting $K = K_+ + \epsilon$, etc., and finding the coefficient of $1/\epsilon$ as $\epsilon \to 0$. If $\mu \ll 1$ and $k > \gamma$, we can use the approximate formulas of Eqs. (10.1.24) and find the residues of $e^{iKx}/[(K^4 - \gamma^4)\sqrt{K^2 - k^2} - \mu\gamma^4]$ to be $(1/4\gamma^2K_+)e^{iK_+x}$ at K_+, $(i/4\gamma^2K_-)e^{-K_-x}$ at iK_-, and $(\mu^2/k\gamma^4)[\gamma^4/(k^4 - \gamma^4)]^3$ at K_s.

Therefore, for $\mu \ll 1$ and for $x > 0$, the integrals around the poles for $a(x)$ are

$$D + D' + D'' \simeq \frac{i\gamma^2F_0}{2h\rho_p\omega^2}\left[\frac{e^{iK_+x}}{4K_+} + \frac{ie^{-K_-x}}{4K_-} + \frac{\mu^2}{k\gamma^2}\left(\frac{\gamma^4}{k^4 - \gamma^4}\right)^3 e^{iK_sx}\right] \qquad \text{for } \gamma < k$$

The first two terms together have zero slope at $x = 0$ and thus fit smoothly onto the symmetrical pair for $x < 0$. The third term, corresponding to a surface wave of small amplitude, does not join symmetrically to its opposite number for $x < 0$; the integral around the branch point cancels out this discontinuity in slope. Remembering that $K^2 - k^2$ is real and positive

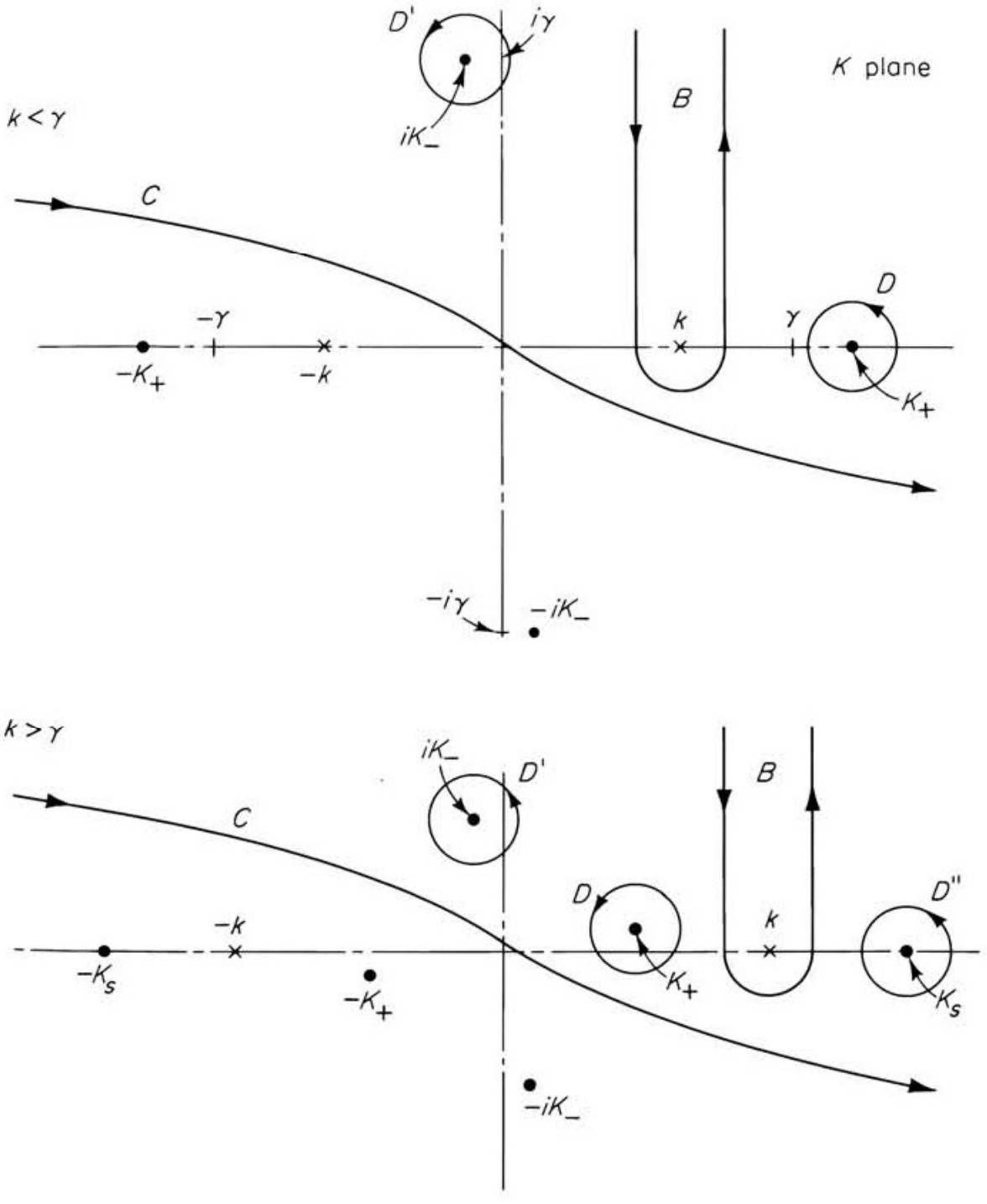


FIGURE 10.3
Contours for the integrals for η and p of Eq. (10.1.27).

when $K \to +\infty$, and thus that $\sqrt{K^2 - k^2} = e^{i\pi/4}\sqrt{u(2k + iu)}$ along the upgoing half of B (with $K = k + iu$) and $\sqrt{K^2 - k^2} = -e^{i\pi/4}\sqrt{u(2k + iu)}$ along the downgoing part, this integral, for $x > 0$, is

$$B_\eta \simeq \frac{-F_0}{2\pi\rho c^2 k^2} e^{ikx} \int_0^\infty e^{-ux} \frac{e^{3\pi i/4}\sqrt{u(2k + iu)}\, du}{1 - i(1/\mu\gamma^4)(k^4 + \cdots + u^4 - \gamma^4)^2 u(2k + iu)}$$

$$\to \frac{2\pi i F_0}{\rho c^2 (2\pi i k x)^{\frac{3}{2}}} e^{ikx} \qquad x \to \infty \qquad \to -\frac{iF_0}{2\rho c^2}\left(\frac{\mu\gamma^4/k}{k^4 - \gamma^4}\right)^3 \qquad x \to 0 \tag{10.1.28}$$

This just cancels the integral D'' at $x = 0$, thus leaving no discontinuity in η there. The asymptotic formula indicates that this part of the wave does not attenuate exponentially, but falls off as $1/x^{\frac{3}{2}}$. This corresponds to a cylindrical wave in the medium, centered at the line source at $x = 0$, as will be demonstrated by computing p.

The integral around the branch point

The residues around the three poles in the integral for p are calculated the same way as were those for η. The integral around the branch cut from k to ∞ requires further discussion. We can write it

$$B_p = \frac{-i\mu\gamma^4 F_0}{4\pi} \int_k^{i\infty} \left[\frac{\exp(iKx + iz\sqrt{k^2 - K^2})}{(K^4 - \gamma^4)\sqrt{k^2 - K^2} - i\mu\gamma^4} + \frac{\exp(iKx - iz\sqrt{k^2 - K^2})}{(K^4 - \gamma^4)\sqrt{k^2 - K^2} + i\mu\gamma^4} \right] dK \tag{10.1.29}$$

When x and/or z is large, both terms in the integrand oscillate rapidly over most of the path of integration. In such cases it is useful to find a point in the allowed region of the K plane, where the argument of the exponential has zero slope; close to this point the integrand ceases to oscillate rapidly, and the majority of the value of the integral arises from this region.

The derivative with respect to K of the exponent of the first term is zero when

$$K = \frac{kx}{\sqrt{x^2 + z^2}} = k \cos\phi$$

where $\phi = \tan^{-1}(z/x)$ is the angle between the radius vector $\mathbf{r}$ from the origin and the x axis. Near this point on the K plane, when $K = k\cos\phi + \epsilon$, the exponent of the first term has the form

$$iKx + iz\sqrt{k^2 - K^2} \simeq ikr - \tfrac{1}{2}i\frac{r^3}{kz^2}\epsilon^2 + \cdots \qquad r^2 = x^2 + z^2$$

If the path of integration is deformed so that it crosses the real K axis at $k\cos\phi$ at an angle of either 45 or 135° (i.e., if ϵ is either $u\sqrt{i}$ or $u\sqrt{-i}$, with u real), the integrand does not change phase, and thus does not oscillate near $u = 0$. But if $\epsilon = u\sqrt{i}$, the exponential increases as $|u|$ is increased; so the point $K = k\cos\phi$ is a minimum for the integrand in this region, whereas if $\epsilon = u\sqrt{-i}$, the point is a maximum. For this reason a point of zero derivative is called a *saddle point* of the function.

If we run the path of integration through the saddle point $K = k\cos\phi$ at 135°, as shown in Fig. 10.4, it is possible to route the rest of the path so that by far the largest contribution to the integral comes from the region close to the saddle point. And if kr is very large, the integrand is non-negligible only for small values of $|u|$. The first approximation to the integral for kr large is then obtained by using the first two terms in the expansion of the exponent and by extending the range of integration to $\pm\infty$. Thus the asymptotic approximation for the first term in B can be written

$$\begin{aligned} B_p &\simeq -\frac{i\sqrt{-i}\gamma^4\mu F_0 e^{ikr}}{4\pi[(k^4\cos^4\phi - \gamma^4)k\sin\phi - i\mu\gamma^4]}\int_{-\infty}^{\infty} e^{-(r/2k\sin^2\phi)u^2}\,du \\ &\simeq -\frac{i\mu\gamma^4 kF_0\sin\phi\, e^{ikr}}{2\sqrt{2\pi ikr}[(k^4\cos^4\phi - \gamma^4)k\sin\phi - i\mu\gamma^4]} \qquad kr, kz \gg 1 \\ &\to \frac{ikF_0\tan\phi}{2\sqrt{\pi ikx}}\, e^{ikx} \qquad kr \to \infty;\ \phi \to 0 \end{aligned} \tag{10.1.30}$$

There is no saddle point in the first quadrant for the second term in the brackets of Eq. (10.1.29); when $kr \gg 1$, the second integrand oscillates continuously, no matter what allowable path is taken from $K = k$ to $K = i\infty$. Thus the expression of Eq. (10.1.30) is the complete expression for the asymptotic form of integral B_p, as written. We note that it is a cylindrical wave, with origin the origin of the x, z coordinates (i.e., the location of the line force). Its amplitude dies out inversely as the square root of r; it is dipole radiation, since its amplitude goes to zero along the plate ($\phi = 0°$ and 180°).

Having reached this stage, we must point out that the arrangement of the branch cut to go through the saddle point $K = k\cos\phi$ has an important effect on the contour integral around the pole at $K = K_+$. If $K = k\cos\phi$ is greater than γ, the cut goes above the pole at K_+, as shown in the upper part of Fig. 10.4. In this case the denominator of the first integral goes to zero at $K = K_+$, as defined in Eq. (10.1.24); therefore K_+ is a pole for the integrand. But if $k\cos\phi$ is smaller than γ, the cut goes below K_+, factor $\sqrt{k^2 - K^2}$ in the numerator changes sign, and K_+ is no longer a pole for the integrand. Therefore the contour integral around K_+ is to be included in the formula for *p only as long as $k\cos\phi$ is larger than γ.*

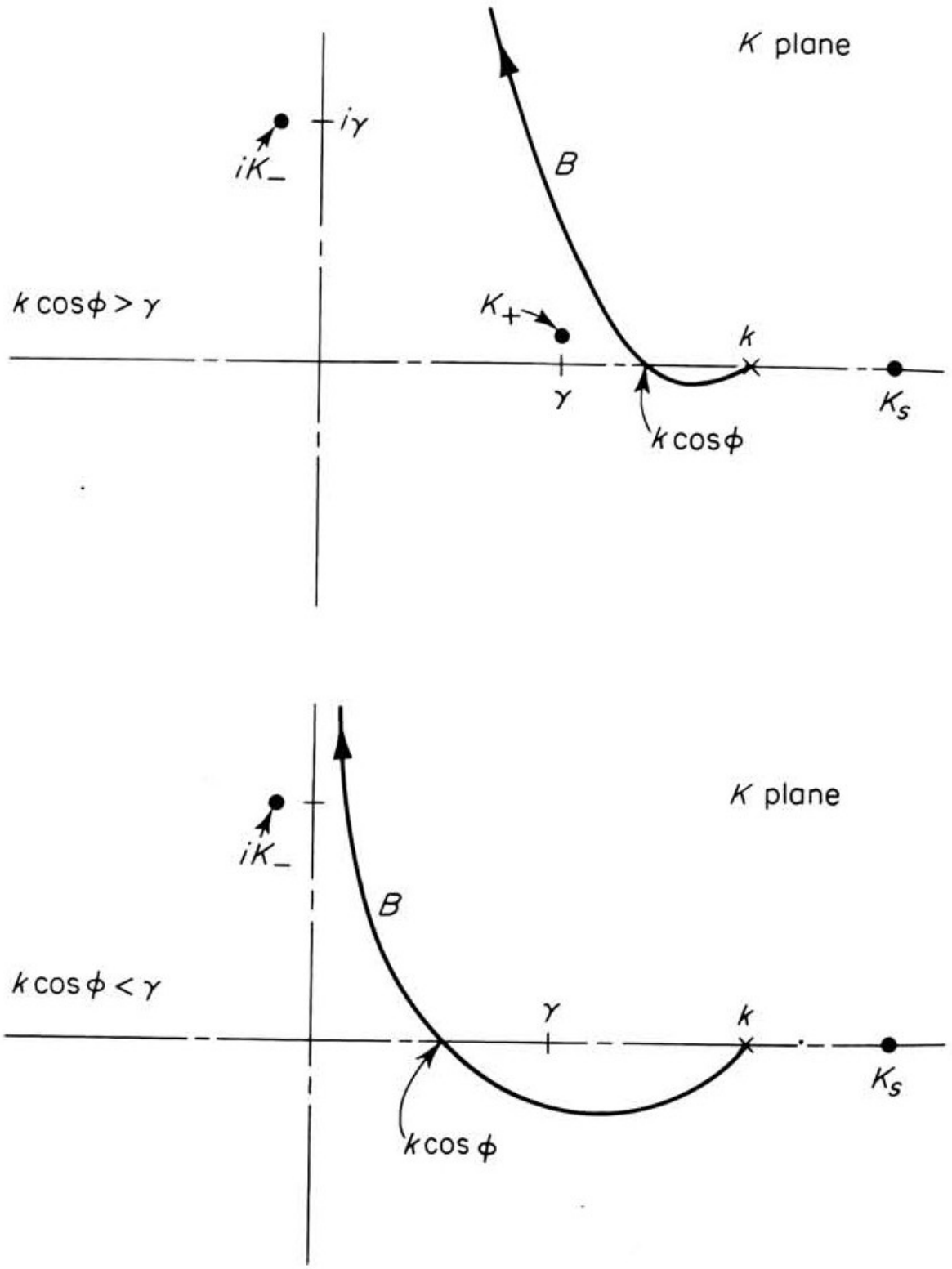


FIGURE 10.4
Path of integration along the branch cut B, going through the saddle point at $K = k \cos \phi$.

Collecting all the expressions and carrying out the contour integrations, we can finally write out the asymptotic formulas for plate displacement and pressure wave, for x and z positive, for μ/γ small, and for $k > \gamma$.

$$\eta_\omega \simeq \frac{i\mu\gamma F_0}{8\rho\omega^2}\left\{\frac{\exp\left[i\gamma x - (\mu\gamma/4k_-)x\right]}{1 + i(\mu\gamma/4k_-)} + \frac{\exp\left[-\gamma x - i(\mu\gamma/4k_+)x\right]}{1 + i(\mu\gamma/4k_+)} + \frac{4\mu^2}{k\gamma}\left(\frac{\gamma^4}{k^4 - \gamma^4}\right)^3 \exp\left(iK_s x\right)\right\} e^{-i\omega t} + B_\eta(x)e^{-i\omega t}$$

$$p_\omega \simeq \frac{\mu\gamma F_0}{8}\left\{\frac{\exp\left[i\gamma x + ik_- z - (\mu\gamma x/4k_-) + (\mu\gamma^2 z/4k_-{}^2)\right]}{k_-[1 + (i\mu/4k_-) + (i\mu\gamma^2/4k_-{}^3)]}\, u(k\cos\phi - \gamma) + i\frac{\exp\left[-\gamma x + ik_+ z - (i\mu\gamma x/4k_+) - (\mu\gamma^2 z/4k_+{}^2)\right]}{k_+[1 + (i\mu/4k_+) + (i\mu\gamma^2/4k_+{}^3)]} + \frac{4i\mu}{k\gamma}\left(\frac{\gamma^4}{k^4 - \gamma^4}\right)^2 \exp\left[iK_s x - \left(\frac{\mu\gamma^4}{k^4 - \gamma^4}\right)z\right]\right\} e^{-i\omega t} + B_p e^{-i\omega t} \tag{10.1.31}$$

where $k_-{}^2 = k^2 - \gamma^2$, $k_+{}^2 = k^2 + \gamma^2$, and $\mu = \rho/h\rho_p$; B_η and B_p are given in Eqs. (10.1.28) and (10.1.30), respectively, and γ is given in Eq. (10.1.21). Function $u(w)$ is the step function, being zero when $w < 0$ and unity when $w > 0$, and K_s is given in Eqs. (10.1.24).

The radiated wave

The radiated wave has an interesting, though not unexpected, shape. Starting from $\phi = 90°$, vertically above the driving force, at first, as we decrease ϕ, the predominant part of p for large r is the cylindrical wave represented by B_p of Eq. (10.1.30). The first term in braces in Eq. (10.1.31) is zero as long as $\cos\phi$ is less than γ/k; the second and third terms are negligibly small for large z. As long as ϕ is larger than $\Psi = \cos^{-1}(\gamma/k)$ [see Eq. (10.1.25)], the wave amplitude is small, because of the factor μ/γ. As ϕ approaches Ψ, this amplitude increases, as will be clear if we rewrite the asymptotic form of $|B_p|^2$ in terms of the angle Ψ.

$$|B_p|^2 \simeq \frac{(\mu k F_0)^2}{16\pi k r}\,\frac{\sin^2\phi}{[(\cos\phi/\cos\Psi)^4 - 1]^2\sin^2\phi + \mu^2}$$

This has a sharp maximum when $\phi = \Psi$, the direction of the pressure wave of Eq. (10.1.25), arising from the free wave in the plate, having wavenumber K_+.

At angle $\phi = \Psi$, the first term in braces for p_ω suddenly enters. Rewriting the mean-square amplitude of this term,

$$|D_p|^2 \simeq \left(\frac{\mu}{8}F_0\right)^2 \cot^2\Psi\,\frac{\exp\left[-(\mu r/2)\cot\Psi\csc\Psi\sin(\Psi - \phi)\right]}{1 + \mu^2\csc^4\Psi} \qquad \phi < \Psi$$

we see that it represents a plane wave going in the Ψ direction, with no attenuation at the angle $\phi = \Psi$, but dropping off rapidly as $\Psi - \phi$ increases. When $\mu r \gg 1$, this drop-off is sharp enough so that $|D_p|^2$ has a delta-function peak in the Ψ direction.

$$|D_p|^2 \to \frac{\mu \cos \Psi \, |F_0|^2}{32r(1 + \mu^2 \csc^4 \Psi)} \, \delta(\Psi - \phi)$$

For ϕ less than Ψ, the B_p term is again the largest, though even its amplitude drops as ϕ diminishes, first rapidly, then more slowly approaching zero as $\phi \to 0$. When ϕ is so close to zero that $\mu z < 1$ (kr still very large), the effect of the surface wave, the third term in braces in the formula for p in Eqs. (10.1.31), begins to be felt. This wave, as mentioned earlier, travels without attenuation, confined roughly to the region $|z| < (h\rho_p/\rho)[(k^4 - \gamma^4)/\gamma^4]$; so its asymptotic, mean-square amplitude is again a delta function

$$\frac{\mu^3 \, |F_0|^2}{8k^2 r} \left(\frac{\gamma^4}{k^4 - \gamma^4} \right)^3 \delta(\phi)$$

Thus, although a small amount of the energy supplied by the linear driving force radiates away in all directions (the cylindrical wave B_p), most of the energy is concentrated into six sharp beams, the more intense being at the angles $\pm\Psi$, $\pm(\pi - \Psi)$ to the plate, the two beams of lesser intensity being along the plate ($\phi = 0, \pi$), corresponding to the surface waves D_p. By the time $|x| \to \infty$, the majority of the energy, which was mostly possessed by the plate when $|x|$ was small, has been transferred to the medium and has radiated away. Involved in this coupled motion are all the waves mentioned at the beginning of this section: the waves with velocity near that of the plate *in vacuo* [the first term in braces in Eq. (10.1.31)]; the bound surface waves with velocity near c (the third term in braces); and the free-wave motion in the medium, caused by the discontinuity at $x = 0$, the line of application of the force (the integrals B). The second term in braces does not constitute a wave; it is the correction term required by the fourth-order equation for the plate, needed only to satisfy boundary conditions at the origin, large only near the origin.

The case of the plate driven by a line force at low frequency, when $k < \gamma$ ($c_p < c$), is much simpler. There is no surface wave, but the wave with wavenumber K_+ propagates along the plate without attenuation. Also, the integral along the branch cut has no high peak, and there is no cutoff of the integral D, since $\Psi = \cos^{-1}(\gamma/k)$ is imaginary. Thus the power radiates fairly uniformly in all directions, but with a peak near the plate surface, and here the majority of the energy is carried by the plate. The details are left as a problem.

Incident plane wave

One way of forcing the plate into motion is to send a wave of sound in its direction. Suppose the wave is plane, propagated at an angle of incidence θ to the z axis, normal to the plate, in the xz plane. This wave will be reflected from the plate, will also set the plate into motion, and thus will generate a transmitted wave on the other side of the plate. The acoustic waves and the induced transverse displacement of the plate will have the forms

$$p = \begin{cases} P_i \exp [ik(x \sin \theta - z \cos \theta) - i\omega t] \\ \qquad + P_r \exp [ik(x \sin \theta + z \cos \theta) - i\omega t] & z > 0 \\ P_t \exp [ik(x \sin \theta - z \cos \theta) - i\omega t] & z < 0 \end{cases}$$

$$\eta = A \exp (ikx \sin \theta - i\omega t)$$

In this case neither the surface wave, of wavenumber K_s, nor the cylindrical wave, represented by the integrals B, is excited.

The equations of motion, with $(p_- - p_+)_{z=0}$ as F_ω and the two equations matching fluid and plate velocities on the two sides of the plate, are

$$-i\omega A = \frac{\cos \theta}{\rho c} (-P_i + P_r) = -\frac{\cos \theta}{\rho c} P_t$$

and

$$(k^4 \sin^4 \theta - \gamma^4)A = \frac{\gamma^4}{2h\rho_p \omega^2} (P_t - P_i - P_r)$$

with solutions

$$P_t = \frac{P_i}{1 - iQ(\theta)} \qquad P_r = \frac{-iQ(\theta)P_i}{1 - iQ(\theta)} \qquad A = \frac{-i(1/\rho c\omega) \cos \theta \, P_i}{1 - iQ(\theta)} \tag{10.1.32}$$

where

$$Q = \left[1 - \left(\frac{k}{\gamma}\right)^4 \sin^4 \theta\right] \frac{k}{\mu} \cos \theta \qquad \mu = \frac{\rho}{\rho_p h}$$

If k/γ (which is roughly equal to $\sqrt{\nu/60{,}000}$ for the 2-mm steel plate in air) is much smaller than unity, then the quantity Q is large for all values of θ, the angle of incidence, except for grazing incidence, $\theta = \frac{1}{2}\pi$. In this case P_t, the amplitude of the transmitted wave, is small; most of the incident wave is reflected. If the frequency is great enough so that $k/\gamma > 1$, then Q will be zero when $\sin \theta = \gamma/k$; at this angle of incidence all the incident energy is transmitted through the plate. Put another way, when the frequency and angle of incidence are such that the horizontal component $k \sin \theta$ of the incident wavenumber is equal to the wavenumber γ of the transverse wave in the plate, the plate is transparent to the incident wave.

Now suppose the incident wave, instead of being sinusoidal, is a pulse, with Fourier transform $P_i(\omega)$.

$$p_i = P_i\delta(T) \qquad T = t - \frac{1}{c}(x\sin\theta - z\cos\theta)$$

$$P_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p_i(T)e^{i\omega t}\,dt = \frac{P_i}{2\pi}\,e^{i(\omega/c)(x\sin\theta - z\cos\theta)}$$

Then the reflected and transmitted waves, and the displacement of the plate, will be corresponding Fourier transforms of the functions given in Eqs. (10.1.32). For example, the transmitted wave, in the region $z < 0$, is

$$p_t = \frac{P_i}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\left[ik(x\sin\theta - z\cos\theta) - i\omega t\right]d\omega}{1 - i(k/\mu)[1 - (\omega/\omega_p)^2\sin^4\theta]\cos\theta} \qquad k = \frac{\omega}{c}$$

$$= \frac{-iP_i}{2\pi}\,\frac{\mu c\omega_p{}^2}{\cos\theta\sin^4\theta}\int_{-\infty}^{\infty} \frac{e^{-i\omega T}\,d\omega}{\omega[\omega^2 - (\omega_p/\sin^2\theta)^2] - i(\mu c\omega_p{}^2/\cos\theta\sin^2\theta)} \tag{10.1.33}$$

where $\mu = \rho/h\rho_p$, and $\omega_p{}^2 = c^4\gamma^4/\omega^2 = 3c^4\rho_p(1 - s^2)/Qh^2$; for the 2-mm steel plate in air, $\mu \simeq 0.0016$ and $\omega_p \simeq 5 \times 10^4$.

There are three poles of this integrand, roots of the denominator; one of them is on the negative-imaginary axis of the ω plane; the other two are above the real axis, symmetrically placed with respect to the origin. We can label the roots of

$$\omega^3 - \omega\left(\frac{\omega_p}{\sin^2\theta}\right)^2 - i\,\frac{\mu c\omega_p{}^2}{\cos\theta\sin^2\theta} = 0$$

as

$$\omega_+ = w + iv \qquad \omega_- = -w + iv \qquad \omega_s = -2iv$$

where

$$w^2 - 3v^2 = \frac{\omega_p{}^2}{\sin^4\theta} \qquad v(w^2 + v^2) = \frac{\mu c}{2\cos\theta}\,\frac{\omega_p{}^2}{\sin^4\theta} \tag{10.1.34}$$

$$w \to \frac{\omega_p}{\sin^2\theta} \qquad v \to \frac{\mu c}{2\cos\theta} \qquad \text{when} \qquad \mu c\sin^2\theta \ll \omega_p\cos\theta$$

the last line giving approximate values when $\mu c/\omega_p \equiv (\rho/c\rho_p)\sqrt{Q/3\rho_p(1 - s^2)}$ is smaller than $\cos\theta/\sin^2\theta$ (it equals about 10^{-3} for the steel plate in air, about 0.4 for the same plate in water). When $\mu c/\omega_p$ is not small, w and v are given by Table 10.1.

The second line of Eq. (10.1.33) has been written so that the denominator of the integrand is

$$(\omega - \omega_+)(\omega - \omega_-)(\omega - \omega_s) = (\omega - iv - w)(\omega - iv + w)(\omega + 2iv)$$

According to Eqs. (1.2.15), the transmitted pressure wave is

$$p_t = \frac{P_i \mu c \omega_p^2}{\cos\theta \sin^4\theta} \begin{cases} \left(\cos wT - 3\frac{v}{w}\sin wT\right)\frac{e^{vT}}{w^2+9v^2} & T<0 \\ \frac{e^{-2vT}}{w^2+9v^2} & T>0 \end{cases} \tag{10.1.35}$$

where $T = t - (1/c)(x\sin\theta - z\cos\theta)$, $(1/c)(x\sin\theta - z\cos\theta)$ being the time the incident pulse-wave would arrive at the point x, z ($z<0$) if the plate were not present.

