13 The thin plate of Eq. (10.1.21) is periodically loaded with line impedances, like the membrane of Sec. 10.3, so that its equation of motion, when driven by a force $F_\varkappa \exp(i\varkappa x - i\omega t)$, is

$$\frac{\partial^4\eta}{\partial x^4} - \gamma^4\eta = \frac{\gamma^4 F_\varkappa}{2\omega^2 h\rho_p}\, e^{i\varkappa x} + \frac{i\gamma^4 Z}{2\omega h\rho_p}\,\eta \sum_n \delta(x - nl)$$

Show that the steady-state displacement of the plate is

$$\eta = \frac{\gamma^4 F_\varkappa/2\omega^2 h\rho_p}{\varkappa^4 - \gamma^4}\, e^{i\varkappa x - i\omega t}\left[1 + \frac{iZ(\gamma l)^4}{2\omega\, h\rho_p}\frac{C}{Q}\sum_{m=-\infty}^{\infty}\frac{e^{(2\pi i/l)x}}{(\varkappa l + 2\pi m)^4 - (\gamma l)^4}\right]$$

where

$$C(\varkappa l) = [\cos(\varkappa l) - \cos(\gamma l)][\cos(\varkappa l) - \cosh(\gamma l)]$$

$$Q(\varkappa l) = \left[\cos(\varkappa l) - \cos(\gamma l) + \left(\frac{i\gamma Z}{8\omega h\rho_p}\right)\sin(\gamma l)\right]$$

$$\times\left[\cos(\varkappa l) - \cosh(\gamma l) - \left(\frac{i\gamma Z}{8\omega h\rho_p}\right)\sinh(\gamma l)\right] - \left(\frac{\gamma Z}{8\omega h\rho_p}\right)^2 \sin(\gamma l)\sinh(\gamma l)$$

Show that the free vibration of the plate occurs when $Q(\varkappa l) = 0$. Sketch the plot, corresponding to Figs. 10.15 and 10.16, for $u_q = \varkappa_q l/\pi$ versus $\pi X/8\omega h\rho_v$ and $v = \gamma l/\pi$, for $Z = R - iX$ purely reactive ($R = 0$). Is the plot periodic in v?

14 Show that the Green's function for the periodically loaded plate of Prob. 13 is

$$G(x \mid x_0) = \frac{l^3}{\pi^4}\sum_m \frac{e^{ik_m x}}{\Delta_m}\,\psi_m(-x_0) = \frac{l^3}{\pi^4}\sum_m \frac{e^{-ik_m x_0}}{\Delta_m}\,\psi_m(x)$$

where

$$\psi_m(x) = e^{ik_n x} + iB\sum_n \frac{e^{ik_n x}}{\Delta_n} \qquad \Delta_n = u_n^2 - v^2 \qquad u_n = u + 2n = \frac{k_n l}{\pi}$$

$$k_n = \varkappa + \frac{2\pi n}{l} \qquad u = \frac{\varkappa l}{\pi} \qquad v = \frac{\gamma l}{\pi}$$

$$B = \frac{v^4 Z}{2\omega l h\rho_p}\frac{C(\varkappa l)}{Q(\varkappa l)}$$

functions C and Q being defined in Prob. 13.

15 Carry through the calculations of Eqs. (10.3.13) to (10.3.18) for the plate of Probs. 13 and 14, to obtain the wave velocities for the periodically loaded plate in contact with the medium.

16 Show that the displacement of the periodically supported membrane of Sec. 10.3, when in contact with the medium and when also acted on by a driving force $F(\varkappa)\exp(i\varkappa x - i\omega t)$, is

$$\eta_\varkappa = \frac{\alpha^2 l F(\varkappa)}{\pi^2\sigma c^2}\left[\frac{w_0 e^{i\varkappa x}}{\Delta_0 w_0 - i\mu l} + \sum_n \frac{iBw_n e^{ik_n x}}{\Delta_0(\Delta_n w_n - i\mu l)}\,\frac{1 + \dfrac{2i\rho l v^2/\pi\sigma}{\Delta_0 w_0 - i\mu l}}{1 - (4i\zeta\rho l v^3/\pi^2\sigma)\Gamma(u)}\right]e^{-i\omega t}$$

where the quantities w_n, Δ_n, B, etc., are as defined in Eqs. (10.3.10), and $\Gamma(u)$ is as defined in Eq. (10.3.15). What is the pressure wave associated with this motion?

17 Plot the mean-square pressure amplitudes $|T_0|^2$ and $|T_m|^2$ of the transmitted and diffracted waves of Eq. (10.3.22), for those waves which go to $z \to -\infty$, for $\zeta = 10$, $2\pi/kl = \frac{1}{3}$, $g = \frac{1}{10}$, and $\alpha = \frac{1}{2}$, as function of the angle of incidence θ.

18 Carry out the calculations resulting in Eqs. (10.3.15) and (10.3.16) for the periodically loaded plate of Probs. 13 and 14.

19 Obtain the equivalent of Eq. (10.3.22) for the diffraction of sound through the periodically supported plate of Probs. 13 and 14.

20 A cylindrical enclosure is bounded by rigid walls at $r = a$, $z = b$, and $z = -d$. It is divided into two parts by a thin plate of the sort described in Sec. 5.3, along the plane $z = 0$. If the coupled standing waves in the two volumes have common angular factors $\cos(m\phi)$ about the cylinder axis, show that the displacement of the plate, $Y_m(r)\cos(m\varphi)$, is related to the pressure difference $(p_- - p_+)_{z=0} = P_m(r)\cos(m\varphi)$ by the equation

$$Y_m(r) = \frac{\gamma^4}{2\omega^2 h\rho_p}\int_0^a G_m(r \mid r_0)P_m(r_0)r_0\,dr_0$$

Write out the series for G_m. Show that a closed form for the integral is possible if P_m were to equal $J_m(qr)$.

$$\int_0 G_m(r \mid r_0)J_m(qr)\,r_0\,dr_0 = \frac{1}{(q^4 - \gamma^4)\Delta_m(\gamma a)}\left\{\Delta_m J_m(qr) + \left[\frac{q}{\gamma}J'_m(qa)I_m(\gamma a) - J_m(qa)I'_m(\gamma a)\right]J_m(\gamma r) + \left[J_m(qa)J'_m(\gamma a) - \frac{q}{\gamma}J'_m(qa)J_m(\gamma a)\right]I_m(\gamma r)\right\}$$

where

$$\Delta_m(\gamma a) = J_m(\gamma a)I'_m(\gamma a) - J'_m(\gamma a)I_m(\gamma a)$$

(resonance of the plate occurs when $\Delta_m = 0$). Show that P_m is given in terms of Y_m by

$$P_m(r) = -\frac{2\omega^2\rho}{a^2}\int_0^a [g_b(r \mid r_0) + g_d(r \mid r_0)]\,Y_m(r_0)r_0\,dr_0$$

$$g_b = \sum_{ij}\frac{\epsilon_j}{b\Lambda_{ij}}\frac{J_m(\pi\alpha_{mi}r/a)J_m(\pi\alpha_{mi}r_0/a)}{(\pi\alpha_{mi}/a)^2 + (\pi j/b)^2 - k^2} \qquad k = \frac{\omega}{c}$$

and g_d has d instead of b. The quantities Λ and α are as defined in Eqs. (9.2.22) and (9.2.23). Suppose that b does not differ much from d, so that the resonance frequencies for $i = n$, $j = l$ are nearly the same for the two cavities. Use the procedures of Sec. 10.4 to show that an approximate expression for the resonance frequencies $kc/2\pi$ closest to $c[(\alpha_{mn}/2a)^2 + (l/2b)^2]^{\frac{1}{2}}$ is

$$\frac{\gamma^4 a^4\rho\epsilon_l}{2h\rho_p(\pi^4\alpha_{mn}^4 - \gamma^4)}\left[\frac{4\gamma^3 a^3 J'_m(\gamma a)I'_m(\gamma a)}{(\pi^4\alpha_{mn}^4 - \gamma^4)(1 - m^2/\pi^2\alpha^2_{mn})\Delta_m(\gamma a)} - 1\right] \times \left[\frac{1/b}{k^2 - (\pi\alpha_{mn}/a)^2 - (\pi l/b)^2} + \frac{1/d}{k^2 - (\pi\alpha_{mn}/a)^2 - (\pi l/d)^2}\right] = 1$$

when $\gamma^4 a^5\rho/2h\rho_p$ is small. Work out expressions for the coupling coefficient, and discuss the effect of plate resonance on the resonances of the coupled rooms. What is the formula for the forced acoustic motion of the two enclosures?

21 Consider a "double wall" consisting of two parallel stretched membranes separated a distance d. A plane wave is incident at an angle of incidence θ on the wall. Determine the transmitted wave and the sound field between the membranes. Show that the transmitted wave amplitude P_t can be written in terms of the incident pressure amplitude P_i as

$$P_t = P_i \frac{1}{(1 + f_1)(1 + f_2) - f_1 f_2 \exp(2ikd \cos\theta)}$$

where

$$f = i \frac{\sigma\omega}{2\rho c}\left[1 - \left(\frac{v}{c}\right)^2 \sin^2\theta\right] \cos\theta$$

where σ = membrane mass per unit area
$v = \sqrt{T/\sigma}$ = velocity of free transverse waves on membrane
ρ = density of air
c = speed of sound in air

The indices 1 and 2 refer to the two membranes which may have different values of ϵ and v.

CHAPTER

11

ACOUSTICS IN MOVING MEDIA

11.1 PLANE WAVES IN UNIFORM FLOW

Most of the material discussed so far has been devoted to the acoustics of a homogeneous fluid at rest, and this indeed covers a substantial portion of the problems of interest in acoustics. The restriction that the fluid be at rest may seem insignificant at first since the question of (uniform) motion or rest is merely a matter of a coordinate transformation. However, when we consider also the sound sources, receivers, and boundaries involved in a specific problem, it is clear that a relative motion between the fluid and the various material bodies involved cannot be eliminated by a coordinate transformation; a transformation that brings the medium to rest sets the sources and the boundaries in motion.

For example, let the fixed boundary be an ordinary straight-tube section open at both ends. If the tube length is L and if there is no flow, the time of travel for a plane-wave pulse back and forth in the tube is $T_1 = 2L/c$, and therefore the fundamental plane-wave resonance frequency of the tube is $f_1 = c/2L$. In the presence of flow, however, the round-trip time for a pulse is $T_1' = [L/(c + V)] + [L/(c - V)] = T_1/(1 - M^2)$, where $M = V/c$, and the corresponding resonance frequency is $f_1' = (1 - M^2)f_1$. In other words, the presence of flow lowers the resonance frequency of the tube. This simple example demonstrates the need for a reexamination of the problems of generation, propagation, and detection of sound when motion relative to the medium is involved. In some sense the motion of the medium makes it nonisotropic, since the speed of sound in the laboratory system depends on the direction of propagation with respect to the direction of motion of the medium. If, in addition, the medium is inhomogeneous, so that the speed of sound and other medium characteristics vary from one point to another as in the atmosphere and in the sea, various refraction and scattering effects must be considered. Some of these problems have already been discussed in previous chapters for a medium at rest. Therefore the main emphasis in this chapter will be on the influence of fluid motion on sound propagation.

Doppler shift

The changing pitch of the sound from, say, a fire-engine siren as it moves by at high speed is familiar to all; this effect, the Doppler effect, is one of the most obvious influences of relative motion between source and medium. Similarly, if a wave train of sound in the medium is received by an observer in motion with respect to the medium, a Doppler change of frequency also will result. If, for example, the observer moves with speed of the wave train, he will find a stationary pattern with zero frequency. The Doppler effect is a purely kinematic phenomenon, and can be studied without resort to the dynamical equations of motion of the wave.

Let us consider, first, the case when an observer A is at rest in the laboratory system S, recording the pressure fluctuations in the sound wave. The medium moves in the x direction with a velocity V and carries a plane sound wave that travels in the medium, also in the x direction, with the speed c of sound. In other words, an observer in a frame of reference S' moving with the medium observes a sound wave traveling in the x direction with a speed c; the frequency to him is, of course, $f' = c/\lambda$, where λ is the wavelength. With respect to an observer at rest, however, the speed of the wave is $c + V$, and since λ is the same to both observers, the observed frequency is $f = (c + V)/\lambda = f'(1 + M)$, where $M = V/c$ is the *Mach number* of the flow. (In what follows we always use *primes* to denote quantities measured by an *observer moving with* the medium; *unprimed quantities* are measured by an *observer at rest* in the "laboratory coordinates.")

Next let us turn to the source that generated the plane wave of wavelength λ. If the source moves with the medium, the source frequency to produce the wave is $f' = c/\lambda$. If the source is at rest in S, however, in the time of one period T_S of the source the wave will travel in the x direction a distance $(c + V)T_S$, which must equal the wavelength λ. The relation between the wavelength and the frequency of the source is then $f_S = 1/T_S = (c + V)/\lambda = f'(1 + M)$, where, as before, $f' = c/\lambda$ is the frequency observed in a coordinate system S' moving with the medium. We recall that the frequency recorded by a stationary observer is also $f'(1 + M)$, the same as the frequency of the stationary source.

If the medium and the source are both at rest in S, but the receiver is moving in the negative x direction with a velocity V, we have a situation that kinematically is equivalent to the previously considered case of a stationary receiver and a medium moving (with the source) in the positive x direction with a velocity V.

It is not difficult to extend this to three dimensions. What needs to be kept in mind is that a steady-state wave, such as a plane wave, is not like a projectile; it is a moving pattern of wavefronts. The distance between these fronts, for example the wavelength λ, and the direction normal to the fronts will be the same for *all* observers, whether they are moving with respect to the medium or not. Other quantities will change, such as measured

frequency, particle velocity (of course), and sometimes pressure, but the geometry of the pattern is invariant.

Thus the wavenumber vector $\mathbf{k}$, of magnitude $2\pi/\lambda$ and direction normal to the wavefronts of a plane wave, is the same in the laboratory system S as it is in the comoving system S', moving with the medium with a velocity $\mathbf{V}$ with respect to S. In the comoving system the pressure wave is proportional to

$$\exp(i\mathbf{k}\cdot\mathbf{r}' - i\omega' t)$$

where $\mathbf{r}'$ and ω' are measured with respect to S', and $k = \omega'/c$. The transformation to the laboratory system is given by

$$\mathbf{r} = \mathbf{r}' + \mathbf{V}t \qquad t = t'$$

so that, to an observer at rest in the laboratory system, the pressure will be proportional to

$$\exp[i\mathbf{k}\cdot\mathbf{r} - i(\omega' + \mathbf{k}\cdot\mathbf{V})t] = \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$$

producing the three-dimensional Doppler formula, relating the frequency $\omega'/2\pi$ in the comoving system to the frequency $\omega/2\pi$ in the laboratory system,

$$\omega = \omega' + \mathbf{k}\cdot\mathbf{V} = \omega'(1 + M\cos\theta) \tag{11.1.1}$$

where θ is the angle between the direction of motion of the medium and the normal to the plane wavefronts. In the laboratory system k is *not* equal to ω/c.

The equations of motion

With the kinematical preliminaries out of the way, we next should examine the equations of wave motion. In the coordinate system S' moving with the fluid, a sound field is described by the usual equations for mass and momentum balance [see Eq. (6.2.7)], which we write in the form

$$\frac{\partial \rho}{\partial t'} + \rho_0 \sum_i \frac{\partial u_i}{\partial x_i'} = 0 \qquad \rho_0 \frac{\partial u_i}{\partial t'} = -\frac{\partial p}{\partial x_i'} \tag{11.1.2}$$

where the subscript i refers to x, y, and z, respectively. If the velocity of the fluid has the components V_i, the transformation of coordinates from the comoving system S' to the laboratory system S is given by

$$x_i = x_i' + V_i t' \qquad t = t' \qquad \mathbf{V} = \text{const}$$

Thus, for any function f of space and time, we have

$$\frac{\partial f}{\partial x_i'} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial x_i'} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x_i'} = \frac{\partial f}{\partial x_i}$$

In other words, the space derivatives are invariant under the transformation; $\partial/\partial x_i' = \partial/\partial x_i$. However, for the time derivative, we get

$$\frac{\partial f}{\partial t'} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial t'} + \sum_i \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t'} = \frac{\partial f}{\partial t} + \sum_i V_i \frac{\partial f}{\partial x_i}$$

or

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \sum_i V_i \frac{\partial}{\partial x_i} \equiv \frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}$$

[compare this with Eq. (6.1.14)]. Using these transformations in (11.1.2), we obtain the equations with respect to the laboratory system S.

$$\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}\right)\rho + \rho_0 \sum_i \frac{\partial u_i}{\partial x_i} = 0 \qquad \rho_0\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}\right)u_i = -\frac{\partial p}{\partial x_i} \quad (11.1.3)$$

From these two equations plus the relationship $dp = c^2\, d\rho$, the wave equation follows.

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}\right)^2 p = \nabla^2 p$$

or

$$\frac{\partial^2 p}{\partial t^2} + 2\sum_i V_i \frac{\partial}{\partial t}\frac{\partial p}{\partial x_i} + \sum V_i V_j \frac{\partial^2 p}{\partial x_i\, \partial x_j} = c^2\nabla^2 p \qquad (11.1.4)$$

For harmonic time dependence Eq. (11.1.4) reduces to

$$\nabla^2 p + \left(\frac{\omega}{c}\right)^2\left(1 + i\frac{1}{\omega}\mathbf{V}\cdot\boldsymbol{\nabla}\right)^2 p = 0$$

For a plane wave traveling in a direction specified by the vector wavenumber $\mathbf{k}$, where $k = \omega'/c$ and ω' is the angular frequency as measured in the comoving coordinate system S', we have $\nabla^2 = -k^2$ and $\mathbf{V}\cdot\boldsymbol{\nabla} = i\mathbf{V}\cdot\mathbf{k}$, again arriving at the relation between ω and ω' given in Eq. (11.1.1).

So far we have considered only the Doppler shift in the case of a plane wave, in which case only the two values of the received frequency $f_1 = f_0/(1 + M)$ and $f_2 = f_0/(1 - M)$ are found. When we consider a spherical wave, emitted by a source that moves by a distant observer, the Doppler-shifted frequency varies continuously between the two values f_1 and f_2. This case will be discussed in some detail in Sec. 11.2.

Fluid motion and transducer response

An important question concerns the measurement of a sound field in a moving fluid. If the transducers used for pressure measurements move with the fluid, the true static P and acoustic pressure p are measured. If the same transducers are kept at rest, however, so that there is relative motion between them and the fluid, the quantities measured by the transducer are not necessarily P and p; the output from the transducers generally will depend on the

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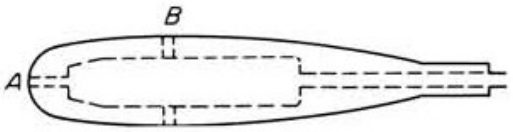


FIGURE 11.1
Streamlined transducer or transducer housing in which the pressure-sensing area is located at *B* for minimum influence of fluid motion and turbulence.

relative motion. Although a detailed analysis of what actually is measured by any one transducer in such a case is quite complex, it nevertheless is possible to say something about this problem, by using some well-known facts about the pressure distribution over a body in a moving fluid.

Suppose the pressure measured by a transducer moving with the medium is P; we wish to measure the pressure when the medium is moving past it with velocity V. If the transducer has a blunt front as illustrated in Fig. 11.1, there will be one point A at the front of the body where the flow velocity is zero. At this point, the stagnation point in the flow, Bernoulli's law [Eq. (6.2.12)] for irrotational fluid motion predicts that the pressure would be

$$P_A = P + \frac{\gamma - 1}{\gamma} \frac{\rho V^2}{2} \tag{11.1.5}$$

where γ is the ratio between the specific heats at constant pressure and volume [see Eq. (6.1.3)].

If, on the other hand, we consider not the stagnation point, but a point B on the side of the body, the flow velocity at this point is approximately equal to the free-stream velocity V (if the body is sufficiently slender), and the pressure is then approximately equal to the free-stream pressure P.

$$P_B \simeq P \tag{11.1.6}$$

In the presence of a plane incident wave with sound pressure p and corresponding fluid velocity $\mathbf{u}$, in the bulk of the fluid away from the body we have $P = P_0 + p$ and $\mathbf{V} = \mathbf{V}_0 + \mathbf{u}$, where P_0 and $\mathbf{V}_0$ are the unperturbed values of the pressure and velocity. If the wavelength of the sound is much larger than the dimensions of the body, the sound pressure about the body is approximately uniform and equal to the incident sound pressure. The stagnation pressure P_A in (11.1.5) can then be expressed approximately as

$$P_A \simeq P_0 + p + \frac{\gamma - 1}{2\gamma} (\rho_0 + \delta) |\mathbf{V}_0 + \mathbf{u}|^2 \tag{11.1.7}$$

If the transducer has its pressure-sensing portion in the stagnation point, we see that, if $V_0 \gg u$, the time-dependent portion of the pressure is

$$p_A \simeq p\left(1 + \frac{\gamma - 1}{\gamma} M_0 \cos \varphi\right)$$

where $M_0 = V_0/c$. We have introduced $\mathbf{V} \cdot \mathbf{u} = Vu \cos \varphi$ and $p = \rho c u$, valid for a plane wave of sound; φ is the angle between $\mathbf{u}$ and $\mathbf{V}$. Thus the

relative motion between the transducer and the medium leads to an increase of the recorded pressure at A by an amount $p[(\gamma - 1)/\gamma]M_0 \cos \varphi$, which depends on the angle between $\mathbf{V}$ and the direction of the sound wave.

If, on the other hand, the pressure-sensing part of the microphone is at point B (assuming that the microphone has a slender shape), the time-dependent part of P_B is approximately equal to the sound pressure p, the same as the pressure measured by a transducer moving with the fluid.

If the medium is turbulent, the relative motion between the microphone and the medium will give rise to an additional time-dependent pressure on the microphone, a pressure which often "masks" the signal produced by the incident sound wave to be detected. The magnitude of these turbulent-pressure fluctuations depends on the location of the pressure-sensing area of the microphone. It is larger at the stagnation point A than at the point B in Fig. 11.1. In the following estimate of the influence of the turbulence we shall assume that the fluctuating turbulent velocity $v(x,y,z,t)$ is much smaller than the mean velocity V of the flow. The turbulent pressure field as recorded by a transducer moving with the average speed of the fluid is of the order of $\rho v^2(x',y',z',t)/2$. The turbulent pressure $p_{B,t}$ recorded at the position B of the stationary transducer will be of this same order of magnitude, $p_{B,t} \simeq \frac{1}{2}\rho v^2(x - Vt, y, z, t)$.