In this case a "precursor" wave is present; the observer at point x, z ($z<0$) hears sound before he would hear the pulse if the plate were not present (i.e., for $T<0$). Because of the factor e^{vT}, the sound is not loud

TABLE 10.1

$\frac{\mu c \sin^2\theta}{\omega_p \cos\theta}$	$\frac{w\sin^2\theta}{\omega_p}$	$\frac{v\sin^2\theta}{\omega_p}$	$\frac{\mu c \sin^2\theta}{\omega_p \cos\theta}$	$\frac{w\sin^2\theta}{\omega_p}$	$\frac{v\sin^2\theta}{\omega_p}$
0.1	1.0037	0.0495	1.2	1.1975	0.3802
0.2	1.0138	0.0965	1.4	1.2312	0.4148
0.4	1.0462	0.1776	1.8	1.2938	0.4740
0.6	1.0848	0.2428	2.2	1.3504	0.5242
0.8	1.1239	0.2961	2.6	1.4026	0.5678
1.0	1.1616	0.3412	3.0	1.4505	0.6067

until $-T$ is small. The frequency heard is inversely proportional to the square of the sine of the angle of incidence (if $\theta \to 0$, $w \to \infty$, and there is no precursor wave). In addition, there is an exponential decay of pressure when T is positive, which attenuates most slowly when $\theta \to 0$ (since $v \to \mu c/2\cos\theta$ as $\theta \to 0$). This result appears to conflict with the discussion of page 479, which indicated that no sound can arrive at the observer faster than the signal velocity can bring it. There is no contradiction, however; the velocity of transverse motion on a plate is proportional to the square root of the frequency ($c_p = \omega/\gamma = c\sqrt{\omega/\omega_p}$), so that there is no limiting velocity as $\omega \to \infty$, and therefore there is no signal velocity (or if you like, the signal velocity is infinite). The pulse, incident on the plate at angle θ, moves along the plate with a velocity $c/\sin\theta$; and $\omega_p/2\pi\sin^2\theta \simeq w/2\pi$ is the *frequency of the plate wave which has this velocity.*

What is happening is also illustrated by the expression for the plate displacement η as function of T. From Eqs. (10.1.32) and (10.1.33) we

have

$$\eta = -\frac{P_i \mu c \omega_p{}^2}{2\pi\rho c \sin^4\theta}\int_{-\infty}^{\infty}\frac{e^{-i\omega T}\,d\omega}{\omega(\omega - w - iv)(\omega + w - iv)(\omega + 2iv)}$$

$$= \frac{-P_i \mu \omega_p{}^2}{\rho \sin^4\theta(w^2 + v^2)(w^2 + 9v^2)}\begin{cases}\dfrac{e^{vT}}{w}\,[(w^2 - 3v^2)\sin wT & \\ \qquad + 4wv\cos wT] & T < 0 \\ \dfrac{1}{2v}\,[(w^2 + 9v^2) - (w^2 + v^2)e^{-2vT}] & T > 0\end{cases}$$

where $T = t - (x/c)\sin\theta$, since $z = 0$ for the plate. Quantity η is continuous in value and slope at $T = 0$; its greatest positive value is just before $T = 0$; after the pulse has passed ($T > 0$), the plate approaches exponentially its final displacement, $-[P_i\mu\omega_p{}^2/2\rho v(w^2 + v^2)\sin^4\theta] = -(P_i\cos\theta/\rho c)$, which is thus its greatest negative displacement. At grazing incidence ($\theta \to \frac{1}{2}\pi$), $v \to \frac{1}{2}(\mu c\omega_p{}^2/\cos\theta\sin^2\theta)^{\frac{1}{3}}$ and $w \to \sqrt{3}v$, both going to infinity as $\cos\theta \to 0$. In this limit all waves become pulse-waves, traveling parallel to the plate surface.

Incident spherical wave

Suppose a simple source of frequency $\omega/2\pi$ and strength S_ω is placed at $x = y = 0$, $z = l$, above a plane plate in the xy plane, the plate having density ρ_p, thickness $2h$, elastic modulus Q, and Poisson ratio s, as before. If the plate were not present, the pressure wave from the source would be

$$p_i = \frac{ik\rho c}{4\pi R}S_\omega e^{ikR - i\omega t} = -ik\rho c S_\omega g_k(r \mid r_0)e^{-i\omega t}$$

$$= \frac{k\rho c}{4\pi}S_\omega\int_0^\infty J_0(\beta r)e^{iw|z-l|}\frac{\beta\,d\beta}{w}\,e^{-i\omega t}$$

in terms of the cylindrical coordinates of Fig. 10.5, where $w = \sqrt{k^2 - \beta^2}$ or $= i\sqrt{\beta^2 - k^2}$. The first formula comes from Eqs. (7.1.5); the last is still another form of the Green's function, appropriate for our present use.

With the source on the z axis, all waves will be symmetric about this axis, independent of angle ϕ. Thus, from Eqs. (1.3.27), the plate displacement and the pressure waves below and above the plate can be written

$$\eta = e^{-i\omega t}\int_0^\infty A(\beta)J_0(\beta r)\beta\,d\beta \tag{10.1.36}$$

$$p = \begin{cases}p_i + p_r = e^{-i\omega t}\displaystyle\int_0^\infty\left[\frac{k\rho c S_\omega}{4\pi w}\,e^{iw|z-l|} + B(\beta)e^{iwz}\right]J_0(\beta r)\beta\,d\beta & z > 0 \\ p_t = e^{-i\omega t}\displaystyle\int_0^\infty C(\beta)e^{-iwz}J_0(\beta r)\beta\,d\beta & z < 0\end{cases}$$

where A, B, and C must be determined from the boundary conditions at

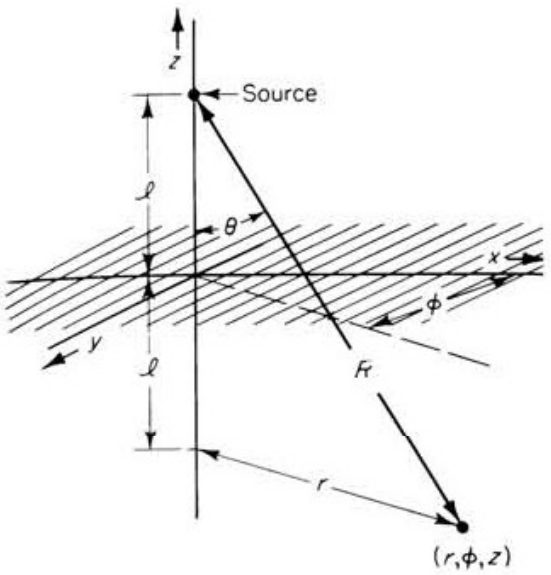


FIGURE 10.5
Angles and distances for waves from a simple source above a plane plate.

$z = 0$. To join velocities at $z = 0$, we set $-i\omega A = -(S_\omega/4\pi)e^{iwl} + (w/\rho ck)B = -(w/\rho ck)C$ or $(\rho ckS_\omega/4\pi w)e^{iwl} = B + C$ and $A = -(iw/\rho c^2k^2)C$. The equation of motion (10.1.21), when applied to the integrands, with $F = p_t - p_i - p_r$, is

$$(\beta^4 - \gamma^4)A = \frac{\mu\gamma^4}{2\rho\omega^2}\left(C - B - \frac{k\rho cS_\omega}{4\pi w}e^{iwl}\right)$$

where $\mu = \rho/h\rho_p$, and $\gamma^4 = 3\omega^2\rho_p(1 - s^2)/Qh^2$, as before.

Solution of the simultaneous equations results in

$$A = -i\frac{w}{\rho w^2}C \qquad B = \frac{\rho\omega}{4\pi w}S_\omega e^{iwl} - C$$

$$C = \frac{-(i/4\pi)\mu\gamma^4\rho\omega S_\omega e^{iwl}}{w^2(\beta^4 - \gamma^4) - i\mu w\gamma^4} \qquad w^2 = k^2 - \beta^2$$

which then can be substituted in the integrals for η, p_r, and p_t. The integrals are similar to those of Eqs. (10.1.27), but the differences merit discussion. In the first place, the integrals over β are from 0 to ∞, not from $-\infty$ to ∞. Since the asymptotic form of J_0 is not simple, it is best to separate it into the two Hankel functions,

$$J_0(\beta r) = \tfrac{1}{2}H_0^{(1)}(\beta r) + \tfrac{1}{2}H_0^{(2)}(\beta r)$$

$$H_0^{(1)}(u) = J_0(u) + iN_0(u) \to \sqrt{\frac{2}{\pi iu}}e^{iu} \qquad u \to \infty \qquad (10.1.37)$$

$$H_0^{(2)}(u) = J_0(u) - iN_0(u) \to \sqrt{\frac{2i}{\pi u}}e^{-iu}$$

in accord with the discussion following Eqs. (7.3.2). The two Hankel functions have a logarithmic singularity at zero, but have simple asymptotic forms, $H_0^{(1)}(u)$ going exponentially to zero when the imaginary part of u is large and positive, $H_0^{(2)}(u)$ going to zero when the imaginary part of u is large and negative.

Thus the integrals of Eq. (10.1.36) can be broken into two parts, the term containing $H_0^{(1)}$ being integrated along contour C of Fig. 10.6, and that

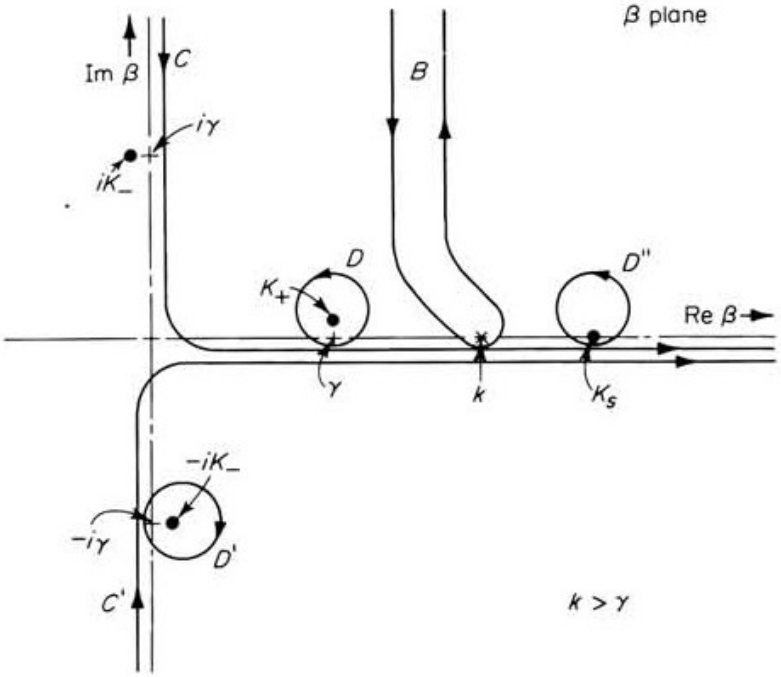


FIGURE 10.6
Contours of integration for Eqs. (10.1.37).

containing $H_0^{(2)}$ going along C'. This is possible because $H_0^{(1)}(iz) = -H_0^{(2)}(-iz)$ (z real), as an examination of Eqs. (5.3.15) will demonstrate. Therefore the integral of the $H_0^{(1)}$ part along the positive-imaginary portion of C is exactly canceled by the integral of the $H_0^{(2)}$ part along the negative-imaginary portion of C'; so these parts can be added to the parts along the 0 to ∞ portions of C and C' without changing the value of the integral sum. Contour C can then be completed by a quarter circle around the first quadrant at ∞, since the integrand is zero there; similarly, contour C' can be completed by surrounding the fourth quadrant. Finally, these closed contours can be shrunk, C reducing to the circles about the poles $\beta = K_+$ and $\beta = K_s$ (if $k > \gamma$) plus the loop from $i\infty$ around the branch point $\beta = k$ and back to $i\infty$, and C' to the circle around the pole $-iK_-$. Approximate values of K_+, K_-, and K_s are given in Eqs. (10.1.24).

The calculation of the integrals proceeds as with the integrals of Eqs. (10.1.27). For example, the wave p_t, transmitted through the plate to the

region $z < 0$, for $k > \gamma$ and $R^2 = r^2 + (l - z)^2$ very large, is

$$p_t \to -\tfrac{1}{8}\rho c k S_\omega \mu \gamma^4 \left\{ u(k \sin\theta - \gamma) \frac{\exp\left[iK_+ r + iq_+(l-z)\right]}{\gamma^2 q_+ \sqrt{k^2 - \gamma^2}\sqrt{2\pi i K_+ r}} \right.$$
$$+ \frac{\exp\left[-K_- r + iq_-(l-z)\right]}{\gamma^2 q_- \sqrt{k^2 + \gamma^2}\sqrt{2\pi K_- r}} - 4iK_s \frac{\exp\left[iK_s r + iq_s(l-z)\right]}{k(k^4 - \gamma^4)\sqrt{2\pi i K_s r}}$$
$$\left. + \frac{4\sqrt{2}}{(k^4 \sin^4\theta - \gamma^4)k\cos\theta - i\mu\gamma^4} \frac{e^{iKr}}{4\pi R} \right\} \quad (10.1.38)$$

Definitions and approximate value of K_+, K_-, K_s, and q_+, etc., are given in Eqs. (10.1.24) for $k > \gamma$.

When μ/γ is small, the exponential of the first term in braces can be rewritten

$$\exp\left\{ i\gamma r + i(l-z)\sqrt{k^2 - \gamma^2} - \frac{\mu\gamma}{4(k^2 - \gamma^2)} \left[r\sqrt{k^2 - \gamma^2} - \gamma(l-z) \right] \right\}$$
$$= \exp\left\{ ikR\cos\left(\tfrac{1}{2}\pi - \theta - \Psi\right) - \tfrac{1}{4}\mu R \sin\left(\tfrac{1}{2}\pi - \theta - \Psi\right)\cot\Psi \csc\Psi \right\}$$

where $r = R\sin\theta$; $l - z = R\cos\theta$; and $k\cos\Psi = \gamma$, Ψ being the angle of Eq. (10.1.25) between the plate and the direction of propagation of the radiation into the medium for the frequency $\omega/2\pi$. Because of the step function $u(k\sin\theta - \gamma)$, this wave has zero amplitude (for R large) for θ smaller than $\frac{1}{2}\pi - \Psi$. At angle $\theta = \frac{1}{2}\pi - \Psi$ to the z axis, there is no exponential attenuation of this part of the wave (though the wave is inversely proportional to $\sqrt{r}$), but the amplitude drops exponentially with $\sin(\frac{1}{2}\pi - \Psi - \theta)$ for larger values of θ. This conical wave arises from the spherical incident wave. Since, as we saw earlier, the plate is transparent, at this frequency, only to waves with angles of incidence close to $\frac{1}{2}\pi - \Psi$, the transmitted wave (this part of it, at any rate) is concentrated on the surface of a cone of angle $\frac{1}{2}\pi - \Psi$ with respect to the z axis and vertex at the source point.

The second term in Eq. (10.1.38) is negligible for large R; it comes from the exponentiallike solution $J_0(i\gamma r)$ required to maintain continuity of slope at $r = 0$ [see discussion of Eqs. (5.3.15)]. The third term corresponds to the surface wave induced by the incident wave; it represents a radial flow of energy outward in a region close to the xy plane of the plate. An incident spherical wave produces such a surface wave (though of small amplitude); an incident plane wave does not, as Eqs. (10.1.32) show.

The last term in Eq. (10.1.38), coming from the integral around the branch cut, is a spherical wave emanating from the source point, with a pronounced dependence on the angle θ. When $\theta = 0$ (along the $-z$ axis) and when $\theta = \frac{1}{2}\pi$ (out along the xy plane), the amplitude is small because of the μ in the factor outside the braces. But the amplitude of this part of

the wave has a sharp maximum, larger by the factor k/μ, at $\theta = \frac{1}{2}\pi - \Psi$, the angle at which a plane wave can penetrate the plate without loss. Thus all the wave types discussed earlier are produced by the incident spherical wave.

At low frequencies, when $c_p < c \quad (k < \gamma)$, there is no surface wave and no direction of strong penetration, since $\Psi = \cos^{-1}(\gamma/k)$ is complex. In the expression for p_t, corresponding to Eq. (10.1.38), the only term which does not die out exponentially as $z \to \infty$ is the integral around the branch point, representing a spherical transmitted wave of small amplitude, increasing monotonically as θ increases from 0 to $\frac{1}{2}\pi$ and having no maximum. Details are left for a problem.

10.2 A FINITE PANEL

When all the coupled systems are infinite in extent, as they were in the problems discussed in Sec. 10.1, the wave motion can extend to infinity, and reflections can be neglected. But when one or more of the systems are finite in size, then standing waves, and therefore resonances, add their complications to an already intricate problem. The calculations required for a solution are often involved, and the solutions, usually approximations, are necessarily complex in order to take into account all the resonances, antiresonances, transparencies, and other effects typical of coupled resonating systems. Space is available, in this chapter and its problems, for only a few examples. In this section a single example is worked out of a triple system, one element of which is finite. A problem deals with a double system, a simplified Helmholtz resonator.

Circular membrane in rigid wall

We consider a rigid wall coincident with the xy plane, with a circular opening in it of radius a, centered at the origin. This opening has a membrane, of areal density σ, stretched across it under tension T, so that the velocity of transverse waves is $c_p = \sqrt{T/\sigma} = c/\alpha$, c being the acoustic velocity in the medium on either side of the wall. The transverse vibration of the membrane is subject to the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \eta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \phi^2} + k^2\alpha^2\eta = -\frac{\alpha^2}{\sigma c^2}P \qquad r < a \qquad (10.2.1)$$

where $\eta = 0$ when $r = a$
$k = \omega/c$
r and ϕ = polar coordinates in xy plane
η = displacement of membrane in z direction
P = pressure difference $p_- - p_+$ tending to move membrane in positive z direction

According to Eq. (5.2.22) [see also Eq. (9.3.31)], the eigenfunctions for free vibration of the membrane, in the absence of the medium, are

$$\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)J_m\left(\frac{j_{mn}r}{a}\right) \qquad \text{where } J_m(j_{mn})=0 \tag{10.2.2}$$

Thus $j_{mn}=\pi\beta_{mn}$ is the nth zero of J_m (see Table X). By methods similar to those used in deriving Eq. (9.4.2), we can show that the Green's function for the membrane, solution of $(\nabla_2^2+k^2\alpha^2)G=-\delta(x-x_0)\delta(y-y_0)$, is

$$G(r,\phi\mid r_0,\phi_0)=\sum_{m=0}^{\infty}\frac{\epsilon_m}{\pi}\cos m(\phi-\phi_0)\sum_{n=1}^{\infty}\frac{J_m(j_{mn}r/a)J_m(j_{mn}r_0/a)}{J_{m-1}^2(j_{mn})[j_{mn}{}^2-(k\alpha a)^2]}$$
$$\text{for } 0\leqslant r,\, r_0\leqslant a \tag{10.2.3}$$

Thus the displacement of the membrane under the influence of a net pressure difference $P(r,\phi)e^{-i\omega t}$ is

$$\eta(r,\phi)=\frac{\alpha^2}{\sigma c^2}\int_0^{2\pi}d\phi_0\int_0^a G(r,\phi\mid r_0,\phi_0)P(r_0,\phi_0)r_0\,dr_0 \tag{10.2.4}$$

times the factor $e^{-i\omega t}$.

The integral is a Fourier series in ϕ, because of the form of G; thus we have a sequence of equations for the Fourier components of η. If we set

$$\eta=\sum_m\cos(m\phi)\,Y_m(r) \qquad \text{and} \qquad P=\sum_m\cos(m\phi)\Phi_m(r)$$

then

$$Y_m(r)=\frac{\alpha^2}{\sigma c^2}\int_0^a G_m(r\mid r_0)\Phi_m(r_0)r_0\,dr_0 \tag{10.2.5}$$

where

$$G_m(r\mid r_0)=\sum_{n=1}^{\infty}\frac{2}{J_{m-1}^2(j_{mn})}\frac{J_m(j_{mn}r/a)J_m(j_{mn}r_0/a)}{j_{mn}{}^2-(k\alpha a)^2}$$

If Φ_m happens to be proportional to the Bessel function $J_m(qr)$, a closed form for Y_m can be found [as was done with Eq. (9.3.33)]. Since the function

$$f(r,\phi)=A\cos(m\phi)\left[J_m(qr)-\frac{J_m(qa)}{J_m(k\alpha a)}J_m(k\alpha r)\right]$$

satisfies the boundary condition $f(a,\phi)=0$ and also satisfies the equation $(\nabla_2^2+k^2\alpha^2)f=A\cos(m\phi)(k^2\alpha^2-q^2)J_m(qr)$, we can show that

$$\int_0 G_m(r\mid r_0)J_m(qr_0)r_0\,dr_0=\frac{1}{q^2-k^2\alpha^2}\left[J_m(qr)-\frac{J_m(qa)}{J_m(k\alpha a)}J_m(k\alpha r)\right]$$
$$=\frac{1}{2k\alpha}\left[rJ_{m-1}(k\alpha r)-\frac{J_{m-1}(k\alpha a)}{J_m(k\alpha a)}aJ_m(k\alpha r)\right] \qquad \text{when } q=k\alpha \tag{10.2.6}$$

These formulas will be useful later.

Reaction of the medium

The air, or other medium, on either side of the wall, has density ρ and transmits acoustic waves with velocity c. The Green's function for the region $z \geqslant 0$ is most appropriately given in Eq. (7.4.8) for cylindrical coordinates r, ϕ, z and for a rigid wall.

$$g_+ = \sum_{m=0}^{\infty} \frac{i\epsilon_m}{2\pi} \cos m(\phi - \phi_0) \int_0^{\infty} J_m(\mu r) J_m(\mu r_0) \frac{\mu\, d\mu}{\varkappa} \begin{cases} \cos \varkappa z\, e^{i\varkappa z_0} & z_0 \geqslant z \geqslant 0 \\ \cos \varkappa z_0\, e^{i\varkappa z} & z \geqslant z_0 \geqslant 0 \end{cases} \tag{10.2.7}$$

where

$$\varkappa = \sqrt{k^2 - \mu^2} \qquad \text{or} \qquad i\sqrt{\mu^2 - k^2}$$

The Green's function g_- for the region $z \leqslant 0$ is obtained from g_+ by reversing the signs of z and z_0.

Motion of the membrane produces waves in the medium, and this produces a reaction back on the membrane. Suppose a plane wave $p = A \exp(ikz \cos\theta + ikr \cos\phi \sin\theta - i\omega t)$, with direction of propagation in the xz plane at angle of incidence θ with respect to the z axis, comes from $-\infty$ to strike the negative side of the wall at $z = 0$. If the whole wall were rigid, this wave would be reflected with no change of amplitude or phase, and the pressure in the region $z < 0$ would be (we omit the time factor)

$$\begin{aligned} p_i &= 2P_i \cos(kz \cos\theta) \exp(ikr \cos\phi \sin\theta) \\ &= 2P_i \cos(kz \cos\theta) \sum_m \epsilon_m i^m \cos(m\phi)\, J_m(kr \sin\theta) \end{aligned} \tag{10.2.8}$$

by using Eq. (1.2.9). If the membrane displacement is expressed as a Fourier series $\eta = \sum \cos(m\phi)\, Y_m(r)$, the pressure wave in region $z < 0$ is modified to become

$$\begin{aligned} p_- &= p_i + \rho\omega^2 \int_0^{2\pi} d\phi_0 \int_0^a g_-(r,\phi,z \mid r_0,\phi_0,0)\eta(r_0,\phi_0) r_0\, dr_0 \\ &\xrightarrow[z\to 0]{} \sum_m \left[2P_1 \epsilon_m i^m J_m(kr \cos\theta) + \rho\omega^2 \int_0^a g_m(r \mid r_0)\, Y_m(r_0) r_0\, dr_0 \right] \cos(m\phi) \end{aligned} \tag{10.2.9}$$

where

$$g_m(r \mid r_0) = i \int_0^{\infty} J_m(\mu r) J_m(\mu r_0) \frac{\mu\, d\mu}{\varkappa} \qquad \varkappa^2 = k^2 - \mu^2$$

as obtained from Eqs. (10.2.7) and (7.4.16), taking into account the fact that the membrane velocity u_z is equal to $-i\omega\eta$, and for the negative side of the wall, the expression of Eq. (7.4.16) for p_- is $+i\rho\omega$, rather than $-i\rho\omega$, times the first integral.

If there is no incident wave in the region $z > 0$, the only sound there will be that produced by the motion of the membrane.

$$p_+ = -\rho\omega^2 \int_0^{2\pi} d\phi_0 \int_0^a g_+(r,\phi,z \mid r_0,\phi_0,0)\eta(r_0,\phi_0) r_0\, dr_0$$

$$\xrightarrow[z\to 0]{} -\rho\omega^2 \sum_m \cos(m\phi) \int_0^a g_m(r \mid r_0)\, Y_m(r_0) r_0\, dr_0 \quad (10.2.10)$$

Taking the second lines of Eqs. (10.2.9) and (10.2.10), we have a second integral equation relating P and η, and thus a second set of equations relating the Φ's with the Y's.

$$P = (p_- - p_+)_0 = p_i + 2\rho\omega^2 \int_0^{2\pi} d\phi_0 \int_0^a g(r,\phi,0 \mid r_0,\phi_0,0)\eta(r_0,\phi_0) r_0\, dr_0$$

or (10.2.11)

$$\Phi_m(r) = 2P_i \epsilon_m i^m J_m(kr \sin\theta) + 2\rho\omega^2 \int_0^a g_m(r \mid r_0)\, Y_m(r_0) r_0\, dr_0$$

for the range $0 \leqslant r \leqslant a$ only. The simultaneous solutions to these and the counterpart equations (10.2.5) will be called $\Phi_m(\theta \mid r)$ and $Y_m(\theta \mid r)$. From them one can compute the energy transmitted through the membrane, as well as the angular distribution of the transmitted wave in region $z > 0$ and the distortion of the reflected wave produced by the membrane motion.

The transmitted wave

Knowing $Y_m(\theta \mid r)$, Eq. (10.2.10) can be used to compute the wave in the region $z > 0$. Its asymptotic form, for z very large, can be computed by using the alternative form for g, given in Eq. (7.4.6). Using the spherical coordinates R, ϕ, ϑ instead of cylindrical coordinates r, ϕ, z, the asymptotic form for g_+ is

$$g_+(R,\phi,\vartheta \mid r_0,\phi_0,0) \to \frac{1}{2\pi} \frac{e^{ikR}}{R} \exp[-ikr_0 \cos(\phi - \phi_0) \sin\vartheta]$$

Inserting this in Eq. (10.2.10) and using Eq. (1.3.11), we obtain

$$p_+ \to \frac{e^{ikR}}{R} \sum_m \Theta_m(\theta \mid \vartheta) \cos(m\phi) \quad (10.2.12)$$

$$\Theta_m(\theta \mid \vartheta) = \rho\omega^2 i^{-m} \int_0^a J_m(kr_0 \sin\vartheta)\, Y_m(\theta \mid r_0) r_0\, dr_0 \qquad 0 \leqslant \vartheta \leqslant \tfrac{1}{2}\pi$$

Thus the sum $\sum \Theta_m(\theta \mid \vartheta) \cos(m\phi)$ gives the asymptotic dependence on the spherical angles ϑ, ϕ of the wave transmitted through the membrane, generated by a plane wave incident at angle θ on the other side of the wall.

The total power transmitted through the membrane could be obtained by integrating $|p_+|^2/\rho c$ over the hemisphere of large radius, from (10.2.12).