At A, on the other hand, the turbulent pressure is $p_{A,t} \simeq \rho \mathbf{V}_0 \cdot \mathbf{v}(x - Vt, y, z, t)$, assuming that $V \gg v$. This pressure is larger than $p_{B,t}$ by a factor of the order of $2V_0/v$.

If, instead of a pressure-sensitive transducer, we have one that responds to a velocity component, say, the x component, a microphone moving with a relative velocity V in the x direction in the turbulent fluid will record a velocity field $v_x(x - Vt, y, z, t)$.

The pressure fluctuations recorded by the moving transducer as a result of the turbulence in the fluid represent undesirable noise which interferes with the measurement of the sound pressure field. If the sound pressure field is p, the corresponding velocity field is $u = p/\rho c$, assuming plane-wave conditions. If we use a microphone with the pressure-sensing element at the point B, the *signal-to-noise ratio* between p and the turbulent noise pressure $p_{B,t}$ is of the order of $p/(\rho v^2/2) \simeq 2\,cu/v^2$. The corresponding signal-to-noise ratio when referred to the stagnation point A is cu/vV. If, instead of a pressure transducer, we use a velocity-sensing transducer, the signal-to-noise ratio is u/v. Since in most cases of interest we have $c > V > v$, it follows that, when operating in turbulent flow, the pressure-type microphone measuring the pressure at point B is least influenced by turbulence, and the velocity transducer is the noisiest.

Most of the standard transducers, such as crystal, condenser, or moving-coil transducers, can be regarded as pressure transducers; when the pressure sensitive membrane points in the direction of the relative motion, the influence of turbulence is the largest. If the transducer is turned 90°, so that

the membrane is parallel with the relative motion, the noise is reduced. The response to the sound pressure, on the other hand, is approximately independent of the orientation.

To get an idea of the order of magnitude of the turbulent-pressure fluctuations, consider a microphone in a turbulent airstream with an average velocity 30 m per sec. The rms value of the turbulent-velocity fluctuation is assumed to be 10 percent of the mean flow speed. At the stagnation point the order of magnitude of the turbulent-pressure fluctuation $\rho V v = 0.1\ \rho V^2$ then will be about 900 dynes per sq cm, which corresponds to a sound-pressure level of 126 db. In other words, only very intense sound waves could be detected under these conditions. At point B the turbulence level would be 13 db below this, still quite loud.

In the discussion so far we have disregarded the pressure fluctuations induced by the transducer because of the turbulent flow it produces. As it moves through the fluid, a turbulent wake is produced behind the transducer, and the corresponding pressure fluctuations produced by the flow will be of the order of $\rho V^2/2$, where, as before, V is the mean relative speed between the microphone and the fluid. Actually, in the range of velocities corresponding to Reynolds numbers between 10^2 and 10^5, these fluctuations are practically periodic. This oscillation, which produces the familiar Kármán vortex street behind the body, has a frequency of the order of $0.2V/d$, where d is the transverse dimension of the wake. In the case of a cylinder at right angles to the flow, the corresponding oscillatory force per unit length of the cylinder, tending to shake the cylinder sidewise, is found to be of the order of $0.5d$ $(\rho V^2/2)$ per unit length of the cylinder, where d in this case is, roughly, the diameter of the cylinder. The corresponding pressure fluctuation recorded by the transducer is then a maximum when the membrane normal points in the direction transverse to the motion (i.e., is in position B). This is in contrast to the case when the pressure fluctuations are due to the turbulence in the incident flow, in which case, as we recall, the maximum noise is obtained when the membrane normal points in the direction of the flow (i.e., at point A). However, the self-induced turbulent-pressure fluctuations by the transducer can be reduced considerably by enclosing the transducer in a screen which alters the character of the wake behind the transducer.

Having made these general comments about the relation between the pressure actually recorded by various transducers, variously oriented and moving at various speeds through the medium, and the pressure P which would be measured at the same point by a transducer moving with the medium, in the rest of this chapter we shall assume that the necessary corrections have been made and will discuss P, the "true" pressure at a given point and time. Of course, we still have the basic difference between the Lagrange and Euler representations portrayed in Fig. 6.2, if the point in question is moving with respect to the medium. The pressure at point $\mathbf{r}$ at time t, in the laboratory system, in a medium moving with respect to this

system, is the pressure measured by a comoving transducer which happens to be at point **r** at time t. If there is no turbulence, this is equal to the pressure measured at point B in a transducer at rest in the laboratory system. When there is turbulence, corrections of the sort just discussed must be made before the measurements will correspond to the pressure given by the formulas in this chapter.

Flow parallel to plane boundary

An important aspect of the effects of moving media is the interaction with the boundary. As a simple example we consider the boundary to be the yz plane, the fluid moving parallel to it with velocity V, as shown in Fig. 11.2. Further, we assume a sinusoidal transverse wave to be traveling along the boundary in the y direction, with velocity c_t. In other words, the boundary can be a membrane or plate, with transverse wave motion in it having wave velocity c_t with respect to its material, or it could be that the boundary is a rigid corrugated surface, in which case c_t would be zero. In any case, the relative velocity $c_t - V$ will produce wave motion in the medium.

We have already seen that velocities and frequencies are altered by relative motion, but that lengths and directions of normals to wavefronts are unaltered. This suggests that we deal with displacements and pressure (or density changes) in fitting boundary conditions, rather than joining velocities at the boundary, as has been done heretofore. As a first step, we describe the wave in the boundary surface in terms of its displacement from the yz plane and its wavelength L (or wavenumber $k_t = 2\pi/L$). Then the speed of this wave and its frequency in the laboratory system S will be c_t and $\omega/2\pi = c_t/L$, respectively. The transverse displacement of the boundary can thus be written, in the laboratory coordinates, as

$$\xi = \xi_0 \exp\,[ik_t(y - c_t t)] = \xi_0 \exp\,[i(\omega/c_t)y - i\omega t]$$

The relative motion between the wave in the boundary and that in the medium produces a plane wave in the medium. In the laboratory

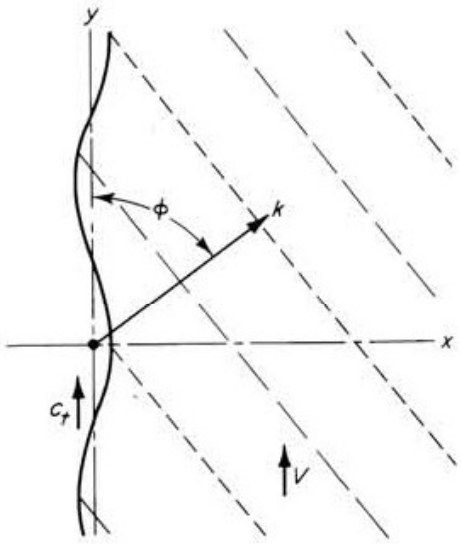


FIGURE 11.2
A transverse wave in the boundary at $x = 0$ moves with velocity c_t; the medium to the right of the boundary moves with velocity V with respect to the laboratory coordinates x, y. Vector **k** is normal to the plane wavefronts in the medium.

coordinates x and y, the sound pressure (see the last paragraph under the preceding heading) will have the same frequency, $\omega/2\pi$, as that of the transverse wave in the boundary (in the same coordinates). Thus the form of the pressure wave, in terms of x, y, and t, will be

$$\begin{aligned} p &= P \exp (ik_x x + ik_y y - i\omega t) \\ &= P \exp [ik(x \sin \phi + y \cos \phi) - i\omega t] \end{aligned} \tag{11.1.8}$$

where $\omega = 2\pi c_t/L$, and k_y must equal $k_t = \omega/c_t$ in order that the sound wave fit the boundary displacement wave along the xz plane.

For comparison, let us write down the results if the medium were at rest, if V were zero. In this case, k would equal ω/c, where c is the speed of sound in the medium. Equating $k \cos \phi$ to k_t yields

$$\cos \phi = \frac{c}{c_t} \qquad \text{and therefore} \qquad k_x = k \sin \phi = \frac{\omega}{c}\sqrt{1 - \left(\frac{c}{c_t}\right)^2} \qquad V = 0$$

which determines the angle ϕ of the radiated wave. As we noted in Sec. 10.1, if the speed c of sound in the medium is greater than the speed c_t of the boundary waves, no true wave is propagated into the medium; k_x is imaginary.

The pressure amplitude P is related to the boundary displacement by using the equation $\rho(\partial^2\xi/\partial t^2) = -(\partial p/\partial x)$, obtained from Eqs. (6.2.3). The result is (to first approximation—see page 238)

$$P = \frac{\rho\omega^2}{ik_x}\xi_0 = \frac{\rho c}{\sqrt{1 - (c/c_t)^2}}(-i\omega\xi_0) \qquad V = 0$$

the impedance load of the medium per unit area of vibrating boundary surface thus being $\rho c/\sqrt{1 - (c/c_t)^2}$.

If, however, the medium is moving in the y direction with velocity V, these results must be modified. The form of the pressure wave in terms of the laboratory coordinates x and y is the same, that given in Eq. (11.1.8), but the values of P, ϕ, and k are changed by the motion. Instead of the simple wave equation $\partial^2 p/\partial t^2 = c^2\nabla^2 p$, the pressure must satisfy Eq. (11.1.4), which in this case is

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y}\right)^2 p = c^2\nabla^2 p$$

Substitution of Eq. (11.1.8) in this yields $(\omega - Vk \cos \phi)^2 = (ck)^2$, which, together with $k_y = k \cos \phi = \omega/c_t = 2\pi/L$, serves to determine both k and ϕ.

$$\begin{aligned} k &= \frac{\omega}{c}\left(1 - \frac{V}{c_t}\right) = \frac{\omega}{c + V \cos \phi} \\ k_x &= \frac{\omega}{cc_t}\sqrt{(c_t - V)^2 - c^2} \qquad k_y = \frac{\omega}{c_t} \\ \cos \phi &= \frac{c}{c_t - V} \qquad \omega = \frac{2\pi c_t}{L} \end{aligned} \tag{11.1.9}$$

Thus k is reduced by the factor $1 - (V/c_t)$, involving the difference $c_t - V$, and $\cos\phi$ is increased by the reciprocal of the same factor (since k_y cannot be changed by the motion of the medium). Thus, also, no energy is radiated away from the surface if c is greater than the velocity difference $|c_t - V|$. For example, if the "wave" in the boundary is motionless (the boundary being corrugated and $c_t = 0$), only if $V > c$ (i.e., only if the flow past the boundary is supersonic) will this flow produce a "wave."

Having determined the angle of propagation ϕ and the wavelength $\lambda = 2\pi/k$ of the wave, we next must determine its amplitude. As mentioned above, we must modify Eqs. (6.2.3) to take the fluid motion into account. The time-derivative terms in these equations are really total derivatives, the time rate of change of velocity or displacement measured in comoving coordinates. As long as the medium is at rest, the correction term $\mathbf{u}\cdot\nabla$ is negligible because the $\mathbf{u}$ is the velocity caused by the sound wave, which is quite small. If the fluid moves with velocity $\mathbf{V}$, on the other hand, we cannot neglect the term $\mathbf{V}\cdot\nabla$ of Eq. (6.1.14). Thus the equation relating P to ξ at the boundary, in laboratory coordinates, for the present case, is

$$\rho\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial y}\right)^2 \xi_x = -\frac{\partial p}{\partial x} \tag{11.1.10}$$

Inserting the expressions for ξ and for p, we obtain

$$pk_t^2(c_t - V)^2\xi_0 = ikP\sin\phi = i\frac{k_t}{c}(c_t - V)P\sin\phi$$

or

$$P = \rho c(-i\omega\xi_0)\frac{(c_t - V)^2}{c_t\sqrt{(c_t - V)^2 - c^2}} \qquad \omega = \frac{2\pi c_t}{L}$$

$$= \frac{-i\omega\rho c\xi_0}{(1 + M\cos\phi)\sin\phi} \qquad M = \frac{V}{c} \tag{11.1.11}$$

which goes to zero when the medium moves as fast as the wave on the boundary ($V = c_t$), and to infinity when $|c_t - V| = c$.

The ratio $P/-i\omega\xi_0$ is the impedance load z of the medium on the boundary. [We use the Eulerian $\partial\xi/\partial t$, rather than the Lagrangian $d\xi/dt$, for the denominator because the velocity in question is the rate of change of the point $(0,y)$ *on the boundary*, disregarding the sideward motion of the fluid; slippage of the fluid past the boundary is supposed to produce no force.] Thus

$$z = \frac{\rho c(c_t - V)^2}{c_t\sqrt{(c_t - V)^2 - c^2}} = \frac{\rho c}{(1 + M\cos\phi)\sin\phi} \tag{11.1.12}$$

where M is the Mach number V/c. This impedance is purely resistive when $|c_t - V|$ is greater than the acoustic velocity c; it is negative-imaginary (masslike) when the differential speed $|c_t - V|$ is subsonic. The acoustic

power radiated per unit area of boundary is

$$\Upsilon = \mathrm{Re}\left(p^* \frac{\partial \xi}{\partial t}\right)_{x=0} = z\,|\omega \xi_0|^2$$

$$= 4\pi^2 \rho c \frac{c_t (c_t - V)^2}{L^2 \sqrt{(c_t - V)^2 - c^2}} \qquad \text{when} \qquad |c_t - V| > c \quad (11.1.13)$$

where L is the wavelength of the sinusoidal wave in the boundary, c_t its velocity of propagation, and V the velocity of the fluid. We now see that when the "wave" in the boundary is a stationary corrugation ($c_t = 0$), no power is radiated, although a pressure "wave" is present in the medium, if $|V| > c$. In this case the "wave" is a pattern of alternately greater and lesser pressure which is stationary with respect to the laboratory system S, and the boundary is stationary, so that no power can be radiated. Also, when $|c_t - V| < c$, no power is radiated; in this case z is reactive.

Refraction at an interface

Next let us consider the transmission of a plane acoustic wave through an interface between two media in relative motion, as shown in Fig. 11.3. We start by investigating the kinematical conditions to be satisfied, from which we shall find a law of refraction analogous to Snell's law.

Consider two semi-infinite fluid regions separated by the yz plane and moving in the y direction with velocities V_1 and V_2, respectively. For simplicity let the propagation vector of the sound wave lie in the xy plane. The plane wave originates in region 1 and travels in a direction which makes an angle ϕ_1 with the y axis, as shown in Fig. 11.3. The sound is partially

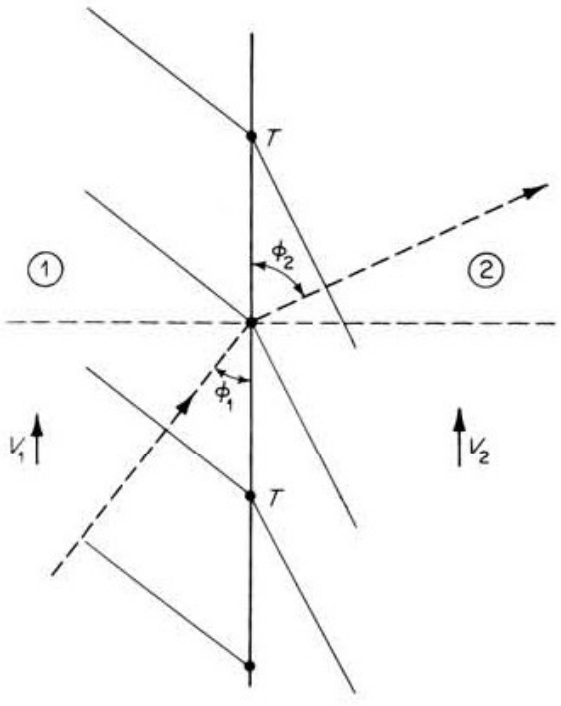


FIGURE 11.3
The trace velocity of the wave along the boundary between 1 and 2 (i.e., the velocity of T) must be the same in both media 1 and 2. $(c_1/\cos\phi_1) + V_1 = (c_2/\cos\phi_2) + V_2$.

transmitted into region 2 in a direction of propagation that makes an angle ϕ_2 with the y axis. To obtain the angle ϕ_2 in terms of the angle ϕ_1, we make use of the fact that the spatial variation of the fields along the interface must be the same in both regions. It follows that the motion of the intersection between a phase surface and the boundary must be the same regardless of whether the phase surface belongs to region 1 or 2. The velocity of such an intersection, indicated by T in Fig. 11.3, can be expressed in two ways, either as $(c_1/\cos\phi_1) + V_1$ or as $(c_2/\cos\phi_2) + V_2$; these, then, must be equal.

$$\frac{c_1}{\cos\phi_1} + V_1 = \frac{c_2}{\cos\phi_2} + V_2$$

or

$$\cos\phi_2 = \frac{c_2\cos\phi_1}{c_1 - \Delta V\cos\phi_1} \tag{11.1.14}$$

Here $\Delta V = V_2 - V_1$ is the relative velocity between the two regions, i.e., the velocity *increase* encountered when crossing the boundary in the direction of the sound.

If $V_2 > V_1$, so that ΔV is positive, which includes the case when region 1 is at rest and region 2 in motion, we note that $\phi_2 < \phi_1$. The critical angle of incidence ϕ_{1c}, corresponding to a refracted wave traveling along the boundary in region 2 (so that $\phi_2 = 0$), is given by

$$\cos\phi_{1c} = \frac{c_1}{c_2 + V} \qquad \phi_2 = 0$$

For angles ϕ_1 smaller than this critical angle, no plane traveling waves will penetrate into region 2, and total reflection occurs. (Under these conditions the sound pressure in region 2 decays exponentially with the distance from the boundary, as will be discussed later.) For the largest value of $\phi_1 = 180°$, the angle ϕ_2 is given by

$$-\cos\phi_{2,\max} = \frac{c_2}{c_1 + \Delta V} \qquad \phi_1 = 180°$$

If $c_2 = c_1$, we note that $180 - \phi_{2,\max} = \phi_{1c}$. The region $\phi_2 > \phi_{2,\max}$ is

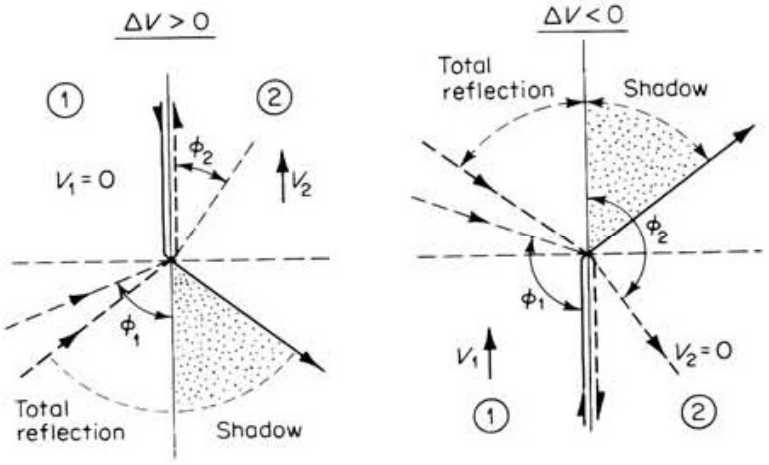


FIGURE 11.4
Wave transmission into a moving fluid, and from a moving fluid into one at rest. $\Delta V = V_2 - V_1$.

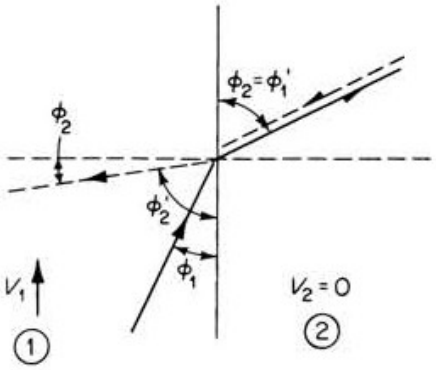


FIGURE 11.5
Reciprocity does not apply for refraction in moving media.

inaccessible for plane sound waves from region 1 and represents an acoustic shadow region. The behavior of the sound wave at the interface when traveling in a direction of increasing flow velocity is illustrated schematically in the first part of Fig. 11.4 for the case $c_2 = c_1$.

If, on the other hand, $V_1 > V_2$, so that ΔV *is negative*, which includes the case of transmission from a moving fluid into one at rest, we find from (11.1.14) that $\phi_2 > \phi_1$. The smallest possible value of ϕ_2 is obtained for $\phi_1 = 0$, and we find $\cos \phi_{2,\min} = c_2/(c_1 + |\Delta V|)$, and the region $\phi_2 < \phi_{2,\min}$ represents a shadow. The largest possible angle, $\phi_2 = 180°$, is obtained for the critical angle of incidence given by $-\cos \phi_{1c} = c_1/(c_2 + |\Delta V|)$. For angles of incidence larger than ϕ_{1c}, total reflection will occur, as indicated schematically in the right-hand part of Fig. 11.4.

It is important to note that if a wave that has been transmitted from region 1 to 2 is reversed in direction, it will not follow along the path of the initial ray; this is illustrated schematically in Fig. 11.5.

Reflection and transmission at the interface

We now turn to the dynamics of the problem in order to calculate the *reflection* and *transmission coefficients* at the interface between the two fluids in relative motion. As before, the interface is the yz plane, and the fluids move with uniform velocities V_1 and V_2 parallel with the yz plane. Let us first note that only the velocity components that lie in the plane of the incident wave (the plane defined by $\mathbf{k}$ and the normal to the interface) will affect the propagation and refraction of sound. This follows directly from the wave equation

$$\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \boldsymbol{\nabla}\right)^2 p = c^2 \nabla^2 p \qquad (11.1.15)$$

For a plane wave of the form $\exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$ we have $\mathbf{V} \cdot \boldsymbol{\nabla} = i\mathbf{V} \cdot \mathbf{k} = iV_{11} \cdot k$, where V_{11} is the component in the plane of the wave. Since the main object now is to investigate the effect of flow, we therefore can, without lack of generality, let $\mathbf{V}_1$ and $\mathbf{V}_2$ be placed along the y axis in the plane of incidence of the wave. Then $\mathbf{k} = (k_x, k_y, 0)$, $\mathbf{V} = (0, V, 0)$, and $\mathbf{V} \cdot \mathbf{k} = Vk \cos \phi = Vk_y$, where ϕ is the angle between $\mathbf{V}$ and $\mathbf{k}$. With $k_y = k \cos \phi$ and

$k_x = k \sin \phi$, it follows from Eq. (11.1.15) that, for either medium,

$$k = \frac{\omega}{c + V \cos \phi} \qquad (11.1.16)$$

$$k_x = k \sin \phi \qquad k_y = k \cos \phi$$

Using this expression for k_y in the two regions 1 and 2 (Fig. 11.3), together with the boundary condition that $k_{1y} = k_{2y}$,

$$\frac{c_1}{\cos \phi_1} + V_1 = \frac{c_2}{\cos \phi_2} + V_2 \qquad (11.1.17)$$

as found previously in Eq. (11.1.14).