But we can also calculate it by integrating $\mathrm{Re}\,(p_+u_z^*)$ over the surface of the membrane, with $u_z = -i\omega\eta$. Since, for $z = 0, p_- - p_+ = P$ and $p_- + p_+ = p_i$, we can write

$$p_+ = \tfrac{1}{2}p_i - \tfrac{1}{2}P = -\rho\omega^2 \sum_m \cos(m\phi) \int_0^a g_m(r \mid r_0)\, Y_m(\theta \mid r_0) r_0\, dr_0$$

from Eq. (10.2.10). Since $u_z = -i\omega \sum \cos(m\phi)\, Y_m(r)$, we have for the total power transmitted

$$\begin{aligned}\Upsilon &= 2\pi\rho\omega^3\, \mathrm{Re}\left[\sum_m \frac{-i}{\epsilon_m} \int_0^a Y_m(\theta \mid r) r\, dr \int_0^a g_m(r \mid r_0)\, Y_m^*(\theta \mid r_0) r_0\, dr_0\right] \\ &= 2\pi\rho\omega^3\, \mathrm{Re}\left[\sum_m \frac{1}{\epsilon_m} \int_0^a Y_m(\theta \mid r) r\, dr \right. \\ &\qquad \left. \times \int_0^a Y_m^*(\theta \mid r_0) r_0\, dr_0 \int_0^\infty J_m(\mu r) J_m(\mu r_0) \frac{\mu\, d\mu}{\sqrt{k^2 - \mu^2}}\right] \qquad (10.2.13)\end{aligned}$$

A possible method of solution is to expand Y_m into a series of the eigenfunctions $J_m(j_{mn}r/a)$ $(n = 1, 2, \ldots)$.

$$Y_m(\theta \mid r) = \sum_{n=1}^{\infty} A_n{}^m(\theta) J_m\left(\frac{j_{mn}r}{a}\right)$$

This can be inserted in the left-hand side of Eq. (10.2.5), to equate with the coefficients of the expansion of G_m, obtaining

$$A_n{}^m(\theta) = \frac{2\alpha^2}{\sigma c^2 J_{m-1}^2(j_{mn})} \frac{1}{j_{mn}{}^2 - (k\alpha a)^2} \int_0^a \Phi_m(\theta \mid r_0) J_m\left(\frac{j_{mn}r_0}{a}\right) r_0\, dr_0$$

Substituting Eq. (10.2.11) for Φ_m and again using the expansion of Y_m produces a set of simultaneous equations from which the coefficients $A_n{}^m$ can be computed. Using the equation for g_m, we have

$$\begin{aligned}A_n{}^m(\theta) = \frac{\alpha^2 a^2/\sigma c^2}{j_{mn}{}^2 - (k\alpha a)^2} \Big\{ & 2P_i \epsilon_m i^m F_n{}^m(ka \sin\theta) \\ & + \tfrac{1}{2} i\rho\omega^2 a \sum_\nu A_\nu{}^m(\theta) J_{m-1}^2(j_{n\nu}) [\theta_{n\nu}{}^m(ka) - i\chi_{n\nu}{}^m(ka)]\Big\} \qquad (10.2.14)\end{aligned}$$

where

$$F_n{}^m(x) = \frac{2j_{mn}J_m(x)}{J_{m-1}(j_{mn})(x^2 - j_{mn}{}^2)} \to 1 \qquad x \to j_{mn}$$

and

$$\theta_{n\nu}{}^m(x) - i\chi_{n\nu}{}^m(x) = \int_0^\infty F_n{}^m(\mu) F_\nu{}^m(\mu) \frac{\mu\, d\mu}{\sqrt{x^2 - \mu^2}}$$

We have used the last of Eqs. (5.2.21), which in the present case becomes

$$\int_0^a J_m\left(\frac{j_{mn}r}{a}\right) J_m(\mu r) r\, dr = \tfrac{1}{2}a^2 J_{m-1}^2(j_{mn}) F_n{}^m(\mu a)$$

to carry out the integrations over the Bessel-function products.

The functions $F_n^m(ka \sin \theta)$ measure the direct effect of the incident wave on the mth component of the transmitted wave. Some of these functions are plotted in Fig. 10.7; in general, they are large only in the range $j_{m,n-1} < x < j_{m,n+1}$, and only those for $m = 0$ are large near $x = 0$. The

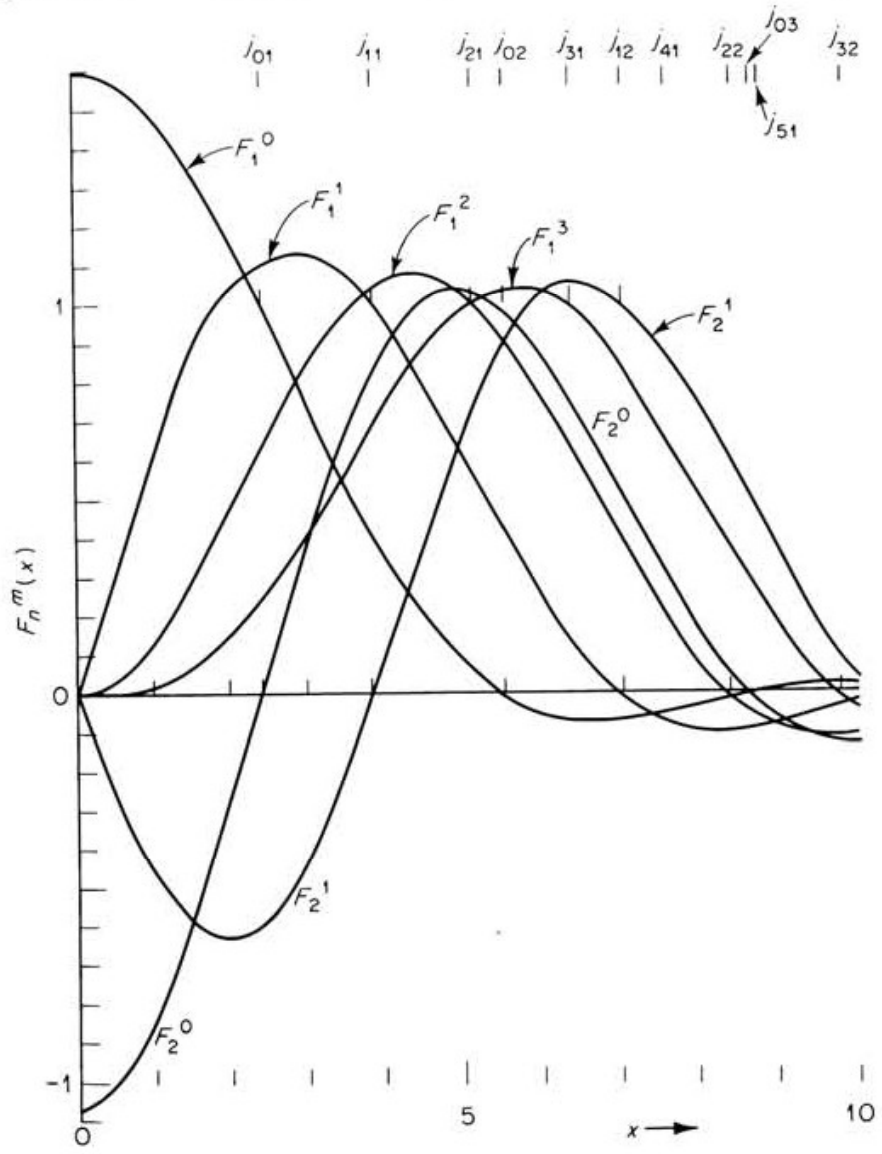


FIGURE 10.7
Angle-dependence functions $F_n^m(x)$ for wave transmitted through circular membrane. See Eqs. (10.2.14) and (10.2.15) and Table XI.

quantities $\theta_{n\nu}^m(x) - i\chi_{n\nu}^m(x)$ measure the interaction between the various eigenfunctions of free-membrane vibration, via the reaction of the medium on both sides of the membrane. The θ and χ are resistance and reactance factors analogous to the factors of Fig. 7.9 for the circular piston. Some of these factors are plotted in Fig. 10.8, more are given in Table XII, the F's in Table XI. In general, they have large magnitudes only near j_{mn} and $j_{m\nu}$, for n different from ν, and the diagonal terms θ_{nn}^m and χ_{nn}^m are larger than the nondiagonal terms, for $n \neq \nu$.

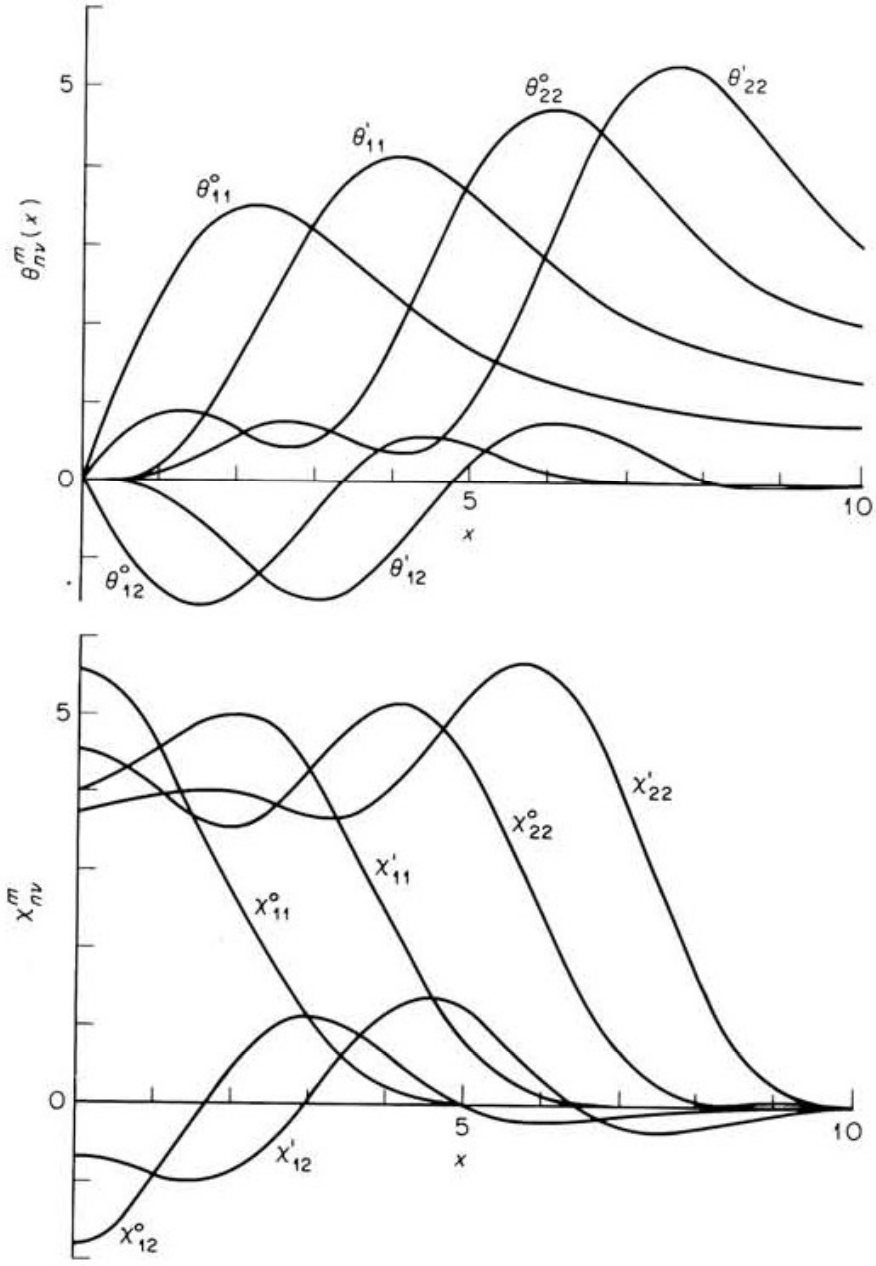


FIGURE 10.8
Resistance and reactance factors for sound transmission through a circular membrane. See Eq. (10.2.14) and Table XII.

Once the A's are determined, the items of physical interest can be computed. For example, the asymptotic form for the transmitted wave, on the side opposite the incident wave, is given in terms of the functions Θ_m of Eqs. (10.2.12).

$$\Theta_m(\theta \mid \vartheta) = \tfrac{1}{2} i^{-m} \rho \omega^2 a^2 \sum_{n=1}^{\infty} A_n{}^m(\theta) J^2_{m-1}(j_{mn}) F_n{}^m(ka \sin \vartheta) \quad (10.2.15)$$

The power transmitted through the membrane becomes

$$\Upsilon = \tfrac{1}{2}\pi \rho \omega^3 a^3 \sum_{m=0}^{\infty} \frac{1}{\epsilon_m} \sum_{n,\nu} A_n{}^m A_\nu{}^m J^2_{m-1}(j_{mn}) J^2_{m-1}(j_{m\nu}) \theta_{n\nu}{}^m(ka) \quad (10.2.16)$$

But to solve for the A's by successive approximations involves the summation of many series. The variational procedure will usually achieve satisfactory accuracy and will result in more compact formulas. First, however, we should see what Eq. (10.2.14) predicts in various limiting situations.

Approximate solutions

When $k\alpha a$ is close in value to j_{mn}, the (m,n)th normal mode of the membrane is strongly excited, and $A_n{}^m(\theta)$ becomes the dominant coefficient in the expansion for η (as long as $-\rho\omega^2 a$ is small compared with $\sigma c^2/\alpha^2 a^2$). For frequencies well below the lowest resonance ($k\alpha a \ll j_{01}$), only A_1^0 is non-negligible; the pressure near the membrane is nearly independent of position and of angle of incidence. Approximate formulas for the quantities of interest, in this case, are

$$\eta \simeq 4\frac{\alpha^2 a^2}{\sigma c^2}\frac{P_i}{j_{01}{}^3 J_1(j_{01})} J_0\left(\frac{j_{01} r}{a}\right)$$

$$p_+ \to \frac{4\rho\alpha^2 k^2 a^4}{\sigma j_{01}{}^4} P_i \frac{e^{ikR}}{R} \qquad R \to \infty \tag{10.2.17}$$

$$\Upsilon \simeq 8\pi a^2 \left(\frac{\rho a}{\sigma}\right)^2 k^3 a^3 \alpha^4 \frac{|P_i|^2}{\rho c}\frac{J_1{}^2(j_{01})}{j_{01}{}^6}\theta_{11}^0(ka)$$

The power transmitted equals $2\pi R^2$ times $1/\rho c$ times the square of the asymptotic amplitude of p_+ (as it should), since $\theta_{11}^0(ka) \to 4ka/j_{01}{}^2 J_1{}^2(j_{01})$ when $ka \to 0$. We note that the transmitted power is proportional to the square of the ratio $(\rho a/\sigma)$ between the density of the medium and that of the membrane; at these low frequencies the transmission is mass-controlled.

As the frequency of the incident wave is increased to the resonance frequency of the (0,1)st normal mode of the membrane, A_1^0 remains the dominant coefficient,

$$A_1^0 \simeq \frac{(2\alpha^2 a^2 P_i/\sigma c^2) F_1^0(ka \sin\theta)}{j_{01}{}^2 - (k\alpha a)^2 - (\rho k^2 a^3/2\sigma) J_1{}^2(j_{01})[\chi_{11}^0(ka) + i\theta_{11}^0(ka)]} \tag{10.2.18}$$

becoming quite large, but not infinite, at resonance ($k\alpha a = j_{01}$). The asymptotic form of the transmitted wave is, approximately,

$$p_+ \to \frac{aP_i F_1^0(ka\sin\theta) F_1^0(ka\sin\vartheta)(e^{ikR}/R)}{[2\sigma/\rho k^2 a^3 \alpha^2 J_1{}^2(j_{01})][j_{01}{}^2 - (k\alpha a)^2] - \chi_{11}^0(ka) - i\theta_{11}^0(ka)} \tag{10.2.19}$$

when $R \to \infty$. The symmetry between the angle of incidence θ and the angle of transmission ϑ is a consequence of the principle of reciprocity.

Close to the (1,1) resonance, the coefficient $A_1{}^1$ predominates, and when $k\alpha a$ is exactly equal to j_{11}, the membrane displacement is

$$\eta \simeq \frac{4P_i F_1{}^1(ka\sin\theta)}{\rho\omega^2 a J_1{}^2(j_{01})[\chi_{11}{}^1(ka) + i\theta_{11}{}^1(ka)]}\cos\phi\, J_1\left(\frac{j_{11} r}{a}\right)$$

and so on.

Variational formulas

For purposes of computation, the variational procedure is more useful than the series expansion. As indicated in connection with Eq. (9.3.28), we can work either with the shape of Φ_m, the pressure component, or with that of Y_m, the membrane-displacement component. We have just seen that Y_m changes shape radically as the frequency approaches one of the resonance frequencies of the membrane. The driving pressure Φ_m also alters near membrane resonance, but if ρa is small compared with σ, the first term on the right-hand side of Eq. (10.2.11), representing the direct effect of the incident wave, remains the dominant term (or at least does not become negligible compared with the second term) at resonance. Thus it will be easier to devise a form for Φ_m which is satisfactory over a wide range of frequency than it will be to discover and work with one for Y_m.

We start by multiplying Eq. (10.2.11) by $Y_m(\theta \mid r)$ and integrating.

$$0 = 2P_i\epsilon_m i^m \int J_m(kr \sin\theta)\, Y_m(\theta \mid r) r\, dr - \int \Phi_m Y_m(\theta \mid r) r\, dr + 2\rho\omega^2 \int Y_m(\theta \mid r) \int g_m(r \mid r_0)\, Y_m(\theta \mid r_0) r_0\, dr_0\, r\, dr$$

This can be changed to an expression in terms of the Φ_m's by using Eq. (10.2.5) several times. From Eq. (10.2.12) we see that the integral in the first term is proportional to the function Θ_m, which determines the asymptotic form of the transmitted wave,

$$\int J_m(k \sin\theta)\, Y_m(\theta \mid r) r\, dr = \frac{1}{\rho\omega^2} i^m \Theta_m(\theta \mid \theta)$$

Adding this to the preceding equation and using Eq. (10.2.5) to eliminate the functions Y_m, we obtain a symmetric expression which can be used to obtain a variational solution,

$$(-1)^m \frac{2P_i\epsilon_m}{\rho\omega^2} \Theta_m(\theta \mid \theta) = 4P_i\epsilon_m i^m \frac{\alpha^2}{\sigma c^2} \int \Phi_m \int G_m J_m(kr \sin\theta) - \frac{\alpha^2}{\sigma c^2} \int \Phi_m \int G_m \Phi_m + \frac{2\rho\omega^2\alpha^4}{\sigma^2 c^4} \int \Phi_m \int G_m \int g_m \int G_m \Phi_m \quad (10.2.20)$$

in our shorthand notation. As usual, we check this by noting that if the shape of Φ_m is varied (inserting $\Phi_m + \delta\Phi_m$ for Φ_m), the coefficient of $\int G_m\, \delta\Phi_m$, inside the integral, is zero if Φ_m satisfies Eqs. (10.2.5) and (10.2.11).

Thus, if we choose to approximate $\Phi_m(\theta \mid r)$ by the trial function $B_m U_m(r)$, the optimal value of B_m will be

$$B_m = \frac{2P_i\epsilon_m i^m \int J_m(kr \sin\theta) \int G_m U_m}{\int U_m \int G_m U_m - (2\rho k^2\alpha^2/\sigma) \int U_m \int G_m \int g_m \int G_m U_m} \quad (10.2.21)$$

from which we can calculate $\Theta_m(\theta \mid \vartheta)$, the transmitted power and any other

desired part of the solution. If U_m is well chosen, the resulting quantities will be close to the exact solution. Moreover, if U_m is a Bessel function $J_m(qr)$, we can take advantage of Eq. (10.2.6) to obtain a closed form, rather than a series, for the integral of $G_m U_m$. For example, we define the function $H_m(uvw)$, finite for all real values of u, v, w and symmetric to interchange of u, v, w, as

$$\begin{aligned}
H_m(uvw) &= \frac{J_m(w)}{a^4}\int_0^a J_m\left(u\frac{r}{a}\right) r\,dr \int_0^a G_m(r \mid s) J_m\left(v\frac{s}{a}\right) s\,ds \\
&= -\frac{uJ_{m-1}(u)J_m(v)J_m(w)}{(u^2-v^2)(u^2-w^2)} - \frac{vJ_{m-1}(v)J_m(u)J_m(w)}{(v^2-u^2)(v^2-w^2)} \\
&\quad - \frac{wJ_{m-1}(w)J_m(u)J_m(w)}{(w^2-u^2)(w^2-v^2)} \\
&\xrightarrow[uvw\to 0]{} \frac{(uvw)^m}{2^{3m+3}m!\,(m+1)!\,(m+2)!} \\
&\xrightarrow[v\to u]{} \frac{J_m(w)}{2(u^2-w^2)}\left[J_m{}^2(u) - J_{m-1}(u)J_{m+1}(u)\right] \\
&\quad + \frac{J_m(u)}{(u^2-w^2)^2}\left[uJ_{m-1}(u)J_m(w) - wJ_{m-1}(w)J_m(u)\right] \\
&\xrightarrow[u\to 0]{} \frac{u^{2m}J_{m+2}(w)}{w^2 2^{2m+1}m!\,(m+1)!} \\
&\xrightarrow[uv\to w]{} \frac{1}{4w^2}\{J_m(w)[mJ_m{}^2(w) - (m-1)J_{m-1}(w)J_{m+1}(w)] \\
&\quad - wJ_{m-1}(w)[J_m{}^2(w) - J_{m-1}(w)J_{m+1}(w)]\}
\end{aligned} \tag{10.2.22}$$

where $u = ka\sin\theta$ and $w = \alpha ka$, as indicated in Eq. (10.2.6). This provides a closed form, rather than an infinite series, for the integral in the numerator of B_m (if U_m is a Bessel function) and also for the first term in the denominator. Functions H_m are plotted in Fig. 10.9, given in Table XIII.

The same procedure may be used for the second integral in the denominator of B_m, but this would result in a set of integrals over the variable μ in g_m, different from those of Eq. (10.2.14), and thus would require a lot of additional numerical integration. Therefore the series expansion for the G's is preferable here, especially since the result,

$$\begin{aligned}
S_m(uw \mid x) &= \frac{J_m{}^2(w)}{a^7}\int J_m\left(u\frac{r}{a}\right)\int G_m \int g_m \int G_m J_m\left(u\frac{s}{a}\right) \\
&= \sum_{n,\nu}\frac{[J_{m-1}(j_{mn})J_{m-1}(j_{m\nu})]^3}{16 j_{mn}j_{m\nu}} F_n{}^m(u)F_\nu{}^m(u)F_n{}^m(w)F_\nu{}^m(w) \\
&\qquad \times [\chi_{n\nu}{}^m(x) + i\theta_{n\nu}{}^m(x)] \\
&\equiv I_m(uw \mid x) + iQ_m(uw \mid x)
\end{aligned} \tag{10.2.23}$$

involves functions already defined and tabulated, converges rapidly, and is

symmetric to interchange of u and w. These functions are plotted in Fig. 10.10.

Setting $U_m = J_m(qr/a)$ for trial function yields

$$B_m = \frac{2P_i\epsilon_m i^m J_m(w)H_m(uqw)}{J_m(w)H_m(qqw) - 2\gamma w^2 S_m(qw \mid x)}$$

where $w = \alpha ka$, $x = ka$; and $\gamma = \rho a/\sigma$ measures the coupling between membrane and medium. Since both γ and S_m are usually small, the second

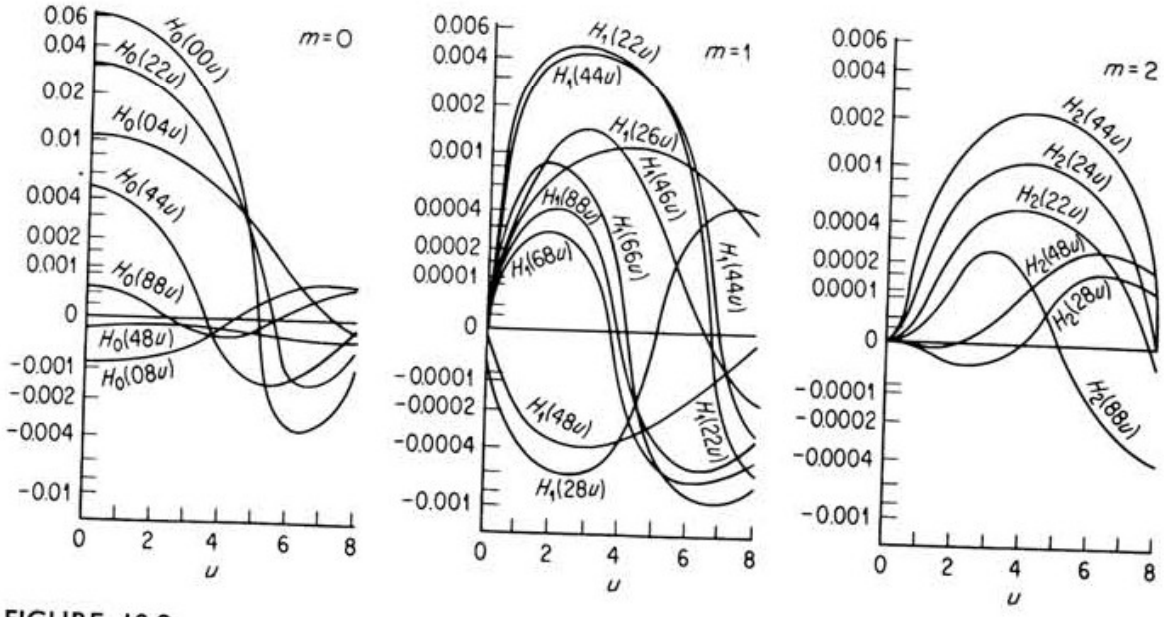


FIGURE 10.9
Typical plots of the angle-distribution function H_m. Note that $H_m(uvw) = H_m(vuw) = H_m(wvu)$. See Eq. (10.2.22) and Table XIII.

term in the denominator is usually much smaller than the first, except when $J_m(w) = 0$ (membrane resonance) or $H_m(qqw) = 0$ (antiresonance). Determining the "best" value of q cannot be done analytically, but graphical determination indicates that $q = u$ is ordinarily a good choice. In nearly all cases the variational expression of Eq. (10.2.20) changes slowly near $q = u$; the exact value at which its derivative with respect to q is zero depends a little on the value of γ, but as long as the second term in the denominator is not large, the choice $q = u = ka \sin\theta$ is quite satisfactory. To obtain a decidedly better trial function, we should have to use a two-term $BJ_m(ur/a) + CJ_m(qr/a)$, with $q \neq u$. We shall show later that this complication is seldom necessary.

When we set $q = u$, the variational solution for the mth component of the pressure difference across the membrane becomes

$$\Phi_m(\theta \mid r) \simeq 2P_i\epsilon_m i^m J_m(w)\frac{iJ_m(kr\sin\theta)}{Z_m(uw \mid x)}$$

where

$$Z_m(uw \mid x) \equiv R_m(uw \mid x) - iX_m(uw \mid x) = |Z_m|\, e^{-i\Omega_m} \qquad (10.2.24)$$

$$= iJ_m(w) - 2i\gamma w^2 \frac{S_m(uw \mid x)}{H_m(uuw)}$$

may be considered the impedance of the mth component of the membrane-plus-radiation load to the incident plane wave. As before, $u = ka \sin \theta$,

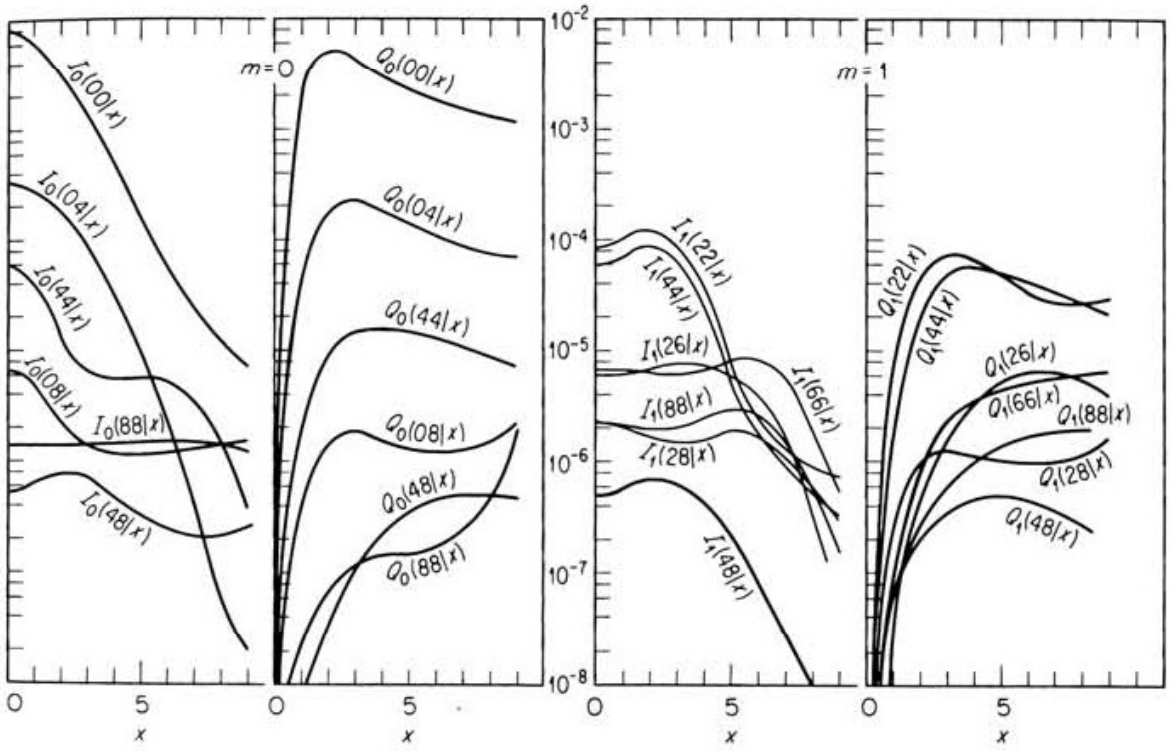


FIGURE 10.10
Typical curves of the radiation resistance and reactance functions I_m and Q_m. Note that $I_m(uw \mid x) = I_m(wu \mid x)$ and $Q_m(uw \mid x) = Q_m(wu \mid x)$.