If the incident and reflected pressure fields are p_i and p_r, the total field in region 1 is

$$p = p_i + p_r = P_1[e^{i(k_{1x}x + k_{1y}y)} + Re^{i(-k_{1x}x + k_{1y}y)}]e^{-i\omega t} \qquad (11.1.18)$$

Region 2 contains only a transmitted field, which is of the form

$$p_2 = P_1 T e^{i(k_{2x}x + k_{2y}y) - i\omega t} \qquad (11.1.19)$$

With reference to the previous discussion of boundary conditions in the problem of sound emission from a plane, the boundary conditions to be applied here are continuity of pressure and displacement ξ_x at the interface $x = 0$. The particle-displacement fields corresponding to (11.1.18) and (11.1.19) are obtained by integrating Eq. (11.1.10) and use of Eqs. (11.1.16) and (11.1.17). For example, the x displacement of medium 1 is

$$\xi_{1x} = \frac{iP_1 \sin \phi_1}{\rho_1 k_1 c_1^2} (e^{i(k_{1x}x + k_{1y}y)} - Re^{i(-k_{1x}x + k_{1y}y)})$$

Equating pressure at $x = 0$ gives us $1 + R = T$; equating displacements gives us $(\rho_1 c_1^2 k_1 \sin \phi_2)T = (\rho_2 c_2^2 k_2 \sin \phi_1)(1 - R)$. From these, plus the requirement that $k_1 \cos \phi_1 = k_2 \cos \phi_2$, an expression for the reflection coefficient is derived.

$$R = \frac{\rho_2 c_2^2 \sin (2\phi_1) - \rho_1 c_1^2 \sin (2\phi_2)}{\rho_2 c_2^2 \sin (2\phi_1) + \rho_1 c_1^2 \sin (2\phi_2)} \qquad (11.1.20)$$

The corresponding transmission coefficient is $T = 1 + R$.

Generally, in discussions of refraction, one defines the angle of incidence and refraction as $\phi_1' = 90 - \phi_1$ and $\phi_2' = 90 - \phi_2$. The form of R is not altered by introducing these angles, but in problems involving flow it is more convenient to use the angles ϕ_1 and ϕ_2. For normal incidence, $\phi_1 = \phi_2 = 90°$, this relation reduces to the familiar result $(\rho_2 c_2 - \rho_1 c_1)/(\rho_2 c_2 + \rho_1 c_1)$ because it follows from (11.1.14) that, for normal incidence, we have $\sin (2\phi_1)/\sin (2\phi_2) = c_1/c_2$. In Fig. 11.6 we have plotted the reflection coefficient for the special case $\rho_1 = \rho_2$, $c_1 = c_2$, $V_2 = 0$, with $V_1 = c_1/2$ and c_1.

The corresponding relation between the angles ϕ_2 and ϕ_1 is also given. There are three angles at which there is no reflection from the interface. These angles are determined by $\sin(2\phi_1) = \sin(2\phi_2)$. With $\phi_1 = \phi_2$, from Eq. (11.1.17), we get $\phi_1 = 90°$. The two remaining angles, corresponding to $\phi_2 = 90 - \phi_1$ and $\phi_2 = 270 - \phi_1$, are determined from

$$\pm\cot\phi_1 = 1 + M_1\cos\phi_1$$

One of these angles is less than 90°; the other, larger than 90°.

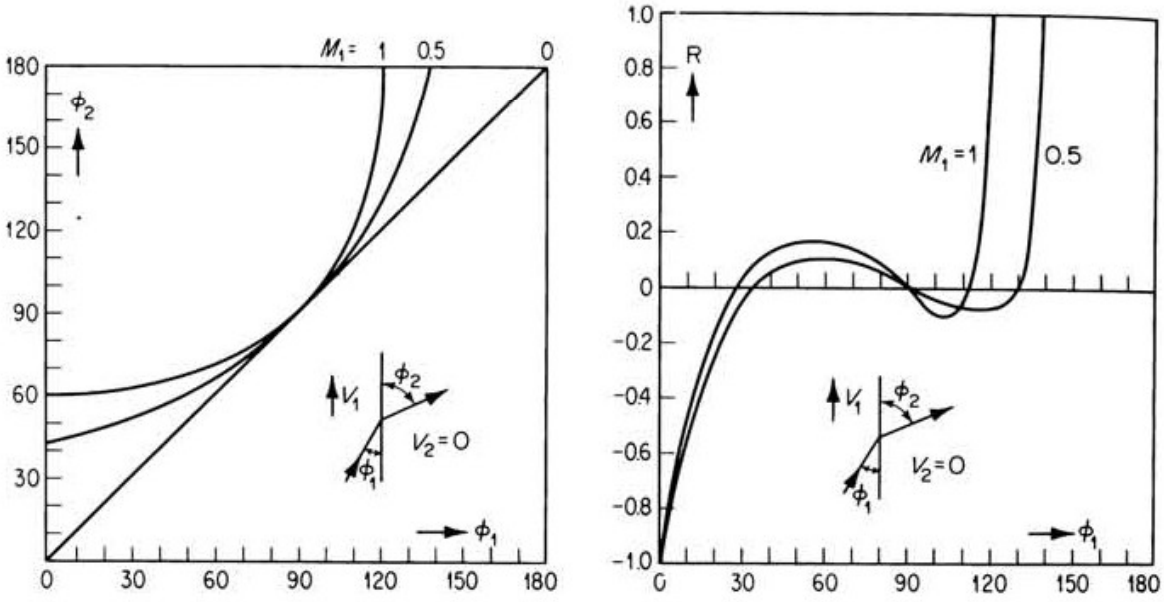


FIGURE 11.6
Angle of refraction and reflection coefficient for a plane wave going from a moving fluid to one at rest.

Reflection from a surface

The reflection coefficient of Eq. (11.1.20) may be derived in another way. According to Eq. (11.1.12), the impedance of medium 2 to being driven by medium 1 is

$$z_2 = \frac{\rho_2 c_2}{(1 + M_2\cos\phi_2)\sin\phi_2}$$

Suppose we simply replace medium 2 by an equivalent impedance and calculate the reflection coefficient by requiring that the ratio between p_1 and $\partial\xi_1/\partial t = -i\omega\xi_1$ be z_2 at every point of the surface $x = 0$. Inserting the expressions for p_1 and ξ_1 produces

$$1 + R = \frac{z_2}{\rho_1 c_1}(1 + M_1\cos\phi_1)(\sin\phi_1)(1 - R)$$

$$R = \frac{(z_2/\rho_1 c_1)(1 + M_1\cos\phi_1)\sin\phi_1 - 1}{(z_2/\rho_1 c_1)(1 + M_1\cos\phi_1)\sin\phi_1 + 1} \qquad (11.1.21)$$

where z_2 is the impedance of the boundary surface as it appears to medium 1.

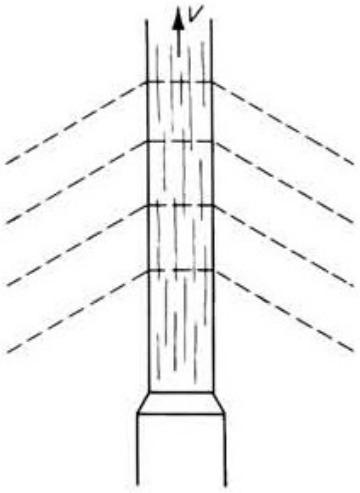


FIGURE 11.7
Sound emitted by a jet is refracted away from the jet at the jet boundary.

If, now, we insert the expression for the z_2 of medium 2, we obtain Eq. (11.1.20), after applying Eqs. (11.1.16) and (11.1.17). We may, however, use Eqs. (11.1.21) for the reflection of a plane wave from any plane surface of effective impedance z_2, when medium 1 is moving past the surface with relative velocity V (where V is in the same direction as the y component of the incident wave). Thus this is the generalization of Eq. (6.3.5) for the reflection coefficient (there called C_r, here called R) when relative motion of the medium is taken into account (we remember that $\cos \vartheta = \sin \phi_1$).

The boundaries between fluid regions in relative motion encountered in practice are of course not so well defined as in the idealized situations treated above. Nevertheless, the results obtained should be useful as a first approximation in more complex problems, such as are encountered, for example, in jets and in atmospheric sound "ducts." Thus, on the basis of the previous analysis, the sound field emitted from sources within a jet is expected to have an acoustic shadow inside a cone in the forward direction with an apex angle $2 \cos^{-1}[c_2/(c_1 + V)]$, where V is the velocity of the jet, and c_1 the speed of sound in the jet. In practice, one indeed finds a region in the forward direction of considerably reduced intensity. To obtain the actual field in this region, it is essential to account for diffraction effects, which of course do not enter when infinite plane boundaries are involved.

Another important problem concerns the propagation of sound in "ducts" formed by velocity and temperature gradients in the atmosphere. Such ducts can trap sound and serve as waveguides for sound over large

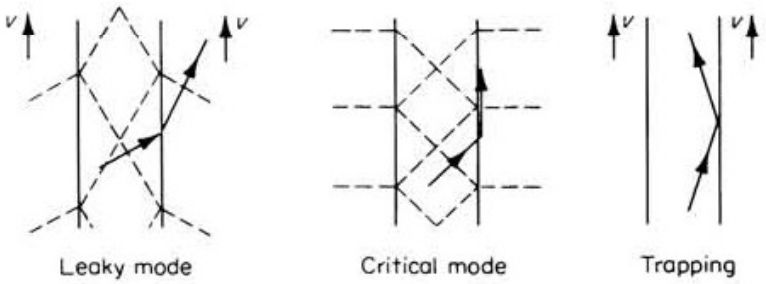


FIGURE 11.8
Acoustic waveguide formed by flow.

distances. Using the results indicated schematically in Fig. 11.4, the qualitative behavior of sound in these two problems of sound emission from a jet and wave trapping in an atmospheric duct is shown schematically in Figs. 11.7 and 11.8.