$w = \alpha ka$, and $x = ka$; $\gamma = \rho a/\sigma$ is the coupling constant. When γ goes to zero there is no radiation load, Z_m becomes simply $iJ_m(w)$, and Φ_m reduces to the mth component of the incident wave. The real part of Z_m, proportional to the θ's of Eq. (10.2.14) and Fig. 10.8, is small if γ is small. The reactive part, X_m, includes the reactance $-iJ_m(w)$ of the unloaded membrane, in addition to the radiation reactance.

The mth component of the membrane's displacement is, from Eqs. (10.2.5) and (10.2.6),

$$Y_m(\theta \mid r) \simeq \frac{2a^2\epsilon_m i^m P_i}{i\sigma c_p{}^2(u^2 - w^2)Z_m(uw \mid z)} \left[J_m(w)J_m\left(u\frac{r}{a}\right) - J_m(u)J_m\left(w\frac{r}{a}\right)\right] \qquad (10.2.25)$$

This function goes to zero at $r = a$, as it should; it does not become infinite at $u = w$, as the limiting form of Eq. (10.2.6) shows. Its amplitude is inversely proportional to the impedance Z_m. The angle-dependence factor Θ_m for the transmitted wave and the resulting transmitted intensity are then

$$\Theta_m(\theta \mid \vartheta) \simeq 2a\gamma\epsilon_m P_i w^2 \frac{H_m(uvw)}{iZ_m(uw \mid x)}$$

$$\frac{1}{\rho c}|p_+|^2 \xrightarrow[R\to\infty]{} \frac{4a^2}{R^2}\gamma^2|P_i|^2\frac{w^4}{\rho c}\sum_{m,l}\epsilon_m\epsilon_l\frac{H_m(uvw)H_l(uvw)}{|Z_m(uw \mid x)Z_l(uw \mid x)|} \times \cos(m\phi)\cos(l\phi)\cos(\Omega_m - \Omega_l) \quad (10.2.26)$$

The transmitted intensity varies with the azimuth ϕ and with the zenith angle ϑ through the variable $v = ka\sin\vartheta$; thus the functions $H_m(uvw)$ determine the dependence of the transmitted wave on angle of incidence θ, on the zenith angle ϑ of the radius to the observer, on the wave-velocity ratio $\alpha = c/c_p$, and on the coupling constant γ. Because of the symmetry of the functions $H_m(uvw)$, all three factors, $\sin\theta$, $\sin\vartheta$, and α, have the same effect. The symmetry between θ and ϑ is another way of stating the reciprocity principle; the symmetry between $\sin\theta$ and α is related to the transparency of the infinite membrane or plate which occurs when $\sin\theta$ equals α [see discussion of Eq. (10.2.30)].

Finally, the total acoustic power transmitted through the membrane is

$$\Upsilon \simeq 8\pi\alpha a^2\gamma^2\frac{|P_i|^2}{\rho c}\sum_m \epsilon_m \frac{w^3 Q_m(ka\sin\theta,\, ka\alpha \mid ka)}{|Z_m(ka\sin\theta,\, ka\alpha \mid ka)|^2} \quad (10.2.27)$$

where the factors Q_m involve the radiation-resistance factors $\theta_n{}^m(ka)$, plotted in Fig. 10.10. We note that $(\pi a^2|P_i|^2/\rho c)\cos\theta$ is the power incident on the membrane, and thus that the dimensionless quantity

$$T \simeq 2\alpha\gamma^2\sum_m \epsilon_m w^2 \operatorname{Re}\frac{1}{Z_m(ka\sin\theta,\, ka\alpha \mid ka)} \quad (10.2.28)$$

may be called the *transmission factor*, the ratio between the power actually transmitted through the membrane to the power carried in an area πa^2 of the incident wavefront. If the membrane transmitted all the incident power, then T would equal $\cos\theta$. It is usually considerably smaller than this, for many reasons, which our variational solution is accurate enough to demonstrate.

Transmission; resonance and antiresonance

The presence of the membrane impedes the flow of incident energy through the circular aperture in the wall, because of its mass and tension, as indicated by the presence of the factors $\alpha = c/c_p = c\sqrt{\sigma/T}$ and $\gamma = \rho a/\sigma$ in the formula for Υ. The transmission factor is inversely proportional to the

square root of the membrane tension and the three-halves power of the membrane mass density σ. These general effects are modified, as we change frequency and angle of incidence, by the various membrane resonances and by the results of the coupling between membrane and medium.

Well below the lowest membrane resonance ($\alpha ka \ll 1$), all terms for $m > 0$ are negligible, and Eqs. (10.2.25) to (10.2.27) become

$$\eta \simeq \frac{a^2 P_i}{2\sigma c_p{}^2}\left[1 - \left(\frac{r}{a}\right)^2\right]$$

$$p_+ \to \frac{\rho a}{8\sigma}(ka\alpha)^2 P_i \frac{ae^{ikR}}{R} \qquad R \to \infty$$

$$\Upsilon \simeq 8\pi a^2 \left(\frac{\rho a}{\sigma}\right)^2 \frac{|P_i|^2}{\rho c}\left(\frac{ka\alpha}{j_{01}}\right)^4$$

which are similar to, but better approximations than, Eqs. (10.2.17).

To separate the various effects which enter as the incident frequency is increased, let us first look at the case of normal incidence. When $\theta = 0$, parameter $u = ka \sin\theta$ is zero, there is no dependence on the axial angle ϕ, and consequently only the $m = 0$ term in the sums over m differs from zero. The simplified formulas become

$$H_0(0vw) = \frac{vJ_0(v)J_1(w) - wJ_0(w)J_1(v)}{vw(w^2 - v^2)} \xrightarrow[v \to w]{} \frac{1}{2w^2}[J_1{}^2(w) - J_0(w)J_2(w)]$$

$$H_0(00w) = \frac{1}{2w^2}J_2(w) \qquad F_n^0(0) = \frac{2}{j_{0n}J_1(j_{0n})}$$

$$S_0(0w \mid x) = \sum_{n,\nu}\left[\frac{J_1(j_{0n})J_1(j_{0\nu})}{2j_{0n}j_{0\nu}}\right]^2 F_n^0(w)F_\nu^0(w)[\chi_n^0(x) + i\theta_{n\nu}^0(x)]$$

$$\eta = Y_0(0 \mid r) \simeq \frac{2a^2 P_i}{\sigma c_p{}^2}\frac{J_0(wr/a) - J_0(w)}{w^2 Z_0(0w \mid x)} \tag{10.2.29}$$

$$p_+ \to \Theta_0(0 \mid \vartheta)\frac{e^{ikR}}{R} \simeq 2\gamma w^2 P_i \frac{H_0(0vw)}{Z_0(0w \mid x)}\frac{ae^{ikR}}{R} \qquad R \to \infty$$

$$\Upsilon \simeq 2\pi a^2 \alpha \gamma^2 \frac{|P_i|^2}{\rho c}\,\mathrm{Re}\,\frac{1}{Z_0(0w \mid x)}$$

where

$$Z_0(0w \mid x) = iJ_0(w) + 4\gamma \frac{w^4}{J_2(w)}\sum_{n,\nu}\left[\frac{J_1(j_{0n})J_1(j_{0\nu})}{2j_{0n}j_{0\nu}}\right]^2 \times F_n^0(w)F_\nu^0(w)[\theta_{n\nu}^0(x) - i\chi_{n\nu}^0(x)]$$

for $v = ka \sin\vartheta$, $w = \alpha ka$, and $x = ka$.

In Fig. 10.11 four curves for $T = \rho c \Upsilon / \pi a^2 |P_i|^2$ are plotted as functions of ka for two different values of $\alpha = c/c_p$ and of $\gamma = \rho a/\sigma$. Increase of γ increases the coupling between the medium and the membrane, thus increasing "throughput" and raising T closer to unity, the value for a completely

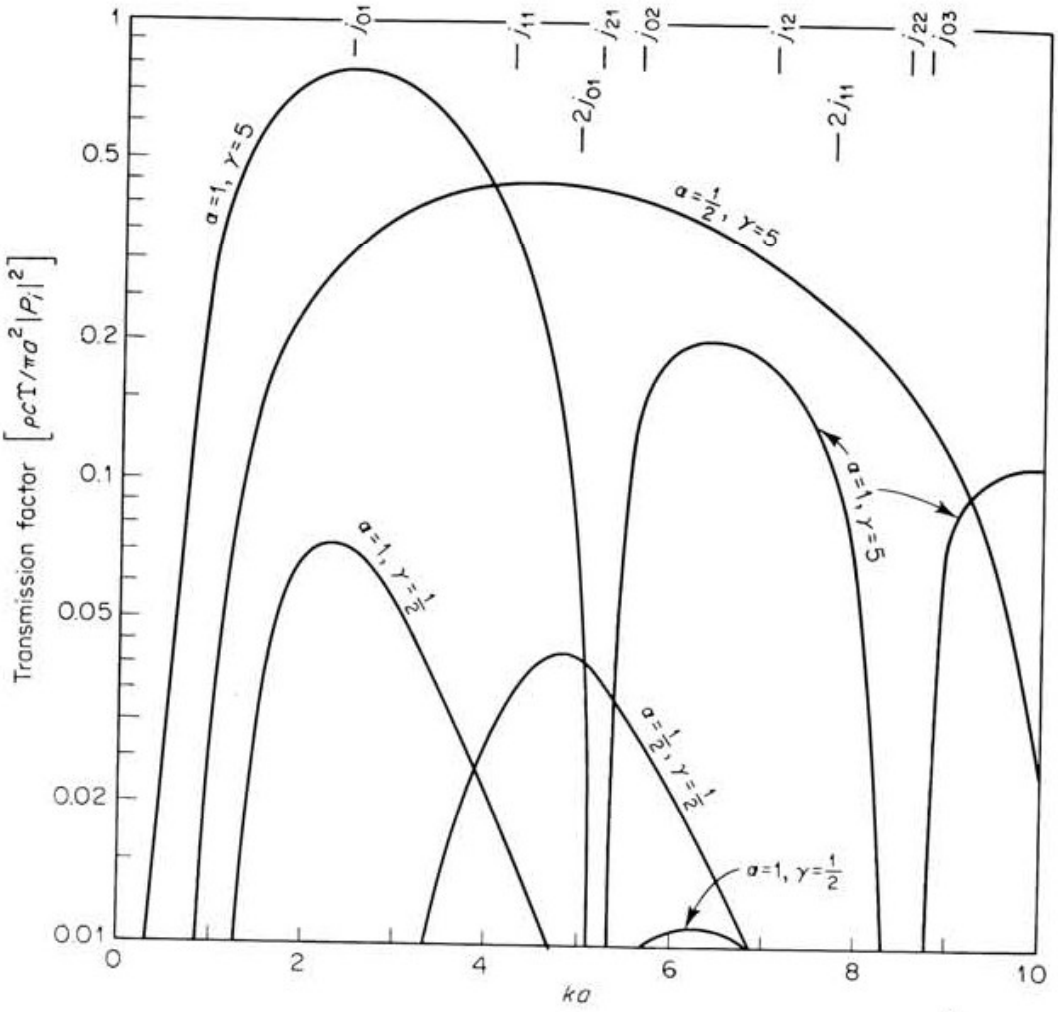


FIGURE 10.11
Power Υ transmitted through a circular membrane of radius a in a rigid wall, from a normally incident plane wave of frequency $kc/2\pi$, for different membrane-medium coupling constants $\gamma = \rho a/\sigma$ and wave-velocity ratios $(c/c_p) = \alpha$.

"transparent" membrane. It also broadens the membrane-resonance peaks. Increase of $1/\alpha$ proportionally increases all the membrane resonances in terms of $x = ka$, since a membrane resonance occurs when $w = \alpha ka = j_{0n}$.

Thus, taking the $\alpha = 1$, $\gamma = 5$ curve, T increases from zero at $ka = 0$ to reach a maximum at $ka \simeq j_{01}$, where $X_0(0w \mid x) = 0$. This is the lowest symmetric resonance of the membrane, at which nearly 80 percent of the incident power is transmitted. As ka is further increased, T decreases, until, when $ka = j_{21}$, T is zero again. At this frequency, called the *antiresonance* frequency, $H_0(00w)$ is zero; the membrane shape is such that no energy is

radiated to infinity, and the membrane behaves as though it were rigid. To the degree of approximation of these formulas, the effective impedance $Z_0(0w \mid x)$ becomes infinite. A better trial function, chosen so that not all parts of the membrane would vibrate in phase [as it must with the form $BJ_0(ur/a)$ for Φ_0], would remove the zero, but there still would be a pronounced minimum at $w = j_{21}$. When $\theta > 0$, the other components, which do not have minima at the same place, would cover the minimum; so it is not usually worthwhile to go to the trouble to find an improved value at the minimum for $m = 0$.

Just above this antiresonance frequency the membrane has its next $m = 0$ resonance. Because the antiresonance is close below, the resonance peak is distorted, the maximum coming somewhat above $w = j_{02}$. As ka is still further increased, there comes another antiresonance at $w = j_{22}$; just beyond it X_0 goes to zero at the next $m = 0$ resonance; and so on. At higher frequencies the peaks and valleys come closer together on a logarithmic scale (and with a better trial function, the valleys diminish in depth), until at high frequencies the curve is an irregularly fluctuating one, with a general trend of decrease with increasing frequency, typical of mass-controlled response. At other angles of incidence, the $m > 0$ terms add their differently placed peaks and valleys; so the curve for T is even less irregular, only the first few peaks and valleys being distinguishable.

Thus the membrane resonance maxima and antiresonance minima, both typical of the finite size of the membrane boundaries, serve to differentiate these phenomena from the reflection and transmission of sound through an unbounded membrane. Appropriate modification of the discussion of Eq. (10.1.32) for the case of a membrane, rather than a plate (taken up in Prob. 7), provides the transmission formulas for an infinite membrane. If the membrane has areal density σ and tension $T = \sigma c_p^2$, an incident plane wave of amplitude P_i and angle of incidence θ, coming from $-\infty$, would produce a plane transmitted wave in region $z > 0$, traveling in the same direction as the incident wave, with an amplitude

$$P_t = P_i\left[1 + \tfrac{1}{2}i\frac{\sigma}{\rho a}ka\cos\theta\left(\frac{1}{\alpha^2}\sin^2\theta - 1\right)\right]^{-1}$$

Thus the infinite membrane sends all its transmitted energy in one direction ($\phi = 0$, $\vartheta = \theta$), instead of in all directions, as Eqs. (10.2.26) indicate for the finite membrane. The equivalent, for the infinite membrane, of the transmission factor T of Eqs. (10.2.29) and Fig. 10.11 would be the power transmitted through an area πa^2 of the membrane, divided by πa^2 times the incident intensity,

$$T = \frac{\cos\theta}{1 + (ka/2\gamma)^2\cos^2\theta[(1/\alpha^2)\sin^2\theta - 1]^2} \qquad \text{(unbounded membrane)} \tag{10.2.30}$$

the cosine in the numerator coming from the inclination of the area πa^2 of the membrane to the direction of the incident-transmitted wave. This formula has a strong maximum at $\sin\theta = \alpha$, when the velocity of the pressure wavefronts at the membrane equals the velocity of transverse waves on the membrane.

To compare this with the transmission factor of Eq. (10.2.28), for the bounded membrane, we have plotted Υ as a function of the angle of incidence θ, for several different frequencies. We choose $\alpha = \frac{1}{2}$ so that the peak of transmission for the infinite membrane will come at $\theta = 30°$; we choose the coupling constant $\gamma = 1$ large enough to expect an appreciable fraction of the incident energy to be transmitted at the angle of transparency. The choice of frequency is not easy if we are trying to exhibit a simple relationship between the transmissions for infinite and finite membranes; the finite membrane has too many resonances and antiresonances to be able to avoid them all (except at very low frequencies, where very little energy gets through anyway). We have chosen $ka = 2$, 4, and 8 to illustrate some of the complexities; other frequencies would have exhibited other peculiarities.

Thus Fig. 10.12 has three plots of T, as computed from Eq. (10.2.28), as function of angle of incidence θ, for $ka = 2$, 4, and 8. The dashed lines are T from Eq. (10.2.30) for the unbounded membrane for the same values of α, γ, and ka, for comparison. At $ka = 2$ the driving frequency is well below any resonance or antiresonance, but the incident wavelength $2\pi/k$ is longer than the diameter of the membrane, and one would not expect much directionality, and thus much similarity, with the infinite-membrane behavior. The transmitted wave is more or less independent of ϕ, and less than one percent of the incident power is transmitted—somewhat less at $\theta = 90°$ than at $\theta = 0°$.

The frequency corresponding to $ka = 4$ is near the first resonance for $m = 0$ $(j_{01}/\alpha = 4.8)$. Thus the curve of T for $ka = 4$ reflects the fact that the $m = 0$ component of Υ is exaggerated; the transmitted power is large for $\theta = 0$ and falls off as θ increases. The components for $m > 0$, which would contribute to a "transparency" peak at $\theta = 30°$, are almost drowned out by the strong $m = 0$ contribution; one can see a small "bump" near $\theta = 30°$ where the $m = 1$ and $m = 2$ components have their maxima.

On the other hand, the frequency corresponding to $ka = 8$ is close to a membrane resonance for $m = 1$ $(j_{11}/\alpha = 7.7)$, so that the $m = 1$ component is exaggerated. Also, both $H_0(uuw)$ and $H_1(uuw)$ have zeros, corresponding to antiresonances, the one for H_0 coming at θ about 30° and the one for H_1 at θ about 50°. Thus the $m = 0$ contribution is nearly nonexistent at $\theta = 30°$, the angle at which the infinite membrane is transparent. The peak of Υ which is exhibited, corresponding to an increase in transmission by a factor of nearly 10 from $\theta = 0$ to $\theta = 20°$, comes predominantly from the $m = 1$ term. This term goes to an antiresonance zero at 50°, leaving the small contributions from the $m = 0$, 2, and 3 components.

Beyond 50° the transmission has another small peak; the resulting curve has little resemblance to the dashed curve.

The angle distribution of the transmitted wave also exhibits some of the complexities arising from the wave reflections from the finite-membrane

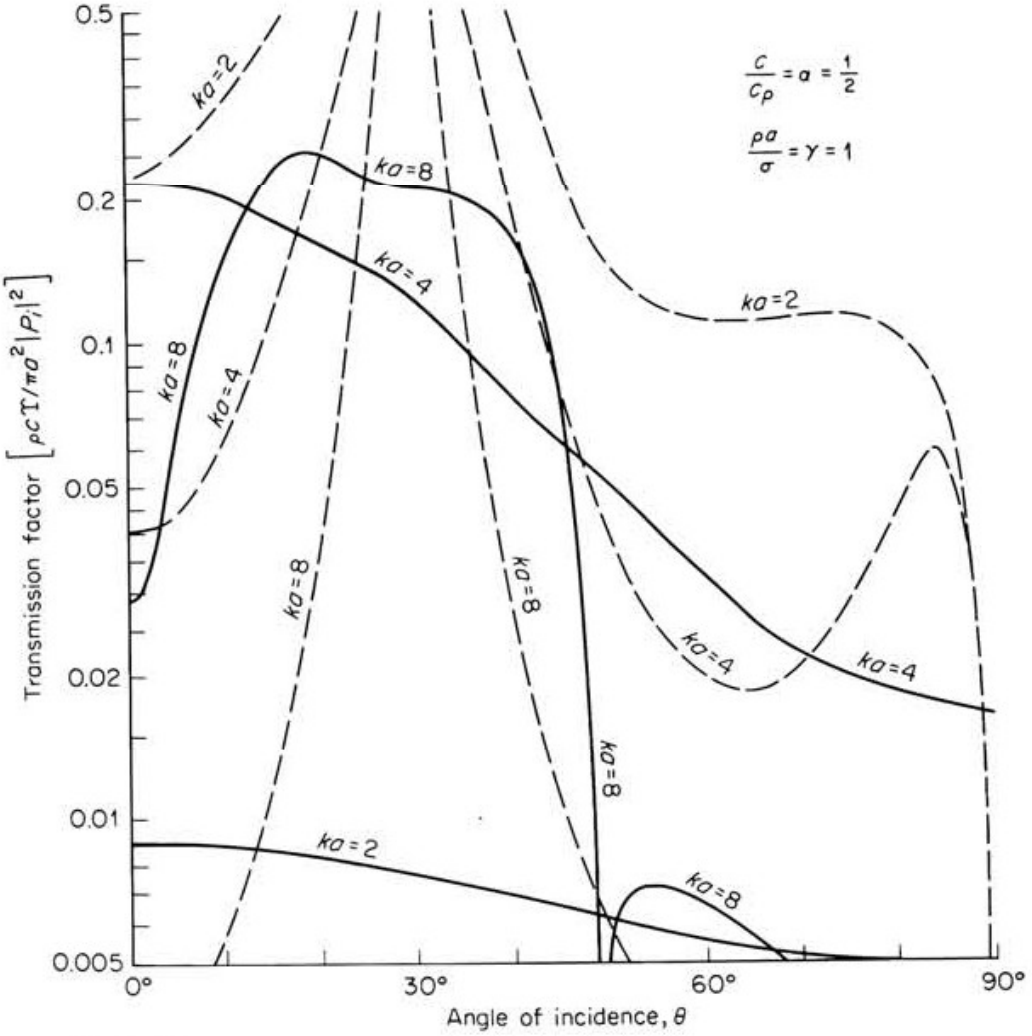


FIGURE 10.12
Power Υ transmitted through a circular membrane of radius a in a rigid wall as a function of angle of incidence θ of a plane wave of frequency $kc/2\pi$. Dashed lines would be the transmission factor if the circular membrane were to behave as though it were of infinite extent, with no wave reflection at its edges.

boundaries, which produce the large differences from the simple behavior of the infinite membrane. In Fig. 10.13 are plotted contours of intensity of the transmitted wave, a large number of wavelengths from the origin on the $z > 0$ side, for a plane incident wave in the direction $\phi = 0$, $\theta = 30°$ (the direction of maximum transmission for an infinite membrane) for the

same values of α, γ, and ka as for Fig. 10.12. The transmitted wave for an unbounded membrane would be a plane wave in the same direction as the incident wave, which would be represented by a delta-function peak at $\phi = 0$, $\vartheta = 30°$, shown by the cross on each contour plot.

In the left-hand plot, for $ka = 4$, we see again the effect of the $m = 0$ resonance. The transmitted energy radiates away in a pattern nearly symmetric with respect to the normal ($\vartheta = 0$); the intensity peak is deflected a

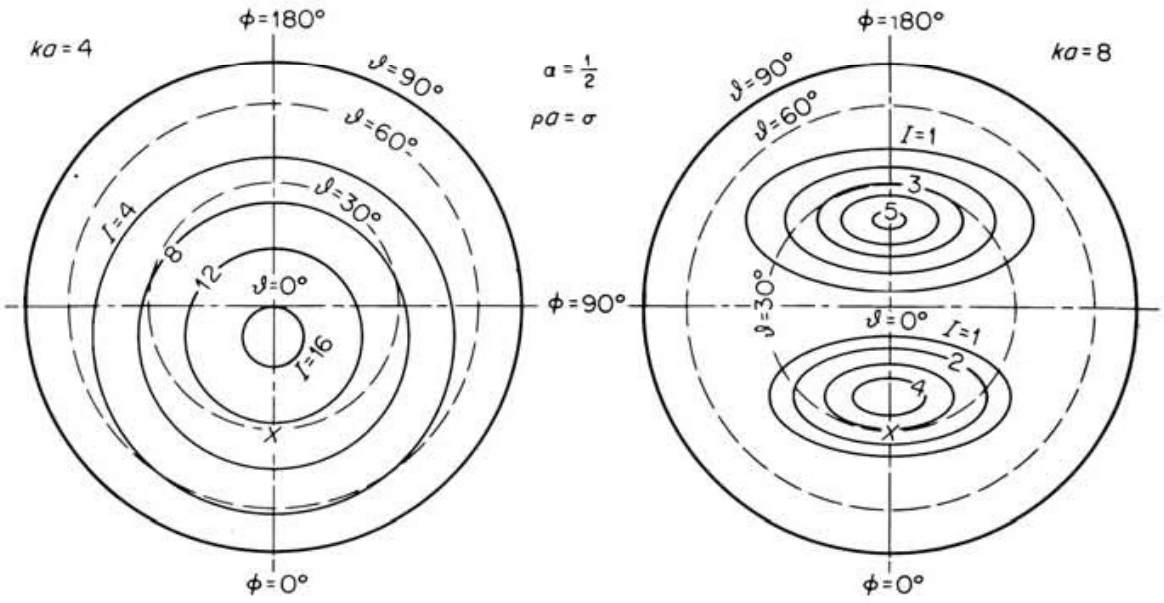


FIGURE 10.13
Angle distribution of transmitted sound for conditions of Fig. 10.12, for angle of incidence of 30°. Contours of I, proportional to $|p_+|^2$, at large distance R from origin, plotted against spherical angles ϑ and ϕ.

small amount away from the zenith, but is certainly not centered on the cross. The peak is quite broad, typical of long wavelengths. In contrast, the radiation pattern for $ka = 8$ is typical of the $m = 1$ resonance. Since the driving frequency is higher than the $m = 1$ resonance, the $m = 1$ component is out of phase with the others, which results in more being radiated in the $\phi = 180°$ direction than in the $\phi = 0$ direction of the incident wave. The $m = 0$ component is very small, as mentioned earlier; the interaction between the $m = 1$ and $m = 2$ components broadens the peaks parallel to the $\phi = 90$–$270°$ axis (makes them "ridges," rather than peaks); it, plus the $m = 0$, $m = 1$ interaction, accounts for the asymmetry with respect to this axis, and the fact that B_1 is negative accounts for the fact that the $\phi = 180°$ ridge is higher than the $\phi = 0$ one. The peak maxima are not at $\vartheta = 30°$, though they are closer than the peak for $ka = 4$.

Other calculations, for other values of ka, α, γ, and θ, show a wide variety of forms, corresponding to the manifold combination of resonances, antiresonances, and transparencies, which can result in all possible combinations of amplitudes and phases for each component. In such an exploration of a truly complex system, the results of a variational calculation are of great value, for they are usually accurate enough to indicate the correct order of magnitude of the various effects, and though the answers may differ by 10 or 20 percent from the correct values, they do not go to infinity or to zero, and thus fail completely, at crucial points.

Somewhat better results can be obtained by using the two-term trial function $AJ_m(ur/a) + BJ_m(qr/a)$, with q adjusted so that $J_m(w)H_m(qqw) = 2\gamma w^2 I_m(qw \mid x)$. In this case the "best" values of A and B turn out to be

$$A_m = \frac{2P_i\epsilon_m i^m J_m(w)H_m(uqw)}{J_m(w)H_m(uqw) - 2\gamma w^2 S_m(uqw \mid x)}$$

$$B_m = 2P_i\epsilon_m i^m J_m(w)H_m(uqw)\frac{-2\gamma w^2 S_m(uw \mid x)}{[J_m(w)H_m(uqw) - 2\gamma w^2 S_m(uqw \mid x)]^2}$$

where

$$S_m(uqw \mid x) = \frac{J_m{}^2(w)}{a^7}\int J_m\left(\frac{ur}{a}\right)\int G_m\int g_m\int G_m J_m\left(\frac{qr}{a}\right)$$

is an obvious generalization of the function of Eq. (10.2.23). When the coupling constant γ is small, B_m is small compared with A_m; so this improvement is slight, except near an antiresonance. Because $S_m(uw \mid x)$ is complex, B_m is out of phase with A_m, so that the resulting shape of η, for each m, has some traveling waves as well as standing waves, as would be required for a damped membrane. As mentioned before, the values of the component terms at their antiresonances are not usually of much interest, so that it is not usually worthwhile to carry out this more arduous calculation. Thus, in practice, as long as γ is less than about 10, the one-term trial function $BJ_m(ur/a)$, with the resulting formulas (10.2.24) to (10.2.27), is sufficient.