Ducts

To indicate at least one aspect of the influence of flow on sound transmission through a duct, we shall consider the simple case of propagation between two plane parallel boundaries. Let the boundary at $y = 0$ be rigid, that is, $z = \infty$, whereas the boundary at $y = d$ has a finite impedance $z = \rho c/\beta$.. The fluid between the boundaries moves in the x direction with a velocity V. The solution to the wave equation,

$$\left(\frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x}\right)^2 p = c^2 \nabla^2 p$$

is then of the form

$$p = P_1 \cos (k_y y) e^{ik_x x} e^{-i\omega t} \tag{11.1.22}$$

where

$$k_x^2 + k_y^2 = k^2 \left(\frac{1 - Mk_x}{k}\right)^2$$

Employing the boundary condition for continuity of the particle displacement ξ_y at $y = d$, we have

$$k_y \tan k_y d = -i\beta \left(\frac{1 - Mk_x}{k}\right)^2 k$$

From these equations we obtain the propagation constants k_x and k_y in the x and y directions. For sufficiently small values of the admittance, we can set $\tan (k_y d) \simeq k_y d$ for the fundamental mode, and the corresponding value of k_x for the fundamental mode can be written in the form

$$k_x = k \frac{1}{1 - M^2 A} (\pm A^{\frac{1}{2}} - MA) \tag{11.1.23}$$

where $A = 1 + (i\beta/kd)$. The resulting pressure attenuation of the fundamental mode for propagation in the direction of the flow is then

$$\alpha \simeq \frac{\xi}{2d} \frac{1}{(1 + M)^2} \qquad \beta \ll 1$$

and for propagation in the opposite direction of the flow,

$$\alpha \simeq \frac{\xi}{2d} \frac{1}{(1 - M)^2} \qquad \beta \ll 1$$

where $\xi = \text{Re}\, \beta$. The same result clearly applies to the case where both planes have the impedance $z = \rho c \beta^{-1}$ if they are separated a distance $D = 2d$. When $|\beta| \ll 1$, the phase velocity downstream and upstream is $c + V$ and $c - V$, and is independent of β to the first order in β.

In this connection it is interesting to consider the case of a rigid duct wall in which $\beta = 0$. Then we obtain, for the two values of k_x corresponding to waves in the direction of the flow and against the flow,

$$k_x^+ = \frac{k}{1 + M} \qquad k_x^- = \frac{k}{1 - M}$$

In other words, the wavelength of the wave in the direction of the flow is stretched to the value $\lambda(1 + M)$, and in the opposite direction the wavelength is compressed to $\lambda(1 - M)$, where $\lambda = 2\pi/k = 2\pi c/\omega$ is the wavelength in a fluid at rest. Since the wave speeds in the direction with and against the flow are $c(1 + M)$ and $c(1 - M)$, respectively, it is clear that the frequency recorded by an observer at rest is the same regardless of the direction of travel of the wave. The evaluation of k_x and k_y under more general conditions requires numerical analysis. The solution presented here is a highly idealized one since it implies a laminar uniform flow in the duct and a point-reacting flexible boundary without turbulence and boundary layers.

In this example it is implied that there is no relative motion between the sound source and the observer; both are stationary in the laboratory system. This is implicit in the assumed form of the pressure wave given in Eq. (11.1.22), in which the frequency is given for the laboratory system S, independent of the flow velocity.

A moving plane-wave source

It was implied in the preceding analysis that the sound source was at rest in the duct. We did not concern ourselves further with the source conditions, but merely studied the behavior of the wave once it had been generated. Although we shall devote Sec. 11.2 to the problem of sound generation by moving sources, it is instructive at this point to consider a simple case involving sound generation in a duct by a plane-wave source when motion between the source and the medium is involved. The relative motion is obtained either by moving the fluid in the tube with the source stationary or by moving the source. Both situations are encountered in practice, particularly when the sound is produced by the heating of the neutral gas in a portion of the duct.

An interesting heating mechanism is encountered in an ordinary electric gas-discharge tube. In such a tube the electrons are kept at a considerably higher temperature than the neutral atoms by the action of the electric field in the tube, and the neutral gas is heated through the encounters with the electrons. If the electric field is time-dependent, the heating of the neutral gas will be time-dependent also, and as a result sound will be produced. Thus, if the gas in the tube is made to flow through the discharge region, we obtain a situation in which the source is stationary and the gas is moving. In a discharge tube, however, the reverse situation is more common. More often than not, there exist, in a gas discharge, traveling-wave disturbances,

known as traveling striations, which represent perturbations of electron density and electron temperature, moving with a certain speed, generally from the anode to the cathode, in the discharge tube. With reference to the neutral gas, these traveling waves represent a periodic source distribution with a frequency $\omega/2\pi = V/\lambda$, where V is the speed of the traveling wave and λ its wavelength. The phase of the source varies along the tube in accordance with the time of travel of the source, and is expressed by the factor $\exp(i\omega x/V)$.

There are many similar situations in which the sound is produced by traveling disturbances, but we limit the discussion here to the one-dimensional problem in which the source function is a uniform monopole distribution traveling with the speed V in the x direction.

$$Q(x,t) = Q \exp\left[i\left(\frac{\omega}{V}\right)x - i\omega t\right] \tag{11.1.24}$$

The amplitude Q of the source will be assumed constant in the regions $-L < x < L$ and zero outside, and the tube will be chosen long enough so that reflected sound can be ignored. The wave equation $c^{-2}\,\partial^2 p/\partial t^2 - \partial^2 p/\partial x^2 = \partial Q/\partial t$, in the case of harmonic time dependence, reduces to $\partial^2 p/\partial x^2 + (\omega/c)^2 p = i\omega Q(x,\omega)$, where Q equals ρ times the s of Eq. (7.1.23), being the mass, rather than the volume generation. We have seen in Eq. (4.4.11) that the Green's function $G(x,x_0,\omega)$ satisfying the equation $\partial^2 G/\partial x^2 + (\omega/c)^2 G = -\delta(x - x_0)$ is $G(x,x_0,\omega) = (i/2k)\exp(ik\,|x - x_0|)$, where $k = \omega/c$. With the source function (11.1.24), the acoustic pressure field in the present case is

$$p(x,\omega) = \frac{-i}{2k}\,Q\int_{-L}^{L}(i\omega)e^{i\omega x_0/V}e^{ik|x-x_0|}\,dx_0$$

In the region $-L > x > L$, outside the source, where we have only outgoing waves, we get

$$p(x,\omega) = QcL\,\frac{\sin\{kL[1 \pm (1/M)]\}}{kL[1 \pm (1/M)]}\,e^{ikx - i\omega t} \tag{11.1.25}$$

where $M = V/c$. The minus sign refers to the region $x > L$ and represents the forward radiated sound from the traveling source. We note that the pressure amplitude contributed per unit length of the source is a maximum when $V = c$, in which case the sound-pressure amplitude in the forward direction is QcL. Under these conditions the sound from all regions of the source arrives in phase at the point of observation. Such constructive interference cannot occur in the backward direction, for $x < -L$.

The sound pressure field inside the source region is made up of waves traveling in both positive and negative x directions. We leave the calculation of this field as one of the problems.

11.2 SOUND EMISSION FROM MOVING SOURCES

The main object of the preceding section was to demonstrate the principal effects of fluid motion on sound transmission, and for this purpose only plane waves of sound were considered. The sound field from an actual sound source, however, is ordinarily not a plane wave; we must therefore inquire about the influence of flow on the sound field from such a source. The problem is not simple. For one thing, the flow field about a source of finite dimensions is generally turbulent, and the interaction of sound with this flow field must be accounted for in a detailed analysis. Furthermore, the corresponding static-pressure distribution and the turbulent-pressure fluctuations on the source in motion may influence the operation of the source considerably. To illustrate this point, we mention a design of a train whistle, which was found to operate perfectly well when tested in the laboratory, but which ceased to operate at all on the train at speeds exceeding 50 mph.

We shall not attempt an explanation of this and similar peculiar effects in moving media; we wish to study more general properties. Even then we have to idealize our problem by considering the radiation from point sources in relative motion with the medium. This avoids the question of the flow distortion about the source and the question of turbulence. In applying the results obtained to an actual source, however, we clearly have to be cautious, in particular at high frequencies, when the wavelength is of the order of the source dimensions.

Practical problems, where the influence of relative motion between the source and the medium is of interest, are found in numerous applications, particularly in aviation acoustics, and in treating some of these problems later in this chapter it is useful to be familiar with the idealized situations to be discussed in this section.

We start by considering the sound radiation from an acoustic point monopole source that moves along a trajectory in a fluid at rest. Our discussion is therefore an extension of the analysis in Chap. 7, to include relative motion between the source and the surrounding medium.

Kinematics

Before turning to a calculation of the sound pressure field, it is instructive to establish some purely kinematic properties of the problem, without using the dynamical equations of motion.

We consider a sound source moving along a path specified by the position vector $\mathbf{r}_s(t)$ with the components $x_s(t)$, $y_s(t)$, and $z_s(t)$. The point of observation O is at the position $\mathbf{r}$ with the coordinates x, y, z at rest with respect to the medium. The sound pressure observed at $\mathbf{r}$ at the time t was emitted by the source at a time $t_e = t - (R/c)$, at which time the source was

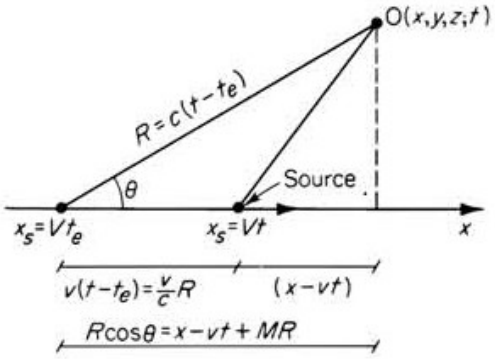


FIGURE 11.9
Source in uniform subsonic motion.

at $\mathbf{r}_s(t_e)$, the emission point E. The distance $\mathbf{R} = \mathbf{r} - \mathbf{r}_s(t_e)$ between E and the point of observation O is determined by the equation

$$R^2 = \left[x - x_s\left(t - \frac{R}{c}\right)\right]^2 + \left[y - y_s\left(t - \frac{R}{c}\right)\right]^2 + \left[z - z_s\left(t - \frac{R}{c}\right)\right]^2 \tag{11.2.1}$$

To solve this equation for R, of course, requires specification of the path $\mathbf{r}_s(t)$. Depending on circumstances, it is possible that more than one value of R is found to satisfy this equation. For example, in the case of supersonic motion of the source along a straight line with constant speed, we shall find that there are two emission points and corresponding emission times which will produce simultaneous contributions to the sound field at the time t at the point of observation.

We shall start with the simplest possible situation, illustrated in Fig. 11.9, in which the source moves along a straight line, the x axis, with a constant speed V smaller than the speed of sound, so that the Mach number $M = V/c < 1$. Then, if we let the source pass the origin of the coordinate system at the time $t = 0$, we have $x_s(t) = Vt$ and $y_s = z_s = 0$. The equation for R in (11.2.1) then reduces to

$$R^2 = \left[x - V\left(t - \frac{R}{c}\right)\right]^2 + r^2 = [(x - Vt) + MR]^2 + r^2 \tag{11.2.2}$$

where $r^2 = y^2 + z^2$. This equation is satisfied by

$$R = \frac{M(x - Vt) \pm \sqrt{(x - Vt)^2 + (1 - M^2)r^2}}{1 - M^2} \tag{11.2.3}$$

If $M < 1$, it is clear from (11.2.3) that only the positive sign will give a positive value for R. If we introduce

$$R_1 = \sqrt{(x - Vt)^2 + (1 - M^2)r^2} \tag{11.2.4}$$

we have

$$R = \frac{M(x - Vt) + R_1}{1 - M^2} \tag{11.2.5}$$

If we denote by θ the angle between the direction of motion and the direction of **R** as indicated in Fig. 11.9, we have $x - x_e = x - Vt_e = R\cos\theta$, and with $R = c(t - t_e)$, we get $M(x - Vt) = M[x - Vt_e - V(t - t_e)] = MR(\cos\theta - M)$. Therefore it follows from (11.2.5) that

$$R_1 = R(1 - M\cos\theta) \tag{11.2.6}$$

a result which will be used frequently later.