A similar calculation could be carried out for a plate set in an otherwise rigid wall. The computations would be somewhat more laborious than the ones sketched here for the membrane, but the general characteristics would turn out to be much the same. The functions F, H, θ, χ, and S would involve combinations of the Bessel functions of real and imaginary arguments required for a disk clamped at its edge, but their general shapes would be similar to those plotted for the membrane, with a different frequency scale to correspond to the different resonance and antiresonance frequencies. A similar series of peaks and valleys would result for the curves of the transmission factor T. Some of this will be demonstrated in the problems.

10.3 PERIODICALLY SUPPORTED SYSTEMS

A system of considerable practical interest consists of a plane partition, strengthened by linear studs or beams, periodically placed. The complete details of finite structures of this type cannot be represented by any soluble model, but many of their properties can be understood by studying a model of infinite extent, with the struts replaced by linear impedance loads, attached at equal spacing to the wall, each load having stiffness, mass, and resistance per unit length. A completely satisfactory model would have a thin, uniform plate for the partition. Such a model can be worked out (some aspects are given as problems), but the algebraic complexities hinder an understanding of the results.

Membrane with regularly spaced supports

We shall, instead, replace the partition by a flexible membrane (see Sec. 5.2) of mass σ per unit area and with transverse wave velocity $c_p = c/\alpha$, c being the wave velocity in the medium (air or water) on both sides of the partition. This is not a bad approximation for simple-harmonic motions, as long as the wavelength of the transverse waves in the wall is smaller than the spacing between struts. In this case the results computed for the membrane model, particularly in regard to the coupled waves in the medium, are much the same as for a model using a plate with equal σ and c_p. Transient effects will differ, of course, because the velocity of transverse waves in a plate is proportional to $\sqrt{\omega}$, instead of being independent of ω, as with a membrane. With this limitation in mind, we proceed with the simpler model, leaving the modifications produced by the stiffness of the plate to be taken up in the problems.

Thus our model is an infinite membrane, the equilibrium plane being the xy plane, of mass σ per unit area, under tension $T = \sigma c_p^2$, with transverse wave velocity $c_p = c/\alpha$ (thus defining the dimensionless constant α). Attached to this membrane, along lines parallel to the y axis, are line impedances, of magnitude Z per unit length (dimensions of force per unit length per unit velocity) along the lines $x = nl$ $(n = \ldots, -1, 0, 1, 2, \ldots)$.

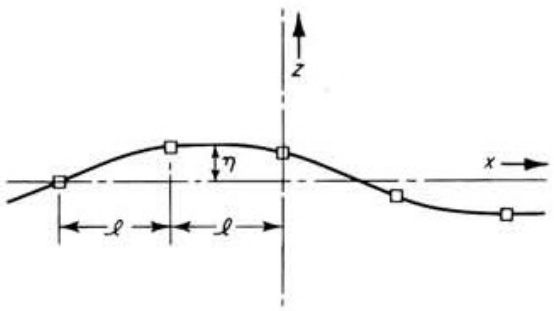


FIGURE 10.14
Transverse displacement of membrane, periodically loaded with linear supports, each having impedance Z per unit length in the y direction.

These are roughly equal to struts or beams, spaced a distance apart. In many cases Z is larger than σc_p, the characteristic impedance of the membrane. Thus our problem, in a sense, involves coupling three systems, the membrane, the set of parallel struts, and the medium on both sides of the wall. Figure 10.14 shows a section of the membrane, with its supporting struts.

Motion in a vacuum

First we deal with the coupling of a membrane-and-strut system, without the added complications of the radiation reaction of the surrounding fluid. We omit also the complication of wave motion in the y direction, since this depends on the longitudinal stiffness of the struts; we consider only wave motion in the x direction. In this case the struts move as a set of coupled oscillators, and the strips of membrane between each pair move in combinations of moving and standing waves coupling the struts. The motion is related to, but more complex than, the motion of coupled oscillators, treated in Sec. 3.3.

As an easy introduction, we can study the behavior of the system under the influence of a transverse driving force $Fe^{i\varkappa x - i\omega t}$ per unit area. The steady-state membrane displacement η must also have the factor $e^{-i\omega t}$ and must repeat itself in each successive strip, with the exception of an additional phase factor $e^{i\varkappa l}$ per strip. In other words, $\eta(x + l)$ should equal $e^{i\varkappa l}\eta(x)$.

In addition to the driving force, there will be, at $x = nl$, the linear reaction of the strut impedance, $-Z(\partial\eta/\partial t) = ikc\eta Z$ times the delta function $\delta(x - nl)$. Thus the equation of motion for the membrane will be

$$\frac{\sigma c^2}{\alpha^2}\frac{\partial^2\eta}{\partial x^2} + \sigma k^2 c^2 \eta = -F_\varkappa e^{i\varkappa x} - ikcZ\eta \sum_{n=-\infty}^{\infty} \delta(x - nl) \qquad (10.3.1)$$

Except at the points $x = nl$, where there is a discontinuity in slope, η must be a combination of a solution of the homogeneous equation plus a solution of the inhomogeneous equation, $-F_\varkappa e^{i\varkappa x}$ being the inhomogeneous term. Thus the solution, except at $x = nl$, will be the combination

$$\frac{\alpha^2 F_\varkappa/\sigma c^2}{\varkappa^2 - k^2\alpha^2}(e^{i\varkappa x} + Be^{ik\alpha x} + Ce^{-ik\alpha x}) \qquad 0 < x < l$$

with constants C and B so adjusted that $Be^{ik\alpha x} + Ce^{-ik\alpha x}$ at $x = l$ is $e^{i\varkappa l}$ times its value $B + C$ at $x = 0$. To achieve this, B must equal $A(e^{i\varkappa l} - e^{-ik\alpha l})$, and C must equal $A(e^{ik\alpha l} - e^{i\varkappa l})$, where A must then be adjusted to satisfy the strut-reaction requirements at $x = nl$.

Thus the solution is

$$\eta = \frac{\alpha^2 F_\varkappa/\sigma c^2}{\varkappa^2 - k^2\alpha^2}(e^{i\varkappa x} + \Omega)$$

where

$$\Omega = A\begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ [\sin \alpha k(x+l) - e^{-i\varkappa l}\sin \alpha kx] & -l < x < 0 \\ [e^{i\varkappa l}\sin \alpha kx + \sin \alpha k(l-x)] & 0 < x < l \\ [e^{2i\varkappa l}\sin \alpha k(x-l) + e^{i\varkappa l}\sin \alpha k(2l-x)] & l < x < 2l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases}$$

with A adjusted so that the discontinuity in slope at $x = 0$ is $-ik\alpha\zeta\eta(0)$, where $\zeta = Z/\sigma c_p = \alpha Z/\sigma c$ is the dimensionless strut-impedance parameter. In other words,

$$A[\alpha k e^{i\varkappa l} - \alpha k\cos(\alpha kl) - \alpha k\cos(\alpha kl) + \alpha k e^{-i\varkappa l}] = -i\alpha k\zeta[1 + A\sin(\alpha kl)]$$

The solution of this is $A = -i\zeta/Q$, where

$$Q = 2\cos(\varkappa l) - 2\cos(\alpha kl) + i\zeta\sin(\alpha kl)$$

The steady-state solution may be written in the form just shown, or as $e^{i\varkappa x}$ times a Fourier series, the coefficients of which may be obtained by substitution in Eq. (10.3.1). Contrary to appearances, η does not become infinite when $\varkappa = k\alpha$. Various forms for η are given here:

$$\eta = \frac{\alpha^2 F_\varkappa/\sigma c^2}{\varkappa^2 - k^2\alpha^2}\begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \left\{e^{i\varkappa x} - i\dfrac{\zeta}{Q}[\sin \alpha k(x+l) + e^{-i\varkappa l}\sin \alpha kx]\right\} & -l < x < 0 \\ \left\{e^{i\varkappa x} - i\dfrac{\zeta}{Q}[e^{i\varkappa l}\sin \alpha kx + \sin \alpha k(l-x)]\right\} & 0 < x < l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases}$$

$$\xrightarrow[\varkappa \to \alpha k]{} \frac{i\alpha F_\varkappa/2k\sigma c^2}{\sin(\alpha k)}\begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \left[\left(x + \dfrac{2l}{\zeta}\right)e^{i\alpha kx}\sin \alpha kl - l e^{i\alpha kl}\sin \alpha kx\right] & 0 < x < l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases}$$

$$= \frac{\alpha^2 F_\varkappa/\sigma c^2}{\varkappa^2 - k^2\alpha^2}e^{i\varkappa x}\left\{1 + \frac{2i\zeta\alpha kl}{Q}[\cos(\varkappa l) - \cos(\alpha kl)] \times \sum_{m=-\infty}^{\infty}\frac{e^{(2\pi i m/l)x}}{(\varkappa l + 2\pi m)^2 - k^2\alpha^2 l^2}\right\}$$

$$\xrightarrow[\varkappa \to \alpha k]{} \frac{\alpha^2 F_\varkappa}{\sigma c^2}e^{i\alpha kx}\left[\frac{1 - \alpha kl\cot(\alpha kl) + 4i(\alpha kl/\zeta)}{4k^2\alpha^2} - \sum_{m=-\infty}^{\infty}(1 - \delta_{0m})\frac{l^2}{2\pi m}\frac{e^{(2\pi i m/l)x}}{2\alpha kl + 2\pi m}\right] \quad (10.3.2)$$

the factor $1 - \delta_{0m}$ in the last form ensuring that the term $m = 0$ is omitted from the sum.

The Fourier series expansions are obtained by substituting $\eta = \sum B_n \exp(i\varkappa x + 2\pi i n x/l)$ in Eq. (10.3.1). To find the coefficients B_m, we multiply by $\exp(-i\varkappa x - 2\pi i m x/l)$ and integrate over x from $-\epsilon$ to $l - \epsilon$ ($\epsilon \to 0$). The integral over the last term on the right is $-ikcZ \sum B_n$, for any value of m; thus we obtain an equation for B_m in terms of $F_\varkappa$ and of $\sum B_n$. Summation over m results in an equation for $\sum B_n$, which involves a sum which can be expressed in closed form.

$$\sum_n \frac{ik\alpha l\zeta}{(\varkappa l + 2\pi n)^2 - (k\alpha l)^2} = -\tfrac{1}{2}i\zeta \frac{\sin(\alpha kl)}{\cos(\varkappa l) - \cos(\alpha kl)}$$

Further algebra results in the series form given in Eq. (10.3.2). Alternatively, it may be obtained by expanding the first form of Eq. (10.3.2) in a Fourier series in the usual manner.

The first form of Eq. (10.3.2) indicates the division between the traveling wave $e^{i\varkappa x}$ and the membrane wave motion, at wavenumber $k\alpha$, induced in each strip by the motion of the struts, each successive strip differing in phase by the factor $e^{i\varkappa l}$. The limiting form shows that η does not become infinite when $\varkappa \to k\alpha$; it becomes infinite only if *both* $\varkappa$ and $k\alpha$ equal $\pi n/l$ or when the quantity Q is zero. The Fourier-series form, valid for all values of x, shows that η is equal to $e^{i\varkappa x}$ times a function which is periodic in x with period l, as required.

Free-wave motion

The free-wave motion of the strut-membrane system is obtained by finding the values of $\varkappa$ for which $\eta/F_\varkappa$ becomes infinite, for then η can be finite even though $F_\varkappa \to 0$. This occurs when Q, defined earlier, becomes zero. Suppose the root of $Q = 0$ with the smallest, positive, real part is $\varkappa_q + i\xi$. Then, because of the periodicity of Q, all the possible roots of $Q = 0$ are

$$\varkappa_n = \varkappa_q + \frac{2\pi n}{l} + i\xi \qquad \text{and} \qquad -\varkappa_q + \frac{2\pi n}{l} - i\xi \tag{10.3.3}$$

where n is any integer, positive or negative. Since the addition or subtraction of $2\pi n/l$ to $\varkappa$ makes no difference to the value of either $e^{i\varkappa l}$ or $\cos \varkappa l$, there are really only two different solutions for the free-wave motion of the system.

$$\eta = \begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A\{\exp[\pm(i\varkappa_q - \xi)l] \sin(\alpha kx) + \sin[\alpha k(l - x)]\} & 0 < x < l \\ A\{\exp[\pm 2(i\varkappa_q - \xi)l] \sin[\alpha k(x - l)] \\ \qquad + \exp[\pm(i\varkappa_q - \xi)l] \sin[\alpha k(2l - x)]\} & l < x < 2l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

$$= -iA \sin(\alpha kl) \sum_{n=-\infty}^{\infty} \frac{\exp\{\pm i[\varkappa_0 + (2\pi n/l)]x\}}{(\varkappa_0 l + 2\pi n)^2 - (\alpha kl)^2} \quad \text{(all values of } x\text{)} \tag{10.3.4}$$

where $\varkappa_0 = \varkappa_q + i\xi$. The solution with the plus signs represents a wave moving to the right; the one with minus signs, a wave going to the left; each with speed $c_q = \omega/\varkappa_q$ and with attenuation constant ξ. The Fourier-series form shows the nature of the periodicity.

A plot of the root $\varkappa_q$ of the equation $Q = 0$ is given in Fig. 10.15 for purely reactive strut impedances, $Z = -iX$. Contours for the wavenumber parameter $\varkappa_q l/\pi = u_q$ are plotted against different values of $v = \alpha kl/\pi$ and

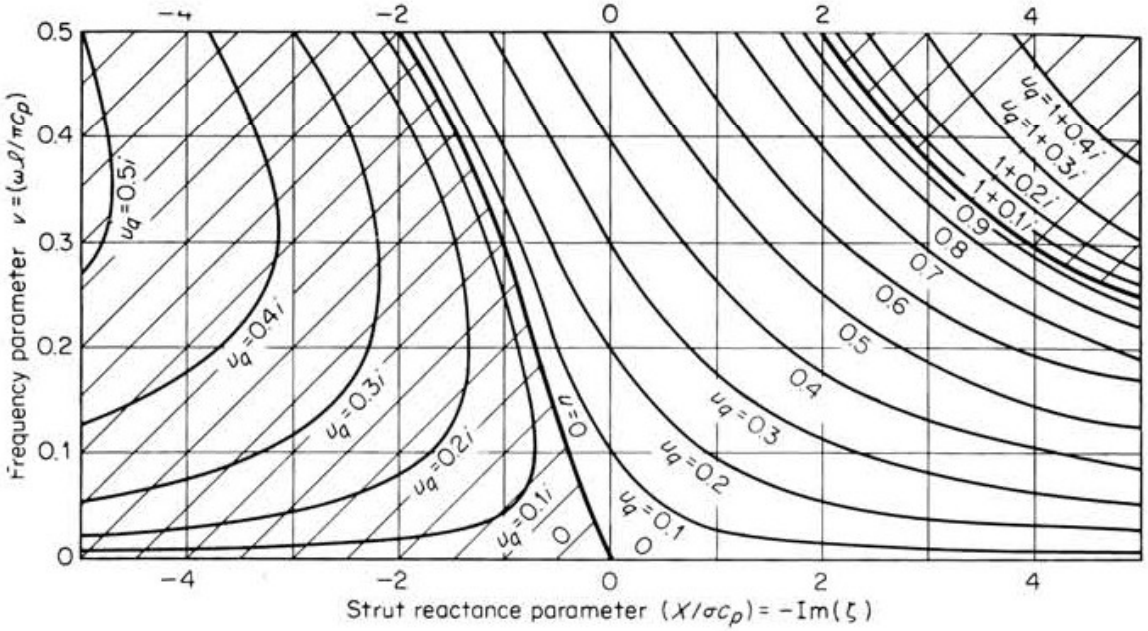


FIGURE 10.15
Relation between system wavenumber parameter $u_q = \varkappa_q l/\pi$, frequency parameter $v = (\omega l/\pi c_p)$, and strut reactance parameter $-\mathrm{Im}\ (\zeta)$ for traveling waves in a periodically supported membrane with purely reactive support struts ($Z = R - iX$, $R = 0$).

of $-\mathrm{Im}\ \zeta$. In this case, when $\mathrm{Re}\ \zeta = 0$, there is a band of values of v and of $-\mathrm{Im}\ \zeta$ for which the parameter u_q is real ($\xi = 0$), and the wave does not attenuate, but travels at a speed $c_q = \omega l/\pi u_q$. Outside this band, in the shaded regions of the plot, $\varkappa$ has an imaginary part ($\xi \neq 0$) and the wave is strongly damped. This is a general property of wave propagation in periodic structures [see, for example, the discussion of Eq. (9.1.35)]; the systems behave like bandpass filters. Because of the periodicity of the equation $Q = 0$, the plot can be extended upward and downward. The range $0 > v > -\frac{1}{2}$ can be covered by changing the signs of u, v, and X, in effect rotating the chart 180° about the origin and changing the sign of u. The range $\frac{1}{2} < v < 1$ is obtained by changing u and v to $1 - u^*$ and $1 - v$ and reversing the sign of X, in effect rotating the chart 180° about the point $X = 0$, $v = \frac{1}{2}$. We note that, for ω small, massy struts ($X > 0$) allow wave propagation, stiff struts ($X < 0$) attenuate the waves.

If Z is not purely reactive, the wavenumber parameter u will be complex over the whole area of Fig. 10.15, but if Re ζ is small, there will be a band, roughly the same as the unshaded area of Fig. 10.15, in which ξ is small and outside of which ξ increases rapidly. When the strut reactance is large compared with the characteristic impedance of the membrane (i.e., when $|\zeta| \gg 1$), the roots of Eq. (10.3.3) are, approximately,

$$\varkappa_q l \simeq \cos^{-1}\left[1 - \tfrac{1}{2}g \sin(\alpha k l)\right] \qquad \xi l \simeq \frac{f \sin(k\alpha l)}{2 \sin(\varkappa_q l)}$$

where $\zeta = f - ig$. These formulas are for the small range of values of $k\alpha l$ for which $\sin(k\alpha l)$ lies between zero and $2/f$ (f is large); for the rest of the range,

$$\varkappa_q l \simeq \frac{g}{f}\,|\cos(k\alpha l)|^{\frac{1}{2}} \qquad \xi l \simeq \ln[g \sin(k\alpha l)]$$

Forced motion *in vacuo*

If, for example, a single transverse line force $F_0 e^{-i\omega t}\,\delta(x)$ acts on the membrane and strut at $x = 0$, the steady-state forced motion is

$$\eta(x) = \begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A\{-e^{(i\varkappa_q - \xi)l} \sin(\alpha k x) + \sin[\alpha k(l + x)]\}e^{-i\omega t} & -l < x < 0 \\ A\{e^{(i\varkappa_q - \xi)l} \sin(\alpha k x) + \sin[\alpha k(l - x)]\}e^{-i\omega t} & 0 < x < l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{10.3.5}$$

where

$$A = \frac{\alpha^2 F_0/\sigma c^2}{2e^{(i\varkappa_q - \xi)l} - 2\cos(\alpha k l) + i\zeta \sin(\alpha k l)}$$

the free wave for positive x going to the right, that for negative x going to the left. If the strut impedance is that of a simple oscillator, $Z = i[(K/\omega) - \omega M] = -iX$, the strut resonance will be at $\omega_0 = \sqrt{K/M}$. Since Z is purely reactive, we can use the plot of Fig. 10.15 to work out the behavior of the system. At very low driving frequencies, X is large and negative and the wave is strongly damped; no power is transmitted to $x = \infty$ by the wall. As ω is increased, the point on the $(X/\sigma c_p, v)$ plane, corresponding to the system, moves along the line $X/\sigma c_p = (M\pi/\sigma l)[v - (v_0^2/v)]$ (the dashed line of Fig. 10.16), where $v_0 = \omega_0 l/c_p \pi$. If $v_0 < \frac{1}{2}$, this line will cross the heavy line marked $u = 0$ at a value of v less than $\frac{1}{2}$, at point a in Fig. 10.16, for example. At this point (for this driving frequency), the wavenumber $\varkappa_0$ is zero, the attenuation constant ξ is zero, and the amplitude A is equal to infinity. For a somewhat higher frequency, $\omega_0/2\pi$, the strut impedance is zero, but $\varkappa_0$ by this time is positive and real (it equals $k\alpha$ at this point), indicating that true wave motion is carrying power away from the driving force to $\pm\infty$. There is no resonance here because the membrane is resisting the force, even if the struts are not.

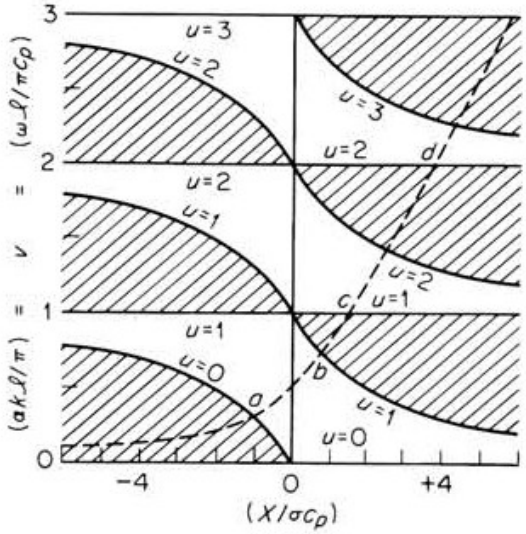


FIGURE 10.16
Illustration of computation of behavior of membrane periodically loaded with simple-oscillator struts. Dashed line is strut reactance curve; when it is in shaded regions the wave motion is attenuated.

As the frequency is further increased, $\varkappa_0$ increases further, until at point *b*, where the dashed line crosses the curve marked $u = 1$, resonance occurs again, with alternate struts moving in opposite directions, instead of being all in phase, as they were when $\varkappa_0 = 0$. Further increase of frequency brings the curve into a shaded region, when the wave motion is attenuated. At still higher frequency, at point *c*, ξ again becomes zero, and the system begins again to carry unattenuated waves. For this case, and for the similar points such as *d*, when *both* u and v equal n, the amplitude of motion does not go to infinity, as it does at points like *a* and *b*, where $\varkappa_0$ equals $n\pi$ but does not equal $k\alpha l$. Of course, the amplitude would never be infinite if we were to take into account the reaction of the air or other medium surrounding the membrane-strut system.

Thus, for some ranges of driving frequency, wave motion can carry power along the membrane; for other ranges, where the curve for strut reactance passes through a shaded region of Fig. 10.16, wave motion is severely attenuated.

Green's functions

To calculate the effect of the presence of an acoustic medium on the system we have just been describing, it will be useful to set up appropriate Green's functions, for both the strut-membrane system and the acoustic media on either side of the membrane. If we are interested in a traveling wave of frequency $\omega/2\pi$, both Green's functions should have a phase factor $e^{ix-i\omega t}$. Because of the periodicity of the system in x, we should expect an additional factor, periodic in x with period l, just as the solution of Eq. (10.3.2) has. We might as well make our Green's functions periodic also, by having them correspond to a regularly spaced sequence of line sources, rather than just one.

Thus the equation for the Green's function for the medium, for two-dimensional motion (still assuming no motion in the y direction), would be for a line source, not only at x_0, z_0, but also ones at $x_0 \pm nl$, z_0, each source

being out of phase by an amount $\varkappa l$ radians from the source next to the left. The equation for such a periodic Green's function is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)g+k^2g=-\delta(z-z_0)\sum_{n=-\infty}^{\infty}e^{in\varkappa l}\,\delta(x-x_0-nl)$$

so that $g(x+l,z\mid x_0,z_0)=e^{i\varkappa l}g(x,z\mid x_0,z_0)$ and also $g(x,z\mid x_0+l,z_0)=e^{-i\varkappa l}g(x,z\mid x_0,z_0)$. Then, instead of integrating $g(\partial p/dn_0)$, etc., over x_0 from $-\infty$ to $+\infty$, we need only integrate from 0 to l, since this moves each one of the series of sources from one end of its period to the other end, thus covering the whole range of x_0.

Put another way, if we multiply the equation for g by p, and the equation for p, $(\partial^2p/\partial x_0^2)+(\partial^2p/\partial z_0^2)+k^2p=0$, by g, subtract, and integrate over x_0 from 0 to l, over z_0 from 0 to ∞, we integrate over but one of the delta functions, the n for which $nl<x<(n+1)l$. Thus the integral reduces to an integral over the delta function and a surface integral, so that

$$e^{in\varkappa l}p(x-nl,z)=\iint\left[g\frac{\partial p}{\partial n_0}-p\frac{\partial g}{\partial n_0}\right]dS_0$$

with the surface integral over the x_0 axis from $x_0=0$ to l, over the line $x_0=l$ from $z_0=0$ to $+\infty$, over the line at $z_0=\infty$ from $x_0=l$ to 0, and back to the origin from $z_0=\infty$ to 0 along the z_0 axis. Since p also satisfies the condition $p(x_0+l,z_0)=e^{i\varkappa l}p(x_0,z_0)$, the $e^{i\varkappa l}$ for the p cancels the $e^{-i\varkappa l}$ for the x_0 dependence of g, and the quantity in brackets is periodic in x_0, with period l. Therefore the integral up the line $x_0=l$ is exactly canceled by the integral down the line $x_0=0$, and if the wave is entirely outgoing the integral over the segment at infinity is zero. What is left is the integral along the x_0 axis from 0 to l.

If now we arrange that $\partial g/\partial z_0=0$ for $z_0=0$, and if we remember that $e^{in\varkappa l}p(x-nl,z)=p(x,z)$, we obtain

$$p_+(x,z)=-\int_0^l g_+(x,z\mid x_0,0)\left(\frac{\partial p}{\partial z_0}\right)_{z_0=0}dx_0$$

Since $(\partial p/\partial z)_{z=0}=i\rho\omega u=\rho\omega^2\eta(x)$, where u is the transverse velocity of the membrane and η is its displacement, we finally arrive at the equation relating the pressure wave in the region $z>0$ to the membrane displacement,

$$p_+(x,z)=-\rho\omega^2\int_0^l g_+(x,z\mid x_0,0)\eta(x_0)\,dx_0 \qquad (10.3.6)$$

if η also satisfies the requirement $\eta(x+l)=e^{i\varkappa l}\eta(x)$. This is, of course, quite similar in form to Eq. (10.2.10); the difference lies in the differences between the Green's functions and the limits of integration.

To find the expression for the function g_+, we note that the series

$$g_+=\sum_{m=-\infty}^{\infty}f_m(z,z_0,x_0)e^{ik_mx}\qquad k_m=\varkappa+\frac{2\pi m}{l}$$

is as easy to handle as a simple Fourier series, for

$$\int_0^l e^{ik_m x} e^{-ik_n x}\, dx = l\delta_{nm}$$

Therefore we can manipulate the equation for g as easily as was done to obtain Eqs. (4.4.14) and (7.4.8), eventually obtaining

$$g_+ = \sum_{m=-\infty}^{\infty} \frac{i \exp\,[ik_m(x - x_0)]}{lq_m} \begin{cases} \cos\,(q_m z)e^{iq_m z_0} & 0 \leqslant z \leqslant z_0 \\ \cos\,(q_m z_0)e^{iq_m z} & 0 \leqslant z_0 \leqslant z \end{cases} \tag{10.3.7}$$

where $q_m = (k^2 - k_m{}^2)^{\frac{1}{2}} = i(k_m{}^2 - k^2)^{\frac{1}{2}}$. The corresponding q_- for the region $z < 0$ is obtained from g_+ by reversing the signs of z and z_0. The integral relation between p_- and η is similar to Eq. (10.3.6), with $-g_-$ substituted for $+g_+$. Therefore the pressure difference at $z = 0$, which produces transverse acceleration of the membrane, is

$$P(x) \equiv p_-(x,0) - p_+(x,0) = 2\rho\omega^2 \int_0^l g(x \mid x_0)\eta(x_0)\, dx_0 \tag{10.3.8}$$

where

$$g(x \mid x_0) = \sum_{m=-\infty}^{\infty} \frac{i}{lq_m} \exp\,[ik_m(x - x_0)]$$

This also satisfies the periodicity condition $P(x + l) = P(x)e^{i\varkappa k}$ if η does.

The Green's function for the membrane-strut system is more complicated. Not only must it satisfy the periodicity condition; it must account for the strut impedances as well as a periodic array of unit sources. The equation is

$$\left(\frac{d^2}{dx^2} + k^2\alpha^2\right) G(x \mid x_0) = -\sum e^{i n\varkappa l}\, \delta(x - x_0 - nl) - \frac{ik\alpha^2 Z}{c} \sum \delta(x - nl) G(x \mid x_0)$$

where $\alpha = c/c_p$. After a great deal of algebra one can write out the appropriate Green's function in any of the following forms:

$$G(x \mid x_0) = \begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{2k\alpha Q}\{2e^{-i\varkappa l} \sin\,\alpha k(x - x_0) - 2 \sin\,\alpha k(x + l - x_0) \\ \qquad + i\zeta[\cos\,\alpha k(x + x_0 - l) - \cos\,\alpha k(x + l - x_0)]\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < x_0 < l \\ \dfrac{1}{2k\alpha Q}\{2 \sin\,\alpha k(x - l - x_0) - 2e^{i\varkappa l} \sin\,\alpha k(x - x_0) \\ \qquad + i\zeta[\cos\,\alpha k(x + x_0 - l) - \cos\,\alpha k(x_0 + l - x)]\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x_0 < x < l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

$$= \frac{l}{\pi^2} \sum_m \frac{e^{ik_m x}}{\Delta_m} \Psi_m(-x_0) = \frac{l}{\pi^2} \sum_m \frac{e^{-ik_m x_0}}{\Delta_m} \Psi_m(x) \tag{10.3.9}$$

where

$$\alpha = \frac{c}{c_p} \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \qquad \zeta = \frac{\alpha Z}{\sigma c} = \frac{Z}{\sigma c_p}$$

$$u = \frac{\varkappa l}{\pi} \qquad v = \frac{k\alpha l}{\pi} = \frac{2\alpha l}{\lambda} \qquad k_m = \varkappa + \frac{2\pi m}{l}$$

$$u_m = \frac{k_m l}{\pi} = u + 2m \qquad \Delta_m = u_m{}^2 - v^2 \qquad \mu = \frac{2\rho}{\pi\sigma} v^2$$

$$q_m = (k^2 - k_m{}^2)^{\frac{1}{2}} \qquad w_m = \frac{l q_m}{\pi} = \left[\left(\frac{v}{\alpha}\right)^2 - u_m{}^2\right]^{\frac{1}{2}} = iW_m \tag{10.3.10}$$

$$W_m = \left[u_m{}^2 - \left(\frac{v}{\alpha}\right)^2\right]^{\frac{1}{2}} \qquad C(u) = \cos(\pi u) - \cos(\pi v)$$

$$Q(u) = 2\cos(\pi u) - 2\cos(\pi v) + i\zeta \sin(\pi v)$$

$$B(u) = \frac{2\zeta v}{\pi} \frac{C(u)}{Q(u)}$$

to list all the abbreviations to be used in the following discussion. The functions

$$\Psi_m(x) = e^{ik_m x} - \frac{i\xi}{Q} \begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \sin[\alpha k(l+x)] - e^{-i\varkappa l} \sin(\alpha k x) & -l < x < 0 \\ e^{i\varkappa l} \sin(\alpha k x) + \sin[\alpha k(l-x)] & 0 < x < l \\ e^{2i\varkappa l} \sin[\alpha k(x-l)] + e^{i\varkappa l} \sin[\alpha k(2l-x)] & l < x < 2l \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{cases}$$

$$= e^{ik_m x} + iB(u) \sum_{n=-\infty}^{\infty} \frac{e^{ik_n x}}{\Delta_n(u)} \tag{10.3.11}$$

are generalizations of the forced waves in vacuum, of Eq. (10.3.2), written in more compact form.

Using this Green's function, the transverse displacement of the membrane-strut system, under the influence of a force $F_\varkappa(x)e^{-i\omega t}$ which satisfies the periodicity condition $F_\varkappa(x + l) = e^{i\varkappa l} F_\varkappa(x)$, is

$$\eta(x) = \frac{\alpha^2}{\sigma c^2} \int_0^l G(x \mid x_1) F_\varkappa(x_1)\, dx_1$$

$$= \frac{\alpha^2}{\pi^2 \sigma c^2} \sum \frac{\Psi_m(x)}{\Delta_m(u)} \int_0^l e^{-ik_m x_1} F_\varkappa(x_1)\, dx_1 \tag{10.3.12}$$

where again the introduction of an array of sources in the equation for G has meant that we need only to integrate over a single period of x_1.

Free-wave motion, fluid present

If we wish to investigate the free-wave motion of the system, we set the force $F_\varkappa$ equal to the pressure P of the medium caused by the membrane motion, given in Eq. (10.3.8). Inserting this for $F_\varkappa$ in Eq. (10.3.12), we obtain the integral equation for free motion of a periodically supported membrane, in contact on both sides with a medium of density ρ and acoustic velocity c,

$$\eta(x) = \frac{2\rho\alpha^2\omega^2}{\sigma c^2}\int_0^l G(x \mid x_1)\,dx_1 \int_0^l g(x_1 \mid x_0)\eta(x_0)\,dx_0$$

$$= i\mu \sum \frac{\Psi_m(x)}{w_m\Delta_m}\int_0^l e^{-ik_m x_0}\eta(x_0)\,dx_0 \qquad (10.3.13)$$

when we substitute the series for G and g and carry out the integration over x_1 [see Eqs. (10.3.10) for definitions of symbols].

In the present case an exact solution can be worked out because the component functions $\Psi_m(x)$ consist of a term $e^{ik_m x}$ plus a term which is independent of m. We assume that $\eta(x)$ is a series $\Sigma A_n e^{ik_n x}$, and proceed to calculate the relative magnitudes of the A's. The procedure is complicated by the fact that the Ψ_n's are *not* orthogonal; so we cannot just multiply by $\Psi_m(-x)$ and integrate, but must expand everything in series of terms $e^{ik_m x}$, which are orthogonal. The steps go as follows:

$$\sum_i A_i e^{ik_i x} = i\mu l \sum_j \frac{1}{w_j\Delta_j}\left[e^{ik_j x} + iB(u)\sum_n \frac{e^{ik_n x}}{\Delta_n}\right]A_j$$

with all summations from $-\infty$ to $+\infty$. Multiplying both sides by $e^{-ik_m x}$ and integrating over x from 0 to l, we obtain

$$A_m = -\frac{w_m}{w_m\Delta_m - i\mu l}\left[\mu l B(u)\sum_n \frac{A_n}{w_n\Delta_n}\right] \qquad (10.3.14)$$

where the quantity in brackets is independent of m but is a function of $u = \varkappa l/\pi$.

This is an infinite homogeneous set of simultaneous equations for the A_m's, which has nonzero solutions only for certain values of u, corresponding to the various sorts of free waves possible for the system. An equation of self-consistency can be obtained by multiplying the equation for A_m by $1/w_m\Delta_m$ and summing, then equating the coefficients of $\Sigma(A_m/w_m\Delta_m)$ on both sides. The result is

$$0 = 1 + \mu l B(u)\sum_m \frac{1}{\Delta_m(w_m\Delta_m - i\mu l)}$$

or

$$1 = \frac{4i\xi\rho l v^3}{\pi^2\sigma}\,\Gamma(u) \qquad (10.3.15)$$

where

$$\Gamma(u) \equiv \frac{C(u)}{Q(u)} \sum_m \frac{1}{\Delta_m(W_m\Delta_m - \mu l)} \equiv \frac{\cos(\pi u) - \cos(\pi v)}{2\cos(\pi u) - 2\cos(\pi v) + i\zeta \sin(\pi v)}$$
$$\times \sum_{m=-\infty}^{\infty} [(u+2m)^2 - v^2]^{-1} \left\{ \left[(u+2m)^2 - \left(\frac{v}{\alpha}\right)^2 \right]^{\frac{1}{2}} \right.$$
$$\left. \times [(u+2m)^2 - v^2] - \left(\frac{2\rho l}{\pi\sigma}\right) v^2 \right\}^{-1}$$

with the proviso that, whenever $v/\alpha > u + 2m$, the radical in the summation becomes $-i[(v/\alpha)^2 - (u+2m)^2]^{\frac{1}{2}}$, *not* $+i$ times the radical.

In our manipulations to obtain Eq. (10.3.15), we have, in effect, multiplied by Δ_m. Thus, when $u_m = v$, *both* $C(u)$ and Δ_m go to zero and, in the limit, Eq. (10.3.15) is satisfied. These roots do not correspond to solutions of Eq. (10.3.13). The other roots of Eq. (10.3.15), however, do correspond to values of $u = \varkappa l/\pi$ (and thus to values of $\varkappa$) which *are* solutions of Eq. (10.3.13), and therefore represent possible free motions of the membrane-strut-medium system. There are two such sets of roots, one for the wave motion which has most of its energy in the membrane-strut part of the system, the other where most of the energy is in the medium.

We note that function $\Gamma(u)$ is periodic in u with period 2, that is, $\Gamma(u + 2n) = \Gamma(u)$. Thus we need determine only those roots of the equation in the first period of u; all other roots are obtained by subtracting or adding an even integer to these. Moreover, an examination of Eq. (10.3.14) shows that the other solutions are really the same, that the series for η for $u = u_0$, one root of Eq. (10.3.15) in the first period, is exactly the same as the series for η for $u = u_0 + 2n$; only the coefficients and terms are relabeled, the $(m+n)$th term of the first series being equal to the mth term of the second series. Thus examination of the four roots of Eq. (10.3.15) in the first period (solutions in pairs, corresponding to the two possible directions of propagation) covers the complete capabilities for wave motion of the triply coupled system.

The membrane-strut waves

When $2\rho l/\pi\sigma$, the mass of a unit area of blanket of the medium, of thickness $2l/\pi$, divided by the mass σ of a unit area of the membrane, is small compared with unity (as it is if the medium is air), then $\Gamma(u)$ must be quite large in order that Eq. (10.3.15) may be satisfied. Thus, to find solutions valid to first order in $2\rho l/\pi\sigma$, we look for singularities of $\Gamma(u)$ in the first period of u. In the range $0 < \mathrm{Re}\, u < 1$, $Q(u)$ has one root, the one plotted in Fig. 10.15 for wave motion of the membrane-strut system *in vacuo* (we have called this root u_q). Therefore, when $2\rho l/\pi\sigma$ is small, there is a solution of Eq. (10.3.15) corresponding to wave motion with velocity near $\omega l/\pi u_q$. These waves involve the greatest amount of motion of the struts; so we call them the membrane-strut waves.

Call the first-period solution of $Q(u) = 0$ by the subscript u_q (if the strut impedance Z is reactive, u_q can be real for certain values of v, as shown in Fig. 10.15). Setting $u = u_q + \epsilon$ and expanding $\Gamma(u)$ to first order in ϵ, we obtain, for one root of Eq. (10.3.15),

$$u_l \simeq u_q - \frac{\zeta^2 \rho l v^3}{\pi^3 \sigma} \frac{\sin(\pi v)}{\sin(\pi u_q)} \left\{ D(u_q) + \frac{1}{(u_q^2 - v^2)[(u_q^2 - v^2)\sqrt{u_q^2 - (v/\alpha)^2} - (2\rho l/\pi\sigma)v^2]} \right\} \quad (10.3.16)$$

where the second term in the braces is the term for $m = 0$ in Γ, and the term $D(u_q)$ is all the other terms, for $u = u_q$. We do this because, if $v = \alpha k l/\pi = 2l/\lambda_m$ (λ_m being the wavelength of waves on the membrane at frequency $\omega/2\pi$) is small compared with unity, the series D can be approximated by

$$D(u) \simeq \frac{1}{32} \sum_{m=1}^{\infty} \left[\frac{1}{(m + \frac{1}{2}u)^5} + \frac{1}{(m - \frac{1}{2}u)^5} \right]$$

which is a small correction to the second term. At higher frequencies several terms in the series must be separated off, and the computation becomes more tedious, but the series always converges satisfactorily.

As with any wave-supporting surface exposed to an acoustic medium, the properties of the surface wave motion depend on whether the wave velocity of the surface *in vacuo* is larger or smaller than the speed c of sound in the medium. In the present case the criterion is whether $v/\alpha = kl/\pi = 2l/\lambda = \omega l/\pi c$ is larger or smaller than $\mathrm{Re}\, u_q = \varkappa_q l/\pi = \omega l/\pi c_q$, where $c_q = \omega/\varkappa_q$ is the effective speed of transverse wave motion in the strut-membrane system *in vacuo*. When $c_q < c$ ($u_q > v/\alpha$) (this occurs to the right of the v axis in Fig. 10.15, for example), the whole quantity in braces in Eq. (10.3.16) is real, and if the strut impedance is either purely reactive or purely resistive, u_l is real to this order of approximation (if ζ is purely reactive, u_l is real to all orders in this case). Thus, as usual, when the velocity c_q is less than c, the acoustic wave motion cannot couple to the wall motion, and the wave propagates along the wall without attenuation [see also Eqs. (10.1.23)]. On the other hand, if $c_q > c$ ($u_q < v/\alpha$), the radical in the second term in braces in Eq. (10.3.16) is $-i[(v/\alpha)^2 - u_q^2]^{\frac{1}{2}}$, u_l has an imaginary part (which turns out to be positive-imaginary), and there is some energy transmitted through the medium to $z = \pm\infty$, and thus some attenuation of the wall wave.

Returning to the equations for η and for p, we see that this sort of wave can be written

$$\eta(x) = A_l \sum_{m=-\infty}^{\infty} \frac{w_m(u_l) e^{i(u_l + 2m)(\pi x/l)}}{w_m(u_l)\Delta_m(u_l) - i\mu l}$$

and

$$p_+(x,z) = -i\omega A_l \frac{\rho ck}{\pi} \times \sum \frac{\exp\{i(u_l + 2m)(\pi x/l) + (i\pi z/l)[(\omega l/\pi c)^2 - (u_l + 2m)^2]^{\frac{1}{2}}\}}{w_m(u_l)\Delta_m(u_l) - i\mu l} \tag{10.3.17}$$

The expression for p_- is obtained by changing the sign of z and also of the whole expression. The series for η is not the same as for $\Psi_0(x)$, the wave *in vacuo*, as given in Eq. (10.3.5), which in our present notation is

$$\sum \frac{\exp[i(u_q + 2m)(\pi x/l)]}{\Delta_m(u_q)}$$

If $\rho l/\pi\sigma$ is small, however, the difference is not great. Root u_l is not much different from u_q, and the effect on the denominator of each term in the series for η lies in the term $i\mu l/\omega_m$, the radiation impedance load on the membrane. Thus, when we include the load of the medium, there will be no infinite amplitudes for η at points such as a and b of Fig. 10.16. The terms in the series for p for which $|u_l + 2m| > \omega l/\pi c$ attenuate as z increases and only serve to modify p near the struts. If $u_l > \omega l/\pi c$, all terms so attenuate, and no energy escapes to $|z| \to \infty$.

There is also a root of $Q = 0$ at $2 - u_l$, within the first period of u; this corresponds to the root $-u_l$, for wave motion in the negative direction. All other roots are for $u = \pm u_l \pm 2m$. However, adding or subtracting $2m$ to u_l in Eqs. (10.3.17) does not change η or p; so really there are only two different membrane-strut waves.

The surface wave

In addition to the wave motion with velocity near that of the wall *in vacuo*, just discussed, there may occur a surface wave which travels at a velocity slightly smaller than the wave velocity c in the unbounded medium, but in which the energy flow is parallel to the surface, being mostly in the part of the medium near the surface. Such a surface wave can exist only if the velocity c_p of transverse wave in the membrane is greater than c [see also the discussion of Eq. (10.1.24)]. The possibility of such a solution arises from the fact that $\Gamma(u)$ has a sequence of poles at points where $[u^2 - (v/\alpha)^2]^{\frac{1}{2}}$ is equal to $(2\rho l/\pi\sigma)[v^2/(u^2 - v^2)]$, the square root being small, not $u^2 - v^2$. Set $u = (v/\alpha) + \epsilon$; if $(v/\alpha)^2 > v^2$, that is, if $c_p > c$, a real value of ϵ is possible. For $\rho l/\pi\sigma$ small, it is approximately given by

$$\sqrt{\frac{2\epsilon v}{\alpha}} \simeq \frac{2\rho l\alpha^2}{\pi\sigma(1 - \alpha^2)} + \frac{4i\zeta\rho l\alpha^4 C(v/\alpha)}{\pi^2\sigma v(1 - \alpha^2)^2 Q(v/\alpha)}$$

leading to

$$u_s \simeq \frac{kl}{\pi} + \frac{\alpha}{2v}\left(\frac{2\rho l}{\pi\sigma}\right)^2\left(\frac{\alpha^2}{1-\alpha^2}\right)^2\left[1 + \frac{2i\zeta}{\pi v}\frac{C(v/\alpha)}{Q(v/\alpha)}\frac{\alpha^2}{1-\alpha^2}\right]^2 \quad (10.3.18)$$

corresponding to a velocity $\omega l/\pi u_s$ a little smaller than c, the acoustic velocity of the medium. If the strut impedance is imaginary, u_s is real.

Since w_0 is small for this wave, we see from Eqs. (10.3.17) (substituting u_s for u_l in the formulas) that η is small and p large, in comparison with the waves for $u = u_l$, when the majority of the energy is carried by the wall. All components of the pressure wave attenuate with increase of $|z|$, because all coefficients of z in the exponent are negative and real. The smallest coefficient, that for $m = 0$, is

$$i\frac{\pi}{l}\left[\left(\frac{v}{\alpha}\right)^2 - u_s^2\right]^{\frac{1}{2}} \simeq -\frac{2\rho}{\sigma}\frac{\alpha^2}{1-\alpha^2}\left[1 + \frac{2i\zeta}{\pi v}\frac{C(v/\alpha)}{Q(v/\alpha)}\frac{\alpha^2}{1-\alpha^2}\right]$$

The disturbance extends out from the wall to a distance of the order of magnitude of the reciprocal of this, a considerable distance if σ is much larger than ρ. All other components, for $|m| > 0$, attenuate more sharply.

When $\alpha > 1$ $(c_p < c)$, ϵ becomes imaginary, and there can be no root of Eq. (10.3.15) which satisfies the physical requirement that p remain finite at $|z| \to \infty$. Thus no surface wave exists when the velocity of transverse waves on the membrane *in vacuo* is less than the acoustic velocity c in the medium.

We can also carry out the calculations for a driving force $Fe^{i\varkappa x - i\omega t}$ applied transversally to the wall, including the reaction of the medium. This can be manipulated, by means of the Fourier integral in $\varkappa$, to obtain the membrane shape and pressure wave arising when the strut at $x = 0$ is driven by a simple-harmonic line force. But this exercise will be left as a problem. The one example of forced motion deserving attention here is when the driving force is produced by an incident plane wave of sound, producing reflected and transmitted waves.

Wave penetration through the wall

If the pressure wave incident on the strut-membrane system is A exp $(ikx \sin\theta - ikz\cos\theta - i\omega t)$, θ being the angle of incidence (in the xz plane), then the integral equation for the pressure above the xy plane is a combination of Eqs. (8.3.2) and (10.3.6).

$$p_+(x,z) = 2Ae^{ikx\sin\theta}\cos(kz\cos\theta) - \rho\omega^2\int_0^l g_+(x,z \mid x_0,0)\eta(x_0)\,dx_0$$

where the $k_0 \equiv \varkappa$ in the g_+ of Eq. (10.3.7) is now $k\sin\theta$. Since this Green's function is chosen to have its normal gradient equal to zero at $z = 0$, the inhomogeneous term in the integral equation must satisfy the same condition, including what would be the reflected wave if the wall were rigid.

We next insert Eq. (10.3.12) for η, with $F_\kappa = (p_- - p_+)_{z=0} \equiv P(x)$. Therefore p_+, p_-, and P are

$$p_+(x,z) = 2Ae^{ikx\sin\theta}\cos(kz\cos\theta) - \frac{\rho k^2\alpha^2}{\sigma}\int_0^l g_+(x,z \mid x_0,0)\,dx_0\int_0^l G(x_0 \mid x_1)P(x_1)\,dx_1$$

$$p_-(x,z) = \frac{\rho k^2\alpha^2}{\sigma}\int_0^l g_-(x,z \mid x_0,0)\,dx_0\int_0^l G(x_0 \mid x_1)P(x_1)\,dx_1 \qquad (10.3.19)$$

$$P(x) = -2Ae^{ikx\sin\theta} + \frac{2\rho k^2\alpha^2}{\sigma}\int_0^l g(x \mid x_0)\,dx_0\int_0^l G(x_0 \mid x_1)P(x_1)\,dx_1$$

where g_+ is given in Eq. (10.3.7), g_- is obtained by reversing the signs of z and z_0, and g is given in Eq. (10.3.8) and G in Eq. (10.3.9). In all these we set $k_m = k\sin\theta + (2\pi m/l)$.

It is obvious from the form of the series for g and G that P will be expressed as a series of the form $\Sigma P_m e^{ik_m x}$. Once this series is known, the series for p_+ and p_- can be determined, for from Eqs. (10.3.19),

$$p_- - p_+ = P \qquad \text{and} \qquad p_- + p_+ = 2Ae^{ik_0 x} \qquad \text{at } z = 0$$

Therefore the wave transmitted through the strut-membrane system into the region $z < 0$ is, at the surface of the membrane,

$$p_-(x,0) = Ae^{ikx\sin\theta} + \tfrac{1}{2}P(x)$$

We can express k_m and $q_m = (k^2 - k_m{}^2)^{\frac{1}{2}}$ in terms of angles θ_m of transmission of the various waves with respect to the negative z axis by defining

$$\sin\theta_m = \sin\theta + \frac{2\pi m}{kl} \qquad \cos\theta_m = (1 - \sin^2\theta_m)^{\frac{1}{2}} \qquad (10.3.20)$$

If $|\sin\theta_m| > 1$, the requirement of finiteness at $z \to -\infty$ is that $\cos\theta_m$ be positive-imaginary; if $|\sin\theta_m| < 1$, $\cos\theta_m$ must be positive-real. In these terms the transmitted wave in region $z < 0$ is expressible in terms of the coefficients of the series for P.

$$p_- = Ae^{ikx\sin\theta - ikz\cos\theta} + \tfrac{1}{2}\sum P_m e^{ikx\sin\theta_m - ikz\cos\theta_m} = \sum_{m=-\infty}^{\infty} T_m \exp(ikx\sin\theta_m - ikz\cos\theta_m) \qquad (10.3.21)$$

Depending on the size of $2\pi/kl = \lambda/l$, there are one or more values of m for which $|\sin\theta_m| < 1$; corresponding to these values there will be one or more plane waves, carrying energy to $z \to -\infty$ at the corresponding angles θ_m (which can be positive or negative). These are the waves diffracted by the periodically spaced struts, according to the usual expression for the transmission angles θ_m from a diffraction grating. The waves for which $|\sin\theta_m| > 1$ will die out exponentially with z; their effect is to adapt the shape of the

wave near $z = 0$ to the motion of the membrane near each strut, as the whole moves in response to the pressure of the incident wave. Thus, if we can solve Eqs. (10.3.19) for P to obtain the coefficient P_m, we can compute the amplitudes of the various diffracted waves.

The equation for P is quickly solved. We substitute Eqs. (10.3.7) and (10.3.9) and the series for P into the last of Eqs. (10.3.19) and integrate over x_0 and x_1 to obtain

$$P_m e^{ik_m x} = -2Ae^{ikx_0} + \frac{2i\rho\alpha^2}{k\sigma} \sum_m \frac{P_m e^{ik_m x}}{(\sin^2 \theta_m - \alpha^2) \cos \theta_m} - \frac{2\rho\zeta\alpha^3 C}{k^2 \sigma l Q} \sum_m \frac{e^{ik_m x}}{(\sin^2 \theta_m - \alpha^2) \cos \theta_m} \sum_n \frac{P_n}{\sin^2 \theta - \alpha^2}$$

which can be solved exactly as Eq. (10.3.14) was. The functions C and Q are defined in Eqs. (10.3.10), with $\pi u = kl \sin \theta$ and $\pi v = k\alpha l$. Referring to Eq. (10.3.21), we see that the amplitude T_0 of the directly transmitted wave, propagating at the angle $\theta = \theta_0$, and the amplitude T_m of the mth diffracted wave, going at angle θ_m as defined in Eqs. (10.3.20), are

$$T_0 = \frac{-igA}{(\sin^2 \theta - \alpha^2) \cos \theta - ig} + B_0 \qquad T_m = B_m$$

where

$$B_m = \frac{A \cos \theta}{[(\sin^2 \theta - \alpha^2) \cos \theta - ig][(\sin^2 \theta_m - \alpha^2) \cos \theta_m - ig]} \times \left\{ \Upsilon \frac{2 \cos (kl \sin \theta) - 2 \cos (k\alpha l) + i\zeta \sin (k\alpha l)}{\cos (kl \sin \theta) - \cos (k\alpha l)} + \sum_n \frac{1}{(\sin^2 \theta_n - \alpha^2)[(\sin^2 \theta_n - \alpha^2) \cos \theta_n - ig]} \right\}^{-1} \tag{10.3.22}$$

and where $\alpha = c/c_p$; $\Upsilon = k^2 l\sigma/2\rho\zeta\alpha^3$, and $g = 2\rho\alpha^2/k\sigma$. The only waves which propagate to $z \to -\infty$ are those for which $|\sin \theta_m| < 1$.

Usually, constant g is small, because $\lambda\rho$ is usually smaller than σ. Where this is the case and where $\alpha < 1 \quad (c_p > c)$, the wall becomes transparent to the incident wave when the angle of incidence θ is equal to $\sin^{-1} \alpha$, that is, when the horizontal phase velocity $(\omega/k) \sin \theta$ is equal to the membrane wave velocity $c_p = \omega/k\alpha$. In this case the amplitude of the directly transmitted wave $(m = 0)$ is A, the incident amplitude; and all the diffracted beams vanish $(B_m = 0)$. When the angle of incidence θ is such that $\sin \theta_m = \alpha \quad (m \neq 0)$, the diffracted beams also disappear, though the directly transmitted wave $(m = 0)$ is in this case not equal to the incident wave; part of the incident wave is reflected to $z \to +\infty$.

If the strut impedance $Z = \sigma c_p \xi$ is zero, all the B_m's for $m \neq 0$ are zero, and the result reduces to that of a wave incident on a uniform membrane. If the struts are very stiff or heavy ($|\xi| \gg 1$), the B_m's are largest when $\sin(k\alpha l) = 0$, which is when the membrane strip between each strut resonates; at these frequencies the diffracted beams are most intense. Except at these resonance angles or frequencies, the transmission through the strut-membrane system is mass-controlled (proportional to $g = 2\rho c^2/k\sigma c_p^2$) for the first part of T_0, and is strut-impedance-controlled (proportional to $\gamma = \Upsilon l/\xi\alpha g$) for the B_m's governing the diffracted waves.

10.4 COUPLED CAVITIES

Finally, we shall work out the phenomena which arise when two resonating cavities are coupled together. Suppose cavity a has volume V_a and boundary surface S_a; cavity b has volume V_a and wall surface S_b. The two are joined by an opening S_0, a surface common to both S_a and S_b, as shown in Fig. 10.17, through which energy can pass from V_a to V_b, and vice versa.