If $M = V/c > 1$, that is, when the speed of the source is supersonic, the plus and the minus signs in (11.2.3) are both acceptable, yielding positive values of R provided that $x - Vt < 0$, that is, when the point of observation is behind the source. We can then write

$$R_\pm = \frac{M(Vt - x) \pm (Vt - x)\sqrt{1 - (M^2 - 1)r^2/(Vt - x)^2}}{M^2 - 1}$$

If we introduce $r/(Vt - x) = \tan\theta_2$ and $\sin\theta_1 = 1/M$ as shown in Fig. 11.10, we get

$$R_\pm = \frac{M(Vt - x) \pm (Vt - x)\sqrt{1 - (\tan\theta_2/\tan\theta_1)^2}}{M^2 - 1} \tag{11.2.7}$$

In other words, R_+ is real only when $\theta_2 < \theta_1$, that is, inside a cone with the vertex at the source and the vertex angle $2\theta_1$. If we consider a point of observation inside the cone as in Fig. 11.10, we can construct the points of emission E_+ and E_- corresponding to R_+ and R_- which contribute to the sound pressure at the point of observation O at a time t. This point O must

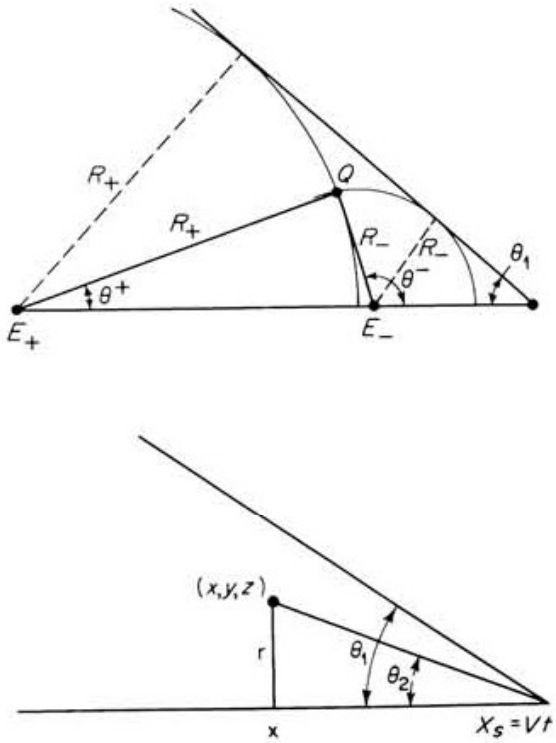


FIGURE 11.10
Kinematics of the wavefield from a point source in uniform supersonic motion.

lie on the intersection between two spherical phase surfaces which are both tangent to the cone and have their centers on the x axis. These centers are the emission points E_+ and E_- that correspond to O. The construction is illustrated in Fig. 11.10.

In the special case when the point of observation O is on the x axis, the distinction between the wave kinematics for subsonic and supersonic sources can be visualized in a simple manner in a space-time diagram as shown in Fig. 11.11. In the subsonic case a point O will be reached at a certain time

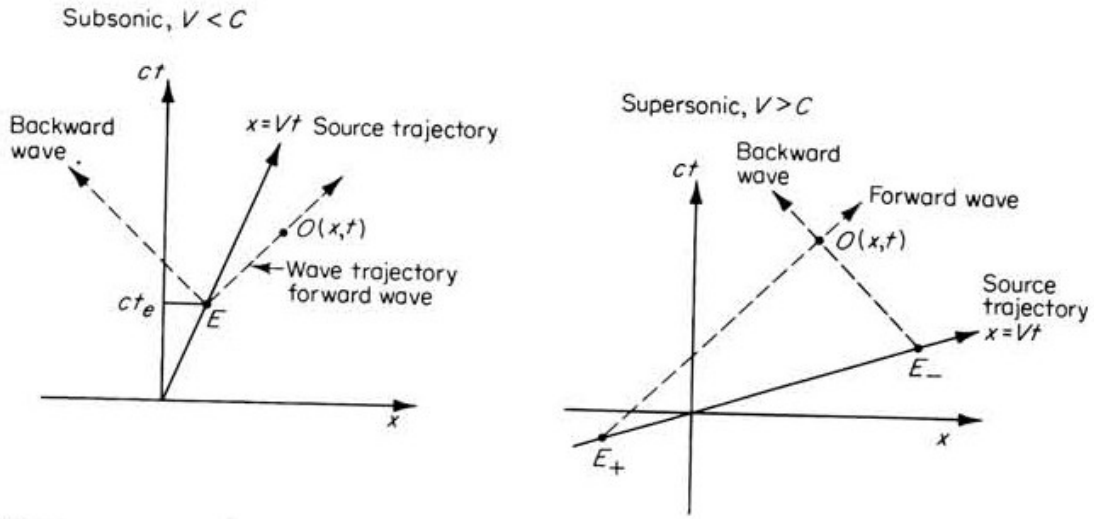


FIGURE 11.11
Space-time diagrams for the wave kinematics of sources in uniform subsonic and supersonic motion.

t only by the forward-going wave from the source corresponding to the emission point E. In the supersonic case, however, there is one forward- and one backward-going wave, reaching O simultaneously, although emitted at different times, corresponding to E_+ and E_-.

Figure 11.12 also gives a hint as to the intensity distribution of the sound. A point O_1 in front of the source receives the energy emitted during the time Δt in the time interval Δt_1 which is smaller than the time interval corresponding to a point of observation O_2 behind the source. In other words, we expect a source which has a symmetric radiation pattern at rest to focus its radiation in the direction of motion. This will be analyzed quantitatively a little later.

By examining the phase surfaces that correspond to a succession of emission points from the moving source, as illustrated in Fig. 11.9, we can easily obtain the Doppler shift noted by an observer at rest. In the forward direction the distance between phase surfaces emitted by the source with a time difference T is $cT - V_sT = T(c - V_s)$. Since the phase surfaces travel with a speed c relative to the observer at rest, he observes a time interval between these phase surfaces $T(c - V_s)/c = T[1 - (V_s/c)]$. The

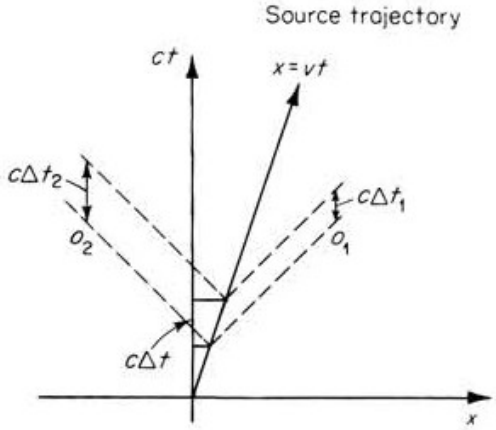


FIGURE 11.12
Relation between emission and reception time intervals for points of observation in front of and behind a moving source.

corresponding observed frequency is then larger than the source frequency by a factor $(1 - M)^{-1}$, where $M = V_s/c$. For an observer at a point corresponding to an angle θ between V and r measured from the emission point, we find that the observed frequency is larger than the source frequency by a factor $(1 - M\cos\theta)^{-1}$. We shall return for a derivation and detailed discussion of this Doppler-shift formula.

The sound pressure field; subsonic motion

Turning now to the *dynamics* of the problem, we consider a monopole point source of sound moving with uniform velocity V with respect to the surrounding fluid. We recall that a monopole point source can be thought of in terms of a small pulsating sphere, the strength of which is specified by q, which represents the total rate of mass flux out of the source (thus q equals ρ times the volume flux S of Sec. 7.1). The source-distribution density can thus be expressed as $Q(\mathbf{r},t) = q(t)\delta(x - Vt)\delta(y)\delta(z)$. We can imagine this point source monopole being produced, for example, by the heating and expansion caused by some modulated radiation focused at a point that is moved through the fluid, or else by the interaction between neutral atoms and electrons, mentioned earlier.

In any event, the wave equation for the sound pressure field, produced by the moving source distribution, is of the form

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\partial}{\partial t} q(t)\,\delta(x - Vt)\,\delta(y)\,\delta(z)$$

Using a velocity potential ψ equaling ρ times the Ψ of Eq. (6.2.9), we have

$$p = \frac{\partial \psi}{\partial t}$$

and we get

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -q(t)\,\delta(x - Vt)\,\delta(y)\,\delta(z) \qquad (11.2.8)$$

This equation can be solved in many different ways. We shall here use a method based simply on a coordinate transformation which enables us to reduce the problem to that of radiation from a stationary source. Later in this chapter we obtain the solution using Fourier transforms.

If we perform a Galilean transformation of coordinates, we reduce the right-hand side of the equation to that corresponding to a stationary source, but then the left-hand side becomes a rather complicated differential form, as we have seen already, in Sec. 11.1. On the other hand, the left-hand side remains unchanged in form if, instead, we use a linear transformation of coordinates analogous to a Lorentz transformation. The transformed right-hand side of the equation hopefully can be written in the form of a stationary source. Thus we attempt the following transformation of coordinates:

$$x' = \gamma(x - Vt) \qquad y' = y \qquad z' = z$$
$$t' = \gamma\left(t - \frac{V}{c^2}x\right) \tag{11.2.9}$$

where $\gamma = \dfrac{1}{\sqrt{1 - M^2}}$ $\qquad M = \dfrac{V}{c}$ $\qquad c =$ speed of sound

If we denote by a prime differential operations with respect to the x_i' system, we obtain

$$\nabla'^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} = -q\left[\gamma\left(t' + \frac{x'V}{c^2}\right)\right]\delta\left(\frac{x'}{\gamma}\right)\delta(y')\,\delta(z')$$

Since $\delta(x'/\gamma) = \gamma\,\delta(x')$, this becomes

$$\nabla'^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} = -\gamma q(\gamma t')\,\delta(x')\,\delta(y')\,\delta(z')$$

To make the equivalence with a stationary source problem complete, we proceed to introduce $t'' = \gamma t'$, $x'' = \gamma x'$, $y'' = \gamma y'$, and $z'' = \gamma z'$, in which case $\nabla'' = \gamma^2\nabla'$, and the new equation becomes

$$\nabla''^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t''^2} = -\gamma^2 q(t'')\,\delta(y'')\,\delta(z'')\,\delta(x'') \tag{11.2.10}$$

The equation is now completely analogous to one representing the radiation from a stationary source with the strength $\gamma^2 q(t'')$. Using the results already obtained in Sec. 7.1, the solution can be written directly in terms of the coordinates (r'',t'').

$$\psi(r'',t'') = \gamma^2\frac{q[t'' \mp (r''/c)]}{4\pi r''} \tag{11.2.11}$$

Transforming back to the initial variables x, t, we obtain

$$t'' \mp \frac{r''}{c} = \gamma t' \mp \frac{\gamma r'}{c} = t - \frac{V(x - Vt)\gamma^2}{c^2} \mp \frac{\gamma}{c}\sqrt{(x - Vt^2)\gamma^2 + y^2 + z^2}$$

$$= t - \frac{M(x - Vt) \pm \sqrt{(x - Vt)^2 + (y^2 + z^2)(1 - M^2)}}{c(1 - M^2)} = t - \frac{R}{c}$$

where

$$R = \frac{M(x - Vt) \pm \sqrt{(x - Vt)^2 + (1 - \beta^2)(y^2 + z^2)}}{1 - M^2}$$

$$= \frac{M(x - Vt) \pm R_1}{1 - M^2} \tag{11.2.12}$$

From the discussion of the kinematics of the problem we recognize R as the distance between the emission point and the point of observation. The minus sign in (11.2.12), which corresponds to the plus sign in (11.2.11), is acceptable only when we have supersonic motion of the source, as discussed earlier.