The integral equations

To solve this problem we first compute the normal modes for each room separately, with the coupling area S_0 rigidly closed; in fact, we assume that the whole boundary area S_a is rigid, and similarly for S_b. In this case the eigenfunctions $\phi_n{}^a$ and eigenvalues $\eta_n{}^a$ and mean-square values $\Lambda_n{}^a$, as defined in Eqs. (9.4.8), for room a, are all real, and the corresponding quantities $\phi_m{}^b(r)$, $\eta_m{}^b$, and $\Lambda_m{}^b$ for isolated room b are also real and independent of frequency. The corresponding Green's functions for driving frequency $\omega/2\pi$,

$$G_a = \frac{1}{V_a}\sum_n \frac{\phi_n{}^a(\mathbf{r})\phi_n{}^a(\mathbf{r}_0)}{\Lambda_n{}^a[(\eta_n{}^a)^2 - (\omega/c)^2]}$$
$$G_b = \frac{1}{V_b}\sum_m \frac{\phi_m{}^b(\mathbf{r})\phi_m{}^b(\mathbf{r}_0)}{\Lambda_m{}^b[(\eta_m{}^b)^2 - (\omega/c)^2]} \qquad (10.4.1)$$

also have no imaginary parts. The series for function G_a is valid only

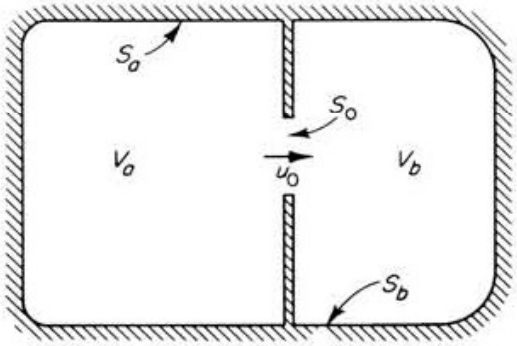


FIGURE 10.17
Two rooms, coupled by a common opening S_0.

within V_a and on S_a; the series for G_b holds only within V_b and on S_b; thus only on S_0 are both series valid.

The problem we wish to solve is that for which S_0 is not rigid but is an open space coupling the two rooms (another example, where S_0 is a flexible membrane, is considered in the problems). Across this surface the normal velocity is $u_0(r^s) = \dfrac{1}{i\omega\rho}\dfrac{\partial p}{\partial n}$ (for convenience, we assume that u_0 is positive for flow from a to b) and the acoustic pressure is $p_0(\mathbf{r}^s)$. Utilizing Eq. (7.1.17) for each room separately, the equations expressing the pressure in each room are

$$p(\mathbf{r}) = \begin{cases} i\rho\omega \iint G_a(\mathbf{r} \mid \mathbf{r}_0^s) u_0(\mathbf{r}_0^s)\, dS_0 & \mathbf{r} \text{ in } V_a \text{ or on } S_a \\ -i\rho\omega \iint G_b(\mathbf{r} \mid \mathbf{r}_0^s) u_0(\mathbf{r}_0^s)\, dS_0 & \mathbf{r} \text{ in } V_b \text{ or on } S_b \end{cases} \tag{10.4.2}$$

where both integrations are over the common surface S_0. Both equations hold simultaneously *only* when $\mathbf{r}$ is on S_0, in which case p is the common pressure p_0. For the solution which approaches the Nth mode in room a as S_0 is reduced to zero, the procedures which produced Eqs. (9.4.11) and (9.4.12) give, for the pressure in the opening and for the eigenvalue when S_0 is open,

$$0 = \Psi_N{}^a(\mathbf{r}^s) + \iint G_{aN}(\mathbf{r}^s \mid \mathbf{r}_0^s) u_N^0(\mathbf{r}_0^s)\, dS_0 + \iint G_b(\mathbf{r}^s \mid \mathbf{r}_0^s) u_N^0(\mathbf{r}_0^s)\, dS_0 \tag{10.4.3a}$$

where

$$G_{aN} = \frac{1}{V_a} \sum_{n \neq N} \frac{\phi_n{}^a(\mathbf{r}) \phi_n{}^a(\mathbf{r}_0)}{\Lambda_n{}^a[(\eta_n{}^a)^2 - (\omega/c)^2]} \tag{10.4.3b}$$

and where

$$\Lambda_N{}^a V_a \left[(\eta_N{}^a)^2 - \left(\frac{\omega}{c}\right)^2 \right] = \iint \phi_N{}^a(\mathbf{r}_0^s) u_N^0(\mathbf{r}_0^s)\, dS_0 \tag{10.4.3c}$$

Equation (10.4.3*a*) is one for the coupling velocity u_N^0 over S_0. Once it is solved, the allowed frequency of free vibration $\omega_N/2\pi$ can be obtained from Eq. (10.4.3*c*), and the pressure distribution in the two rooms can be obtained from Eq. (10.4.2). The integral equation is amenable to variational calculation; the function to be varied is

$$\Lambda_N{}^a V_a \left[(\eta_N{}^a)^2 - \left(\frac{\omega}{c}\right)^2 \right] = 2 \int \phi_N{}^a u_N^0 + \int u_N^0 \int G_{aN} u_N^0 + \int u_N^0 \int G_b u_N^0 \tag{10.4.4}$$

If we use the very simple trial function $u_N^0 = A$, the "best" values for $k = \omega/c$ and for A are given by the equations

$$A = \frac{-\int \phi_N{}^a\, dS}{\int dS \int (G_{aN} + G_b)\, dS_0} \qquad k^2 \simeq (\eta_N{}^a)^2 + \frac{(\int \phi_N{}^a\, dS)^2}{\Lambda_N{}^a V_a \int dS \int (G_{aN} + G_b)\, dS_0}$$

The resonance frequency $c\eta_N{}^a/2\pi$ of room a, uncoupled, may be close to a resonance frequency $c\eta_M{}^b/2\pi$ of room b, uncoupled; if so, it may be useful also to separate off the term in G_b having $(\eta_M{}^b)^2 - (\omega/c)^2$ in the denominator. The equations for k^2 and for A then can be rearranged into the forms

$$\frac{(\int \phi_N{}^a\, dS)^2/\Lambda_N{}^a V_a}{k^2 - (\eta_N{}^a)^2} + \frac{(\int \phi_M{}^b\, dS)^2/\Lambda_M{}^b V_b}{k^2 - (\eta_M{}^b)^2} \simeq \int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0$$

$$G(\mathbf{r} \mid \mathbf{r}_0) = G_{aN} + G_{bM} \tag{10.4.5}$$

$$A \simeq \frac{\Lambda_N{}^a V_a}{\int \phi_N{}^a\, dS}\,[(\eta_N{}^a)^2 - k^2] = \left(\int \phi_N{}^a\, dS\right) \times \frac{\Lambda_M{}^b V_b[k^2 - (\eta_M{}^b)^2]}{(\int \phi_M{}^b\, dS)^2 - \Lambda_M{}^b V_b[k^2 - (\eta_M{}^b)^2] \int dS \int G\, dS_0}$$

where all integrations are over the coupling area S_0, and G_{aN} and G_{bM} are the series of Eqs. (10.4.1), with the Nth and the Mth terms, respectively, omitted. The first equation, for k^2, is now symmetric with respect to the two nearest resonance frequencies, one for each room. The equation for A is not symmetric because the normalizing equation for u_N^0, Eq. (10.4.3c), is in terms of the normal mode $\phi_N{}^a$ of room a, which is not symmetric. The ratios between the pressures in the two rooms, calculated from Eq. (10.4.2), will be symmetric, however.

Analogy with coupled oscillators

The equation for k^2 has the same form as that of two coupled linear oscillators. For example, Eqs. (3.1.1) can be written in the form

$$(\omega_{01}{}^2 - \omega^2)A = \omega_{c1}{}^2(B - A) \qquad (\omega_{02}{}^2 - \omega^2)B = \omega_{c2}{}^2(A - B)$$

where

$$x = Ae^{-i\omega t} \qquad y = Be^{-i\omega t} \qquad \omega_{01}{}^2 = \frac{K_1}{m_1} \qquad \omega_{c1}{}^2 = \frac{K_3}{m_1}$$

$$\omega_{02}{}^2 = \frac{K_2}{m_2} \qquad \text{and} \qquad \omega_{c2}{}^2 = \frac{K_3}{m_2}$$

The natural frequencies of the two oscillators, without the presence of the coupling spring of stiffness K_3, are $\omega_{01}/2\pi$ and $\omega_{02}/2\pi$, respectively, and the quantities ω_1, ω_2, and μ of Eq. (3.1.2) are given by

$$\omega_1{}^2 = \omega_{01}{}^2 + \omega_{c1}{}^2 \qquad \omega_2{}^2 = \omega_{02}{}^2 + \omega_{c2}{}^2 \qquad \mu^2 = \omega_{c1}\omega_{c2}$$

The equation for ω^2, giving the natural frequencies of the coupled oscillators,

is obtained by eliminating $A - B$ from Eqs. (3.1.1). Written in the form

$$\frac{{\omega_{c1}}^2}{\omega^2 - {\omega_{01}}^2} + \frac{{\omega_{c2}}^2}{\omega^2 - {\omega_{02}}^2} = 1$$

it resembles the first of Eqs. (10.4.5). Thus we can write

$$(K_{cN}{}^a)^2 = \frac{(\int \phi_N{}^a\, dS)^2/\Lambda_N{}^a V_a}{\int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0} \qquad \text{and} \qquad (K_{cM}{}^b)^2 = \frac{(\int \phi_M{}^b\, dS)^2/\Lambda_M{}^b V_b}{\int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0}$$

and obtain the equation

$$\frac{(K_{cN}{}^a)^2}{k^2 - (\eta_N{}^a)^2} + \frac{(K_{cM}{}^b)^2}{k^2 - (\eta_M{}^b)^2} \simeq 1 \tag{10.4.6}$$

for the natural frequencies of the coupled rooms which are nearest to the Nth and Mth frequencies of the uncoupled rooms.

The solution of this equation for the natural frequencies of the coupled rooms is $ck_{NM}{}^+/2\pi$ and $ck_{NM}{}^-/2\pi$, where

$$\begin{aligned}(k_{NM}{}^+)^2 &= \tfrac{1}{2}[(\eta_N{}^a)^2 + (\eta_M{}^b)^2] + \tfrac{1}{2}\{[(\eta_N{}^a)^2 - (\eta_M{}^b)^2]^2 + 4{\mu_{NM}}^2\}^{\frac{1}{2}} \\ &\to (\eta_N{}^a)^2 + \frac{{\mu_{NM}}^4}{(\eta_N{}^a)^2 - (\eta_M{}^b)^2} \\ (k_{NM}{}^-)^2 &= \tfrac{1}{2}[(\eta_N{}^a)^2 + (\eta_M{}^b)^2] - \tfrac{1}{2}\{[(\eta_N{}^a)^2 - (\eta_M{}^b)^2]^2 + 4{\mu_{NM}}^2\}^{\frac{1}{2}} \\ &\to (\eta_N{}^a)^2 - \frac{{\mu_{NM}}^4}{(\eta_N{}^a)^2 - (\eta_M{}^b)^2}\end{aligned} \tag{10.4.7}$$

where

$${\mu_{NM}}^2 = K_{cN}{}^a K_{cM}{}^b = \Lambda_N{}^a \Lambda_M{}^b V_a V_b \frac{\int \phi_N{}^a\, dS \int \phi_M{}^b\, dS}{\int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0}$$

The limiting forms, indicated by the arrows, hold when $\mu^2 \ll (\eta_N{}^a)^2 - (\eta_M{}^b)^2$ (we assume, with no loss of generality, that $\eta_N{}^a > \eta_M{}^b$). In the rare cases when $\mu^2 \gg (\eta_N{}^a)^2 - (\eta_M{}^b)^2$, the approximate solution is

$$(k_{NM}{}^\pm)^2 \to \tfrac{1}{2}[(\eta_N{}^a)^2 + (\eta_M{}^b)^2] \pm {\mu_{NM}}^2$$

Thus, for frequencies nearest the Nth resonance ($c\eta_N{}^a/2\pi$), of room a, uncoupled, and the Mth resonance ($c\eta_M{}^b/2\pi$) of uncoupled room b, the coupled rooms behave like two coupled oscillators with coupling constants $K_{cN}{}^a$ and $K_{cM}{}^b$ and ${\mu_{NM}}^2 = K_{cN}{}^a K_{cM}{}^b$. As with the simpler system, the resonance frequencies are pushed apart by the coupling.

When constant ${\mu_{NM}}^2$ is much smaller than $(\eta_N{}^a)^2 - (\eta_M{}^b)^2$, the particular standing wave for the coupled rooms which oscillates with the frequency

$ck_{NM}^{+}/2\pi$ has the following waveforms in the two rooms [obtained by setting the u_0 of Eq. (10.4.2) equal to the A of Eqs. (10.4.5)]:

$$p_a(\mathbf{r}) \simeq (i\rho ck_{NM}^{+})\left[\phi_N{}^a(\mathbf{r}) + \sum_{n\neq N} \frac{\Lambda_N{}^a \int \phi_n{}^a\, dS}{\Lambda_n{}^a \int \phi_N{}^a\, dS} \frac{(\eta_N{}^a)^2 - (k_{NM}{}^{+})^2}{(\eta_n{}^a)^2 - (k_{NM}{}^{+})^2}\, \phi_n{}^a(\mathbf{r})\right]$$

$$p_b(\mathbf{r}) \simeq (i\rho ck_{NM}{}^{+})\left(\frac{\Lambda_N{}^a V_a}{\Lambda_M{}^b V_b}\right)^{\frac{1}{2}} \frac{\mu_{NM}{}^2}{(k_{NM}{}^{+})^2 - (\eta_M{}^b)^2} \tag{10.4.8}$$

$$\times \left[\phi_M{}^b(\mathbf{r}) + \sum_{m\neq M} \frac{\Lambda_M{}^b \int \phi_m{}^b\, dS}{\Lambda_m{}^b \int \phi_M{}^b\, dS} \frac{(\eta_M{}^b)^2 - (k_{NM}{}^{+})^2}{(\eta_m{}^b)^2 - (k_{NM}{}^{+})^2}\, \phi_m{}^b(\mathbf{r})\right]$$

each expression being multiplied by the common time factor $\exp(-ik_{NM}{}^{+}ct)$. Thus, for the resonance frequency nearest $c\eta_N{}^a/2\pi$, the standing wave is largest in room a, the ratio between p_b and p_a being approximately the small quantity

$$\sqrt{\frac{\Lambda_N{}^a V_a}{\Lambda_M{}^b V_b}}\, \frac{\mu_{NM}{}^{+}}{(k_{NM}{}^{+})^2 - (\eta_M{}^b)^2}$$

For the resonance frequency $ck_{NM}{}^{-}/2\pi$, nearest the room b natural frequency, the formulas are reversed, and the sound is mostly concentrated in room b.

When either or both rooms have sound-absorbent walls, the eigenvalues and eigenfunctions $\eta_n{}^a$, $\phi_n{}^a$, etc., become complex and frequency-dependent, so that Eq. (10.4.6) is an implicit equation for k^2. The roots $k_{NM}{}^{+}$ and $k_{NM}{}^{-}$ are then complex, corresponding to the fact that the free vibrations of the air in the coupled rooms lose energy to the walls.

The coupling constant

The quantity most difficult to compute in these formulas is the function

$$\int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0 = \sum_{n\neq N} \frac{[\int \phi_n{}^a(\mathbf{r}^s)\, dS]^2}{\Lambda_n{}^a V_a[(\eta_n{}^a)^2 - k^2]} + \sum_{m\neq M} \frac{[\int \phi_m{}^b(\mathbf{r}^s)\, dS]^2}{\Lambda_m{}^b V_b[(\eta_m{}^b)^2 - k^2]}$$

These series converge, though slowly. If the walls are absorbing, the result has an imaginary, as well as a real, part. If the area S_0 of the opening is small compared with S_a and S_b, we can use the following argument to obtain an approximate formula for $\int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0$, which is at least as accurate as the trial function we have used for u_0.

We first point out that if we add the missing $n = N$ and $m = M$ terms, the expression is proportional to the radiation impedance on a rigid piston placed in the opening S_0, radiating sound into both rooms. Referring to Eqs. (7.4.31) or Eq. (7.4.44), we see that this impedance would be

$$Z_r = \frac{F_a + F_b}{u_0} = -i\rho ck \iint_{S_0} dS \iint_{S_0} [G_a(\mathbf{r}^s \mid \mathbf{r}_0{}^s) + G_b(\mathbf{r}^s \mid \mathbf{r}_0{}^s)]\, dS_0$$

If the rooms are nonabsorbent, Z_r will be pure-imaginary, indicating that all the energy radiated is reflected back to the piston, chiefly in the form of the most strongly resonant waves (the $n = N$ and $m = M$ terms, which were omitted in the sums). But if the room walls are absorbent, so that much of the sound emitted by the piston never returns to the piston, and if the resonant terms are omitted, the resulting expression should be approximately equal to the piston impedance for completely absorbent rooms; and this should be approximately equal to the impedance load on a piston set in a plane wall, radiating into infinite space on both sides of the wall (as long as S_0 is not near a corner of either room).

What has been just said is that if both rooms are at least moderately absorptive (8 Re β greater than the ratio of S_0 to $S_a + S_b$, as will be shown), the quantity $\int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0$ is approximately proportional to the impedances given in Eqs. (7.4.31) or Eq. (7.4.44); in fact, that

$$\int dS \int G(\mathbf{r} \mid \mathbf{r}_0)\, dS_0 \simeq 2i \frac{Z}{\rho c k} = \frac{2S_0}{k}(\chi + i\theta) \tag{10.4.9}$$

where, if the opening S_0 is circular, of radius a, $\chi = \chi_0(2ka)$, $\theta = \theta_0(2ka)$; if the opening is rectangular with sides a and b, $\chi = [ab/(a^2 - b^2)][\chi_\square(ka) - \chi_\square(kb)]$ and $\theta = [ab/(a^2 - b^2)][\theta_\square(ka) - \theta_\square(kb)]$; in any case, χ will be the specific reactance, and θ the specific resistance ($R/\rho c S_0$), per unit area of a piston of the shape and size of the opening, radiating into free space on both sides.

To this approximation the coupling constant becomes

$$\mu_{NM}{}^2 \simeq \frac{\int \phi_N{}^a\, dS \int \phi_M{}^b\, dS}{S_0{}^2 \sqrt{\Lambda_N{}^a \Lambda_M{}^b}} \frac{kS_0/2}{(\chi + i\theta)\sqrt{V_a V_b}} \tag{10.4.10}$$

Thus the coupling constant is roughly equal to the product of the ratio between the mean values of $\phi_N{}^a$ and of $\phi_M{}^b$ across S_0 to the rms values of $\phi_N{}^a$ and $\phi_M{}^b$ over V_a or V_b, and the ratio between the factor $kS_0/2(\chi + i\theta)$ and the geometric mean of the room volumes, a small quantity having the dimensions of k^2. We note that if a nodal surface of either $\phi_N{}^a$ or $\phi_M{}^b$ cuts across S_0, so that either of the area integrals is zero, then, to this degree of approximation, there is no coupling between the rooms for this frequency range. In these cases a better approximation can be worked out (if it is worth the large amount of algebra!) by setting $u_0 = A\phi_N{}^a + B\phi_M{}^b$ in expression (10.4.4) and carrying out the variational calculation for best values of A and B.

Forced motion of the coupled rooms

If an acoustic source of frequency $\omega/2\pi$, representable by a distributed source strength $S_\omega(\mathbf{r})$, is placed in room a, the integral equations for the

pressure in the two rooms are

$$p(\mathbf{r}) = \begin{cases} -ik\rho c \iiint G_a(\mathbf{r} \mid \mathbf{r}_0) S_\omega(\mathbf{r}_0)\, dv_0 \\ \quad + ik\rho c \iint G_a(\mathbf{r} \mid \mathbf{r}_0{}^s) u_0(\mathbf{r}_0{}^s)\, dS_0 & \mathbf{r} \text{ in } V_a \text{ or on } S_a \\ -ik\rho c \iint G_b(\mathbf{r} \mid \mathbf{r}_0{}^s) u_0(\mathbf{r}_0{}^s)\, dS_0 & \mathbf{r} \text{ in } V_b \text{ or on } S_b \end{cases} \tag{10.4.11}$$

The volume integral is the pressure which would be in room a if S_0 were rigidly closed; it could be called the *driving pressure* and could be symbolized as $P(\mathbf{r})$ ($\mathbf{r}$ in V_a or on S_a). We can now set up a variational expression for the integral,

$$T_\omega \equiv \iint u_0(\mathbf{r}_0{}^s) P(\mathbf{r}_0{}^s)\, dS_0 \tag{10.4.12}$$

the real part of which is the power that the driving pressure gives to the coupling area S_0, which is transmitted through to room b. Equating $p(\mathbf{r})$ on S_0 from the two parts of Eq. (10.4.11), multiplying by u_0 and integrating over S_0, and then adding Eq. (10.4.12), we obtain the variational expression

$$T_\omega = 2\int u_0 P\, dS + ik\rho c \int u_0\, dS \int (G_a + G_b) u_0\, dS_0$$

Inserting the trial value $u_0 = A$, as before, we obtain

$$A \simeq \frac{-\int P\, dS}{ik\rho c \int dS \int (G_a + G_b)\, dS_0} \qquad T_\omega \simeq \frac{-(\int P\, dS)^2}{ik\rho c \int dS \int (G_a + G_b)\, dS_0} \tag{10.4.13}$$

with corresponding solutions for the steady-state pressure in the two rooms, obtained from Eq. (10.4.11).

Thus the sound pressure in room a is the pressure P, the pressure present if S_0 were closed, with a correction term because S_0 is open; the pressure in room b is the amount which has come through S_0 from a. If all walls are rigid, the transmission factor T_ω is pure-imaginary; in steady state no further power is being transmitted from room a to room b. If the walls of both rooms have nonzero wall conductance, T_ω will have a real part corresponding to the power which flows through S_0, to be absorbed on the walls of room b.

As an example, suppose both rooms are rectangular, having walls of specific acoustic admittance $\beta_a = \xi_a - i\sigma_a$ and $\beta_b = \xi_b - i\sigma_b$, respectively, with magnitude small but not zero. Then Eq. (9.3.41) indicates that, instead of $\eta_n{}^a$ in Eqs. (10.4.1) et seq., we should use $K_n{}^a$, where

$$(K_n{}^a)^2 \simeq (\eta_n{}^a)^2 - 2(\sigma_a + i\xi_a)\frac{S_n{}^a k}{V_a} \tag{10.4.14}$$

where

$$2S_n{}^a = \epsilon_{n_x} S_x{}^a + \epsilon_{n_y} S_y{}^a + \epsilon_{n_z} S_z{}^a - \epsilon_{n_x} S_0$$

and $\eta_n{}^a$ is as given in Eq. (9.4.29); $S_x{}^a = 2l_y{}^a l_z{}^a$ is the area of the two walls perpendicular to the x axis, etc.; and we assume that opening S_0 is in an x wall. There is a similar expression for $K_m{}^b$.

Now suppose the driving frequency is small, in the range where the resonance peaks are separated, and suppose we excite the Nth mode of room a, ω being set equal to the real part of $K_N{}^a$. Then one term in G_a will be larger than all the rest of $G_a + G_b$ (unless room b happens to have an exactly coinciding resonance). The rest of G_a and all G_b will produce the coupling term of Eq. (10.4.9); so the acoustic wave in room a and the power transmitted through S_0 to room b will be approximately

$$p_a(\mathbf{r}) \simeq B_N \phi_N{}^a(\mathbf{r}) \frac{L_w{}^a}{L_w{}^a + \frac{1}{2}L_c{}^a} \qquad \text{Re } T_\omega \simeq |B_N|^2 L_c{}^a \frac{L_w{}^a}{L_w{}^a + \frac{1}{2}L_c{}^a} \tag{10.4.15}$$

where

$$L_w{}^a = \frac{2\xi_a S_N{}^a \Lambda_N{}^a}{\rho c} \qquad L_c{}^a = \frac{(\iint \phi_N{}^a \, dS)^2 \theta}{2\rho c S_0(\theta^2 + \chi^2)} \qquad B_N = \frac{\rho c \iiint S_\omega \phi_N{}^a \, dv}{2\xi_a S_N{}^a \Lambda_N{}^a}$$

with θ and χ as given in Eq. (10.4.9).

These formulas have a fairly simple physical interpretation. According to Eq. (9.4.32), $L_w{}^a$ is the power which standing wave $\phi_N{}^a$ loses to the walls of room a. Quantity $L_c{}^a$ is the power lost by wave $\phi_N{}^a$ to coupling area S_0, transferred to room b. Thus the energy density in room a (proportional to $|p|^2$) is reduced, below the level $(B_n\phi_N{}^a)^2$ it would have if S_0 were closed, by the ratio between the energy $L_w{}^a$ lost to the walls alone and the total energy lost. The amount transmitted to room b is proportional to $L_w{}^a$, if there is no coincidence of resonance.

Alternatively, when the driving frequency of the source in room a is equal to one of the resonance frequencies of room b but room a does not resonate, the pressure in room b is correspondingly increased. The pressure distribution in room a, when S_0 is closed, can still be called $P(\mathbf{r})$, though it will have a more complex form than $B\phi_N{}^a$, which it had at resonance. In this case the resonant term in G_b is larger than G_a and the rest of G_b, and the wave in room b is

$$p_b(\mathbf{r}) \simeq \phi_M{}^b(\mathbf{r}) \frac{\iint P \, dS}{\iint \phi_M{}^b \, dS} \left[\frac{(\iint \phi_M{}^b \, dS)^2}{2\rho c S_0 \, |\theta - i\chi|} \right] \frac{1}{L_w{}^b + \frac{1}{2}L_c{}^b} \tag{10.4.16}$$

Since the coupling factor in brackets is of the order of magnitude of $L_c{}^b = (\iint \phi_M{}^b \, dS)^2 \theta / 2\rho c S_0 \, |\theta - i\chi|^2$, and thus is much smaller than $L_w{}^b = \xi_b S_M{}^b \Lambda_M{}^b / \rho c$, we see that p_b is smaller than P, but is much larger than the p_b, for the situation of Eqs. (10.4.15).

The analysis starting with Eqs. (10.4.1) can also be carried through, when S_0 is covered by a membrane or thin plate which can vibrate and thus transmit acoustic power, by adapting the procedures discussed in the beginning of Sec. 10.2. The quantity to be varied is not the velocity in S_0, but the pressure difference across S_0, which can be considered constant across

S_0 if S_0 is small. We are led to a double integral, involving two Green's functions, one for the air in the rooms and one for the membrane motion. The formulas are thus more involved, but satisfactory approximations can be developed. This will be left to the problems.

Higher frequencies

When the frequency is high enough for geometrical acoustics to be valid, the formulas of (10.4.15) and (10.4.16) no longer apply. In this frequency range the simple formulation of Sec. 9.5, using energy density w and mean intensity I, applies. The intensity in room a is $I_a = |p_a|^2/4\rho c = cw_a/4$, and that in room b is I_b. Quantity a_a, the wall absorption for room a, is given by Eq. (9.5.2) or (9.5.16); a_b is the corresponding wall absorption for room b. If coupling area S_0 is small, the wavelength of the sound may be small enough, compared with room dimensions, so that geometrical acoustics can be used, yet large enough so that transmission through S_0 is still that of a plane wave striking the opening, and the formulas (7.4.31) or (7.4.44) can still be used in computing the impedance $2\rho c S_0(\theta - i\chi)$ of the opening.

If the pressure is fairly uniform across S_0, the effective acoustic impedance of the air on the other side of the opening is $\rho c S_0(\theta - i\chi)$, and in accord with Eq. (10.4.12), the power removed from room a (and therefore transmitted into room b) is $\epsilon_{n_x} S_0 |p_a|^2 \theta/\rho c(\theta^2 - \chi^2)$. The factor ϵ_{n_x}, which is 2 for most high-frequency modes, is the ratio of the mean-square pressure at the wall containing S_0 to the mean-square pressure throughout the room a. Thus the power transmitted from room a to room b is $2[S_0 |p_a|^2 \theta/\rho c(\theta^2 + \chi^2)]$, and that going in the opposite direction is $2[S_0 |p_b|^2 \theta/\rho c(\theta^2 + \chi^2)]$ [we assume that the phase relations between p_a and p_b are random so that the mean value of Re $(p_a p_b^*)$ is zero]. The total power lost to room a is then $I_a a_a + 8S_0 I_a[\theta/(\theta^2 + \chi^2)]$, and the power gain, by transmission through S_0 from room b, is $8S_0 I_b[\theta/(\theta^2 + \chi^2)]$. The total acoustic energy in room a is $w_a V_a = 4V_a I_a/c$, and if a source of acoustic power Π is located in room a, the equations relating the rate of change of intensity to the power lost and gained in the two rooms are

$$\begin{aligned} \frac{4V_a}{c}\frac{dI_a}{dt} + a_a I_a + 8S_0 C_0 I_a &= 8S_0 C_0 I_b + \Pi \\ \frac{4V_b}{c}\frac{dI_b}{dt} + a_b I_b + 8S_0 C_0 I_b &= 8S_0 C_0 I_a \end{aligned} \tag{10.4.17}$$

where $C_0 = \theta/(\theta^2 + \chi^2)$ is the specific conductance of the opening. For steady state the solution is

$$\begin{aligned} I_a &= \frac{\Pi(a_b + 8S_0 C_0)}{(a_a + 8S_0 C_0)(a_b + 8S_0 C_0) - (8S_0 C_0)^2} \simeq \frac{\Pi}{a_a + 8S_0 C_0} && S_0 C_0 \ll a_a \\ I_b &\simeq \frac{8\Pi S_0 C_0}{(a_a + 8S_0 C_0)(a_b + 8S_0 C_0)} && S_0 C_0 \ll a_b \end{aligned} \tag{10.4.18}$$

When opening S_0 is closed, the intensity in room a is Π/a_a, and the mean-square pressure, which can be written $|P|^2$, is equal to $4\rho c\Pi/a_a$. Thus we can write, to the first order in the small quantity S_0C_0/a_a,

$$|p_a|^2 \simeq |P|^2 \frac{a_a}{a_a + 8S_0C_0}$$
$$|p_b|^2 \simeq |P|^2 \frac{8S_0C_0}{a_b + 8S_0C_0} \frac{1}{1 + (8S_0C_0/a_a)} \tag{10.4.19}$$

for the mean-square pressure in the two rooms. This can now be compared with the formulas of Eqs. (10.4.15) and (10.4.16), obtained for resonance conditions at lower frequencies. Equations (9.5.16) and (10.4.15) indicate that $a_a = 8\xi_a S_N{}^a$ when ξ_a, the specific conductance of the walls of room a, is small. Multiplying top and bottom of Eqs. (10.4.19) by $\Lambda_N{}^a/4\rho c$ and remembering that $2\Lambda_N{}^a \simeq (\iint \phi_N{}^a\, dS)^2$ on the average, we see that, very approximately, the geometrical equation for $|p_a|^2$ is equivalent to

$$|p_a|^2 \simeq |P|^2 \frac{L_w{}^a}{L_w{}^a + L_c{}^a}$$

which is the same as the square of Eqs. (10.4.15) when $L_c{}^a \ll L_w{}^a$. A similar comparison can be made between the second of Eqs. (10.4.19) and the square of Eq. (10.4.16).

Although the subject of coupled systems, and the various tricks useful in working out their behavior, could be continued well-nigh indefinitely, we must turn to other matters. Between the text and the problems of this chapter, the more useful results and techniques have been at least touched on. Next we take up acoustic propagation in moving media.

Problems

1 A flexible tube under tension is filled with fluid of density ρ and adiabatic compressibility $\kappa = 1/\rho c_a{}^2$. It has inside radius a when at equilibrium with the static pressure P of the fluid, the longitudinal tension, and the tube's transverse elasticity. When an acoustic wave travels in the fluid, the acoustic pressure p ($\ll P$) stretches the tube so that its radius is $a + \eta (\eta \ll a)$. Alternatively, a dilation wave can travel along the tube walls; the resulting change of radius η will expand or compress the fluid inside. Show that the first-order equations of motion relating the acoustic pressure p, the net flow of fluid $U = \pi(a + \eta)^2 u$ along the tube, and the change η in radius of the tube, for wave motion of wavelength long compared with a, are

$$2\pi a\epsilon \frac{\partial^2 \eta}{\partial t^2} = 2\pi a T \frac{\partial^2 \eta}{\partial z^2} + \pi a^2 p - 2\pi a K \eta$$

or

$$\frac{\partial^2 \eta}{\partial z^2} = \frac{1}{c_t{}^2} \frac{\partial^2 \eta}{\partial t^2} + \left(\frac{\omega_0}{c_t}\right)^2 \eta - \frac{a}{2T} p$$

$$-\rho \frac{\partial U}{\partial t} = \pi a^2 \frac{\partial p}{\partial z} \qquad -\frac{\partial U}{\partial z} = \pi a^2 \kappa \frac{\partial p}{\partial t} + 2\pi a \frac{\partial \eta}{\partial t}$$

or

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{c_a^2}\frac{\partial^2 p}{\partial t^2} + 2\frac{\rho}{a}\frac{\partial^2 \eta}{\partial t^2}$$

where ϵ, T, and K are the effective mass per unit length, tension, and transverse stiffness, respectively, of the tube walls; $c_t = \sqrt{T/\epsilon}$, the velocity of the dilation wave in an empty tube; $\omega_0 = \sqrt{K/\epsilon}$ is 2π times the frequency of dilation oscillation of the empty tube; and $c_a = \sqrt{1/\rho\kappa}$ is the acoustic velocity of the free fluid. Show that two simple-harmonic waves, of form $e^{i(\omega/c)z - i\omega t}$, are possible in the filled tube; that if $\omega c_t^2/\epsilon\omega^2 \ll 1$, the phase velocities of the two waves are, approximately,

$$c_a\left\{\frac{\omega^2[1 - (c_t/c_a)^2] - \omega_0^2}{\omega^2[1 - (c_t/c_a)^2] - \omega_0^2 \mp (\rho c_a^2/\epsilon)}\right\}^{\frac{1}{2}}$$

Compare this with the formulas of Eq. (9.1.16), when $c_t \ll c_a$. Show that in the case of the minus sign in the denominator, most of the energy is carried by the fluid; for the plus sign, the tube walls carry most of the energy.

2 Suppose the flexible wall of the tube of Prob. 1, in addition to having fluid inside, is in contact with the same fluid outside the tube, extending to infinity. Show that the equation for $\partial^2\eta/\partial z^2$ has an additional term $(a/2T)p_0$, where p_0 is the pressure just outside the tube. Show also that the radial fluid velocity just outside the tube, $\partial\eta/\partial t$, must equal $\frac{1}{i\omega\rho}\frac{\partial p_0}{\partial r}$. For a simple-harmonic wave, such that $p = D\exp[i(\omega/c)z - i\omega t]$, $\eta = E\exp[i(\omega/c)z - i\omega t]$, and $p_0 = GH_0^{(1)}(\varkappa r)\exp[i(\omega/c) - i\omega t]$, where $\varkappa^2 = (\omega/c_a)^2 - (\omega/c)^2$, show that the formulas for phase velocity given in Prob. 1 are modified by changing the terms $1 - (c_t/c_a)^2$ to

$$1 - \left(\frac{c_t}{c_a}\right)^2 + \frac{\rho a^2}{4\epsilon}\left\{\ln\frac{1 - (c_t/c_a)^2 - (\omega_0/\omega)^2}{\rho a^2/\epsilon C^2} + i\pi\right\}$$

or

$$1 - \left(\frac{c_t}{c_a}\right)^2 - \frac{\rho a^2}{4\epsilon}\ln\left\{\frac{C^2\omega^2 a^2}{c_t^2}\,\frac{\omega^2[1 - (c_t^2/c_a^2)] - \omega_0^2 - (\rho c_a^2/\epsilon)}{\omega^2[1 - (c_t^2/c_a^2)] - \omega_0^2} - \frac{C^2\omega^2 a^2}{c_a^2}\right\}$$

for the fluid wave or the tube wave, respectively. [For the value of C, see Eq. (10.1.6).] Explain the presence of the imaginary term in the first expression; why should the fluid wave attenuate, and why should the tube wave not attenuate? Show that there is a third wave, also nonattenuating, thus with an imaginary value for $\varkappa$, with a velocity $c = [(1/c_a)^2 + (x/\omega^2 a^2)]^{-\frac{1}{2}}$, and with x the solution of $x = (1/c)\exp\{(4/\rho a^2)[1 - (c_t/a\omega)^2 x - (c_t/c_a)^2 - (\omega_0/\omega)^2 + (a^2\rho/\epsilon x)]\}$.

3 Show that an alternative solution of Eq. (10.1.16) is

$$\eta(z) = -\frac{\mu\alpha^2 k^2}{2\pi}\int_0^l g(z \mid z_0)\,dz_0 \int_0^l G_\omega(z_0 \mid z_1)\eta(z_1)\,dz_1$$

where

$$G_\omega(z_0 \mid z_1) = -\int_{-\infty}^{\infty} e^{iK(z_0 - z_1)}[1 + a(K^2 - k^2)\ln(Ca\sqrt{K^2 - k^2})]\,dK$$

and $g(z \mid z_0)$ is the Green's function for the string of Eqs. (4.4.14) and (4.5.39) (we have shifted the origin from the center of the string to one end). Show

that this homogeneous integral equation may be turned into a variational expression for the Nth eigenfunction by setting $(\omega l/c_s)^2 - (\pi N)^2 = (\mu\alpha^2 k^2 l/\pi)\Phi_N$, where

$$\Phi_N = \int_0^l dz_0 \sin\left(\frac{\pi N x_0}{l}\right)\int_0^l G_\omega(z_0 \mid z_1)\eta(z_1)\,dz_1$$

Show that the variational expression for Φ_N [see Eq. (9.4.13)] is

$$\Phi_N = 2\int \sin\left(\frac{\pi N z}{l}\right)\int G_\omega(z \mid z_0)\eta(z_0) - \int \eta(z)\int G_\omega(z \mid z_0)\eta(z_0) - \frac{1}{\pi}\mu\alpha^2 k^2 l\int \eta(z)\int G_\omega(z \mid z_1)\int g_N(z_1 \mid z_2)\int G_\omega(z_2 \mid z_0)\eta(z_0)$$

where

$$g_N(z \mid z_0) = \sum_{n\neq N} \frac{\sin(\pi n z/l)\sin(\pi n z_0/l)}{(\pi n)^2 - (\omega l/c_s)^2}$$

Show that, if we choose η to be approximated by $A\sin(\pi N z/l)$ and vary A, the resulting approximate solution is

$$\Phi_N \simeq \int \sin\left(\frac{\pi N z}{l}\right)\int G_\omega(z \mid z_0)\sin\left(\frac{\pi N z_0}{l}\right)\left\{1 + \frac{\mu\alpha^2 k^2 l/\pi}{\int \sin(\pi N z/l)\int G_\omega \sin(\pi N z_0/l)} \times \int \sin\left(\frac{\pi N z}{l}\right)\int G_\omega \int g_N \int G_\omega \sin\left(\frac{\pi N z_0}{l}\right)\right\}$$

Compare this result with Eq. (10.1.19), coming from the Rayleigh-Ritz variational procedure. Would the present method be more dependable for the higher modes?

4 Show that the asymptotic form for the pressure wave radiated by the plate of Eqs. (10.1.27), in the region $x > 0$, $z > 0$, for $k < \gamma$, is

$$p_\omega \to \frac{i\mu\gamma F_0}{8\sqrt{\gamma^2 - k^2}}\left\{\exp(i\gamma x - z\sqrt{\gamma^2 - k^2}) + \sqrt{\frac{\gamma^2 - k^2}{\gamma^2 + k^2}}\exp\left[-\gamma x + iz\sqrt{\gamma^2 + k^2} - \frac{\mu\gamma^2 z}{4(\gamma^2 + k^2)}\right] + \frac{4\gamma^3\sqrt{\gamma^2 - k^2}\,e^{ikr}}{\sqrt{2\pi ikr(\gamma^4 - k^4\cos^4\phi)k\sin\phi}}\right\}$$

Compare this with the expressions of Eqs. (10.1.30) and (10.1.31) for $\gamma < k$. What is the plate displacement for $\gamma > k$?

5 An infinite membrane of areal density σ, under tension T per unit length, is in contact on both sides with a medium of infinite extent, of density ρ, and acoustic velocity c. Show that linear simple-harmonic transverse waves of displacement $\eta = A\exp(iKx - i\omega t)$ are possible on the membrane if the wavenumber K has the value

$$K \simeq k_m\left(1 + \frac{\rho}{\sigma\sqrt{k_m^2 - k^2}}\right) \quad \text{when} \quad \frac{T}{\sigma} = c_m^2 < c^2, \frac{2\rho}{\sigma\sqrt{k_m^2 - k^2}} \ll 1$$

and where $k = \omega/c$, and $k_m = \omega/c_m$. What is the phase velocity of the wave?

What is the pressure distribution in the medium above the membrane? Show that, when $c_m > c$, two waves are possible, with two different wavenumbers,

$$K \simeq k_m\left(1 + \frac{i\rho}{\sigma\sqrt{k^2 - {k_m}^2}}\right) \qquad \text{and} \qquad K_s \simeq k\left[1 + \frac{(2\rho {k_m}^2/\sigma k)^2}{(k^2 - {k_m}^2)^2}\right]$$

What is the physical significance of the imaginary term in the first expression? What is the pressure distribution in the medium in both cases? Why is the K_s possible when $c_m > c$, and not possible when $c_m < c$?

6 The membrane of Prob. 5 is driven by a line force $F_0\delta(x)e^{-i\omega t}$. Show that the asymptotic form for the pressure wave radiated in the region $z > 0$, $x > 0$, when $2\rho/\sigma\sqrt{({k_m}^2 - k^2)} \ll 1$, is

$$p_+ \simeq \frac{ik_m\rho F_0}{\sigma\sqrt{{k_m}^2 - k^2}}\left\{\exp\left(ik_m x - z\sqrt{{k_m}^2 - k^2}\right) + \frac{2k_m\sqrt{{k_m}^2 - k^2}\, e^{ikr}}{\sqrt{2\pi ikr}({k_m}^2 - k^2\cos^2\phi)}\right\}e^{-i\omega t} \qquad c_m < c$$

$$\simeq \frac{k_m\rho F_0}{\sigma\sqrt{k^2 - {k_m}^2}}\left\{u(k\cos\phi - k_m)\exp\left[ik_m x - \frac{\rho k_m x}{\sigma\sqrt{k^2 - {k_m}^2}} + iz\sqrt{k^2 - {k_m}^2} + \frac{\rho {k_m}^2 z}{\sigma(k^2 - {k_m}^2)}\right] - \frac{4i\rho {k_m}^3}{k\sigma(k^2 - {k_m}^2)}\exp\left[iK_s x - \frac{2\rho {k_m}^2 z}{\sigma(k^2 - {k_m}^2)}\right] - \frac{4ikk_m\sqrt{k^2 - {k_m}^2}\sin(\phi)\, e^{ikr}}{\sqrt{2\pi ikr}[(k^2\cos^2\phi - {k_m}^2)k\sin\phi - i(2\rho {k_m}^2/\sigma)]}\right\}e^{-i\omega t} \qquad c_m > c$$

where ${c_m}^2 = T/\sigma$, $c^2 = 1/\rho\kappa$, $k = \omega/c$, $k_m = \omega/c_m$, $r^2 = x^2 + z^2$, and $\tan\phi = z/x$; $u(w) = 0$ when $w < 0$, $= 1$ when $w > 0$; and K_s is as given in Prob. 5. What changes in the formulas are needed for $x < 0$? For $z < 0$? Compute the transverse displacement of the membrane. Discuss the behavior of the radiated wave for r large, $c_m > c$, and ϕ near $\psi = \cos^{-1}(k_m/k)$.

7 The membrane of Prob. 5 has incident on it a plane sound wave

$$p_i = P_i\exp\left[ik(x\sin\theta - z\cos\theta) - i\omega t\right]$$

at angle of incidence θ. Show that the reflected and transmitted wave amplitudes P_r and P_t and the amplitude A of transverse motion of the membrane are given by Eq. (10.1.32), except that now quantity Q is $[1 - (c_m/c)^2\sin^2\theta] \times (\sigma k/2\rho)\cos\theta$. At what angles of incidence is the membrane transparent to the wave?

8 Show that the expression for the p_t of Eq. (10.1.38), for $k < \gamma$, is

$$p_t \to \frac{\rho ckS_\omega\gamma^2}{8(\gamma^2 - k^2)}\left\{\frac{\exp\left[i\gamma r - (l - z)\sqrt{\gamma^2 - k^2}\right]}{\sqrt{2\pi i\gamma r}} + \frac{\gamma^2 - k^2}{\gamma^2 + k^2}\frac{\exp\left[-\gamma r + i(l - z)\sqrt{\gamma^2 - k^2}\right]}{\sqrt{2\pi\gamma r}} + \frac{4\sqrt{2}\gamma^2(\gamma^2 - k^2)(e^{ikR}/4\pi R)}{(\gamma^4 - k^4\sin^4\theta)k\cos\theta}\right\}$$

9 The spherical wave from a simple source S_ω and frequency $\omega/2\pi$ at $z = l$, $x = y = 0$, falls on the membrane of Prob. 5. Show that the asymptotic form of the transmitted wave ($z < 0$) is

$$p_t \to \frac{k_m^2 \rho S_\omega}{\sigma(k_m^2 - k^2)} \left\{ \frac{1}{2\pi i k_m r} \exp\left[ik_m r - (l - z)\sqrt{k_m^2 - k^2}\right] + \frac{(k_m^2 - k^2)(4e^{ikR}/2\pi R)}{(k_m^2 - k^2\cos^2\theta)k\sin\theta + i(2\rho k_m^2/\sigma)} \right\} e^{-i\omega t} \qquad c_m < c$$

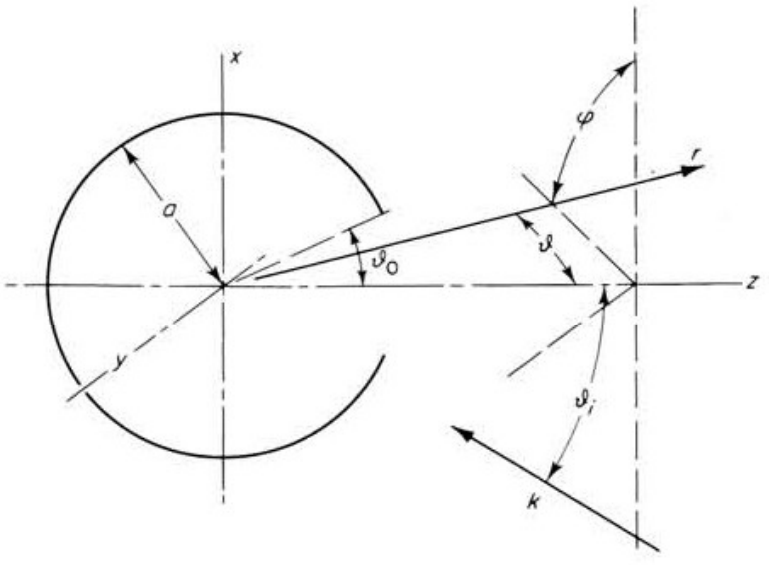
$$\to \frac{-k_m^2 \rho S_\omega}{\sigma(k^2 - k_m^2)} \left\{ \frac{u(k\cos\theta - k_m)}{\sqrt{2\pi i k_m r}} \exp\left[ik_m r - \frac{\rho k_m r}{\sigma\sqrt{k^2 - k_m^2}} + i(l - z)\sqrt{k^2 - k_m^2} + \frac{\rho k_m^2 (l - z)}{\sigma(k^2 - k_m^2)} \right] + \frac{2}{\sqrt{2\pi i k r}} \exp\left[ikr - \frac{2\rho k_m^2 (l - z)}{\sigma(k^2 - k_m^2)} \right] + \frac{4(k^2 - k_m^2)(e^{ikR}/2\pi R)}{(k^2\cos^2\theta - k_m^2)k\sin\theta - i(2\rho k_m^2/\sigma)} \right\} e^{-i\omega t} \qquad c < c_m$$

where $r^2 = x^2 + y^2$, $R^2 = x^2 + y^2 + z^2$, and $\cos\theta = r/R$. Compare this result with the solution of Eq. (10.1.38) for transmission through a plate.

10 Suppose the circular plate of Eq. (5.3.6) is substituted for the membrane of Eq. (10.2.1). Write out the resulting modification of the integral equation and Green's function of Eq. (10.2.5). Use the results of Eq. (5.3.10) to obtain the appropriate modification of Eq. (10.2.6). What are the new expressions for the auxiliary functions H_m and S_m of Eqs. (10.2.22) and (10.2.23)? In terms of them, write out the modified formulas (10.2.24), (10.2.25), and (10.2.27). What are the differences from the membrane in spacing of the first few resonances and antiresonances?

11 For $c/c_p = \frac{1}{2}$, $\gamma = \rho a/\sigma = 1$, plot the transmission factor of Fig. 10.12 against θ for $ka = 3$ and $ka = 6$. Plot the angle distribution of Fig. 10.13 for $ka = 6$. Discuss the effects of the various resonances and antiresonances.

12 A Helmholtz resonator, consisting of a hollow sphere of radius a, centered at the origin, with a circular hole of angular aperture θ_0, is shown in the figure.



A plane wave of frequency $\omega/2\pi$ is incident on the sphere, its direction being in the xz plane, in the negative direction along a line at angle θ_i to the z axis. If the radial component of the air velocity in the hole is $u_a(\vartheta,\varphi)$, show that the pressure inside the sphere is, for $r < a$,

$$p_- = \sum_{m=0}^{\infty} \epsilon_m \cos(m\varphi) \sum_{n=m}^{\infty} (2n+1)\frac{(n-m)!}{(n+m)!} S_n{}^m(\theta_i) P_n{}^m(\cos\vartheta) \frac{j_n(kr)}{j_n'(ka)}$$

where the spherical harmonics $Y_{mn}{}^1(\vartheta,\varphi) = \cos(m\varphi)P_n{}^m(\cos\vartheta)$ are defined in Eqs. (7.2.7). Show that the incident plane wave is

$$p_i = A\sum_m \epsilon_m \cos(m\varphi) \sum_n (-i)^n(2n+1)\frac{(n-m)!}{(n+m)!} P_n{}^m(\cos\theta_i) P_n{}^m(\cos\vartheta) j_n(kr)$$

and thus that the total pressure field outside the sphere is

$$p_+ = \sum_m \epsilon_m \cos(m\varphi) \sum_n (2n+1)(-i)^n \frac{(n-m)!}{(n+m)!} P_n{}^m(\cos\vartheta)$$
$$\times \left\{ AP_n{}^m(\cos\theta_i)\left[j_n(kr) - \frac{j_n'(ka)}{h_n'(ka)} h_n(kr)\right] + i^n S_n{}^m \frac{h_n(kr)}{h_n'(ka)}\right\}$$
$$\to p_i + i\frac{e^{ikr}}{kr} \sum_{m,n} \epsilon_m (2n+1)\frac{(n-m)!}{(n+m)!} \left\{ A(-1)^n \frac{j_n'(ka)}{h_n'(ka)} P_n{}^m(\cos\theta_i) \right.$$
$$\left. - (-i)^n \frac{S_n{}^m(\theta_0)}{h_n'(ka)}\right\} \cos(m\varphi) P_n{}^m(\cos\vartheta)$$

The effect of viscosity is approximately to produce a pressure difference $p_- - p_+ = \rho c k d_v u_a$ across the surface of the opening.

If now we assume that $u_a = (U/\sqrt{2})\sin(\frac{1}{2}\theta_0)(\cos\vartheta - \cos\theta_0)^{-\frac{1}{2}}$ (why make this assumption?), the coefficients $S_n{}^m$ become

$$S_n{}^m \equiv \frac{i\rho c}{4\pi}\int_0^{2\pi} \cos(m\varphi)\, d\varphi \int_0^{\theta_0} u_a(\varphi,\vartheta) P_n{}^m(\cos\vartheta)\sin\vartheta\, d\vartheta$$
$$= \frac{i\rho c U}{2n+1} \sin(\tfrac{1}{2}\theta_0)\sin[(n+\tfrac{1}{2})\theta_0]\,\delta_{m0}$$

The constant U could be determined by a variational procedure, but if we take the cruder approximation, requiring that the equation $p_- - p_+ = \rho c k d_v u_a$ at $\vartheta = 0$, show that the equation determining U is

$$-\frac{\rho c}{k^2a^2} U \sin(\tfrac{1}{2}\vartheta_0) \sum_n \sin[(n+\tfrac{1}{2})\vartheta_0]\left[\frac{1}{j_n'(ka)h_n'(ka)} + k^2a^2kd_v\right]$$
$$= A\Psi(\theta_i) \equiv \frac{iA}{k^2a^2}\sum_n (2n+1)\frac{(-i)^n P_n(\cos\theta_i)}{h_n'(ka)}$$

where

$$\Psi(\theta_i) = \sum_n \frac{2n+1}{k^2a^2}\frac{e^{-i\delta_n(ka)}}{B_n(ka)}(-i)^n P_n(\cos\theta_i) \xrightarrow[ka\to 0]{} 1 - \tfrac{3}{2} ika\cos\theta_i$$

Quantities B_n and δ_n are defined in Eqs. (7.2.15) and given in Table VIII. Series $A\Psi(\theta_i)$ is the pressure which would be present at the point $r = a$, $\vartheta = 0$, if there were no hole in the sphere. It can be considered to be an approximate expression for the driving pressure, forcing the flow U through the hole.

Therefore the coefficient of $-U$, on the first line, is a rough approximation to the impedance $\rho c\zeta$, per unit area, of the hole. Using the comparison series

$$\sum_n \sin\left[(n+\tfrac{1}{2})\theta_0\right] = \frac{1}{2\sin(\frac{1}{2}\theta_0)}$$

$$\sum_n \frac{(2n+1)\sin(n+\frac{1}{2})\theta_0}{(n+\delta_{0n})(n+1)} = (\pi-\theta_0)\cos(\tfrac{1}{2}\theta_0)$$

to approximately cancel out the terms in $1/j_n(ka)h_n'(ka)$, show that the impedance of the opening is $3(\rho c/ka)\sin^2(\frac{1}{2}\theta_0)(R - iX)$, where

$$X = \tfrac{1}{3}k^2a^2(\pi-\theta_0)\cot(\tfrac{1}{2}\theta_0) - \operatorname{Re} K$$

and

$$R = \tfrac{1}{6}k^2 a d_v \csc^2(\tfrac{1}{2}\theta_0) - \operatorname{Im} K$$

where

$$K = \frac{1}{3ka}\sum_{n=0}^{\infty}\frac{\sin(n+\frac{1}{2})\theta_0}{\sin(\frac{1}{2}\theta_0)}\left[\frac{\cot\delta_n}{B_n^2} + (ka)^3\frac{2n+1}{(n+1)(n+\delta_{0n})} - \frac{i}{B_n^2}\right]$$

is a small quantity when $2\pi ka \ll 1$. Show that, at low frequencies, $K \simeq 1 - \frac{1}{15}k^2a^2 - \frac{i}{3}k^3a^3$, so that the impedance becomes

$$4\pi a^2\sin^2(\tfrac{1}{2}\theta_0)\left[R_s - i\omega L_s + i(1/\omega C_s)\right]$$

where

$$R_s = \frac{\rho c}{4\pi a^2}\left[\tfrac{1}{2}kd_v\csc^2(\tfrac{1}{2}\theta_0) + k^2a^2\right] \qquad C_s = \frac{4\pi a^3}{3\rho c^2}(1 + \tfrac{1}{15}k^2a^2)$$

$$L_s = \frac{\rho}{4\pi a}(\pi-\theta_0)\cot(\tfrac{1}{2}\theta_0)$$

Point out the physical significance of each term in these analogous impedances. At what value of ω is the resonance? Is this the only resonance of the resonator (i.e., is this the only value of ω for which $\operatorname{Re} K = 0$)? Since the formulas for the velocity in the opening result in

$$S_n{}^m \simeq -\frac{Aka\Psi(\theta_i)}{3(2n+1)}\frac{\sin(n+\frac{1}{2})\theta_0}{\sin(\frac{1}{2}\theta_0)}\frac{\delta_{m0}}{X - iR}$$

write out the expression for the pressure amplitude at the center of the sphere. At what frequency is this maximum? What is its value then? Show that the scattering and absorption cross sections of the resonator at longer wavelengths are

$$\Sigma_s \simeq \tfrac{1}{3}\pi a^2(ka)^4(1 - \tfrac{3}{5}k^2a^2) + \frac{\pi a^2}{R^2+X^2}\left(\tfrac{2}{9}(ka)^5 X\frac{\sin(\frac{3}{2}\theta_0)}{\sin(\frac{1}{2}\theta_0)} + (ka)^6\left\{1 + \tfrac{1}{5}X^2 + \frac{1}{27}\left[\frac{\sin(\frac{3}{2}\theta_0)}{\sin(\frac{1}{2}\theta_0)}\right]^2\right\} - \tfrac{1}{3}R^2\right)$$

$$\Sigma_a \simeq 4\pi a^2\frac{kd_v|\Psi(\theta_i)|^2k^2a^2}{9(R^2+X^2)}\csc^2(\tfrac{1}{2}\theta_0) \qquad ka \ll 1$$

Again explain the origin of each term. What are the peak values of Σ_s and Σ_a at resonance, when $X = 0$? Compare them with the geometrical cross section πa^2.