In the subsonic case, only the plus sign can be used in (11.2.12). The corresponding value of R is labeled R^+, and the subsonic solution for ψ is then

$$\psi(\mathbf{r},t) = \frac{q[t - (R^+/c)]}{4\pi R_1} \tag{11.2.13}$$

where we have used $r'' = \gamma^2 R_1$.

$$R_1 = \sqrt{(x - Vt)^2 + (1 - M^2)(y^2 + z^2)}$$

Using the expression $R_1 = R(1 - M\cos\theta)$ given in (11.2.6), we can express this solution as

$$\psi(\mathbf{r},t) = \frac{q[t - (R/c)]}{4\pi R(1 - M\cos\theta)}$$

where, from Eq. (11.2.6) and Fig. 11.9, θ is the angle between the x axis and $\mathbf{R}$. The corresponding sound pressure field, as we recall from (11.2.8), is $p = \partial\psi/\partial t$. Thus

$$p = \frac{\left(1 - \frac{1}{c}\frac{dR}{dt}\right) q'\left(t - \frac{R}{c}\right)}{4\pi R_1} - \frac{q\left(t - \frac{R}{c}\right)}{4\pi R_1^2}\frac{dR_1}{dt}$$

With R given in (11.2.12), we obtain

$$\frac{1}{c}\frac{dR}{dt} = -\frac{1}{1 - M^2}\left[M^2 + \frac{M(x - Vt)}{R_1}\right]$$

$$= -\frac{M}{1 - M^2}\left[M + \frac{R(\cos\theta - M)}{R(1 - M\cos\theta)}\right] = -\frac{M\cos\theta}{1 - M\cos\theta} \tag{11.2.14}$$

where we have used $x - Vt = x - Vt_e - V(t - t_e) = R\cos\theta - MR$ and $R = c(t - t_e)$. Hence

$$p = \frac{1}{4\pi}\frac{q'[t - (R/c)]}{R(1 - M\cos\theta)^2} + \frac{q}{4\pi}\frac{(\cos\theta - M)V}{R^2(1 - M\cos\theta)^2} \qquad (11.2.15)$$

The first of the two terms in (11.2.15) represents the radiation field with the corresponding pressure decreasing as $1/R$. We note that when $\theta = 90°$, the pressure in the far field is the same as when the source is at rest. As expected from the qualitative considerations in connection with Fig. 11.12, we now see that the pressure in the forward direction ($\theta = 0$) is larger than the pressure in the backward direction ($\theta = \pi$) by a factor $(1 + M)^2/(1 - M)^2$.

Simple-harmonic source

We now consider harmonic time dependence so that $q(t) = q_0 \sin(\omega_0 t)$. The pressure field, which contains the factor $q_0\omega_0 \cos[\omega_0(t - R/c)]$, then has the phase

$$\phi = \omega_0\left(t - \frac{R}{c}\right)$$

which no longer is simply proportional to time since R now is time-dependent.

If the point of observation is on the line of motion of the source in front of the source, so that $x - Vt > 0$, we have $R = x - Vt$, and

$$\phi = \frac{\omega_0}{1 - M}t - \frac{\omega_0}{c(1 - M)}x$$

The observed frequency is then

$$\omega_1 = \frac{\omega_0}{1 - M}$$

and the corresponding wavelength obtained from the spatial dependence of the phase is $(1 - M)\lambda_0$.

Similarly, if the point of observation is behind the source, so that $x - Vt < 0$, we obtain in a similar manner the received frequency and wavelength

$$\omega_2 = \frac{\omega_0}{1 + M} \qquad \text{and} \qquad \lambda_2 = (1 + M)\lambda_0$$

When the point of observation is not on the line of motion, the frequency clearly will vary continuously as a function of time between the limits $\omega_0/(1 - M)$ and $\omega_0/(1 + M)$, which then correspond to $t = -\infty$ and $t = +\infty$. Actually, the concept of frequency is not so well defined, when, as in our case, the phase no longer is proportional to time. However, if we generalize the

concept of frequency and define it as the time derivative of the phase, we obtain

$$\omega = \frac{d\phi}{dt} = \omega_0\left(1 - \frac{1}{c}\frac{dR}{dt}\right) = \omega_0\left(1 + \frac{M\cos\theta}{1 - M\cos\theta}\right) = \frac{\omega_0}{1 - M\cos\theta} \tag{11.2.16}$$

where we have used the expression for $(1/c)(dR/dt)$ in Eq. (11.2.14). This is

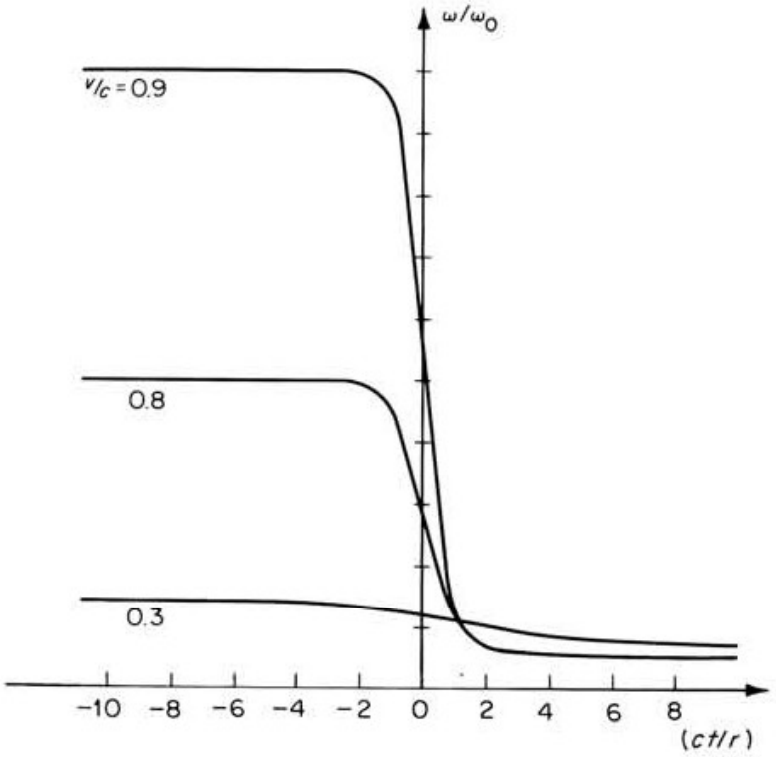


FIGURE 11.13
Time dependence of frequency $\omega/2\pi$ at point of observation located a distance r from the line of motion of the source. Frequency of source $= \omega_0/2\pi$, speed $= V$.

the general Doppler formula. As the source moves past an observer, the angle θ varies between $\theta = \pi$ and $\theta = 0$, and the frequency varies between the values $\omega_0/(1 - M)$ and $\omega_0/(1 + M)$.

Thus we have seen that both the pressure amplitude and the frequency can be expressed in a very simple manner in terms of the angle θ as measured with respect to the source position at the retarded time $t - (R/c)$. If, instead, we wish to express these quantities explicitly in terms of the time t, we have to introduce

$$(1 - M\cos\theta)^{-1} = \frac{R}{R_1} = (1 - M^2)^{-1}\left[1 - \frac{M\tau}{\sqrt{(1 - M^2) + \tau^2}}\right]$$

where $\tau = t/(r/V)$. R_1 and R are given in (11.2.12) and (11.2.13). In Figs. 11.13 and 11.14 we have shown the time dependence of the frequency and the amplitude at the position $x = 0$ a distance r from the line of motion of the source. We note that the main changes in the frequency and the amplitude both take place in a time of the order of r/V.

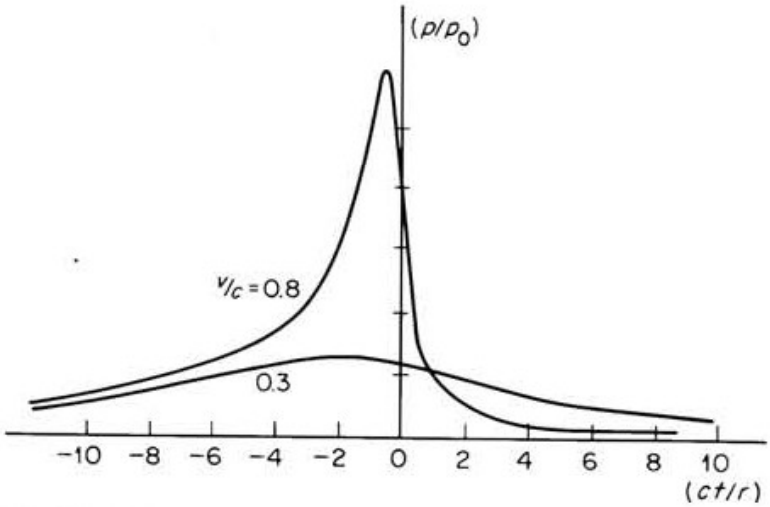


FIGURE 11.14
Time dependence of the pressure amplitude at the point of observation described in Fig. 11.13. The reference pressure p_0 is the pressure that would be observed when $V = 0$.

Supersonic motion

The sound pressure field in the case of supersonic motion is obtained by adding the contributions from the two source points E_+ and E_-.

$$p = \frac{\partial}{\partial t}\left\{\frac{q[t - (R^+/c)]}{4\pi R_1} + \frac{q[t - (R^-/c)]}{4\pi R_1}\right\} \tag{11.2.17}$$

where

$$R^{\pm} = \frac{M(Vt - x) \pm R_1}{M^2 - 1}$$
$$R_1 = \sqrt{(Vt - x)^2 - (M^2 - 1)r^2} \tag{11.2.18}$$

Introducing $Vt - x = V(t - t_e) - (x - Vt_e) = MR - R\cos\theta$, we obtain

$$R_1 = R^+(M\cos\theta^+ - 1) = -R^-(M\cos\theta^- - 1)$$

where the θ^+ and θ^- indicate the directions of R^+ and R^- from the emission points E^+ and E^- to the point of observation, as previously indicated in

Fig. 11.10. The time dependence of the two angles is given by

$$(M\cos\theta^+ - 1)^{-1} = \frac{R^+}{R_1} = \frac{1}{M^2-1}\frac{M\tau}{\sqrt{\tau^2-(M^2-1)}} + 1$$

$$(M\cos\theta^- - 1)^{-1} = -\frac{R^-}{R_1} = -\frac{1}{M^2-1}\frac{M\tau}{\sqrt{\tau^2-(M^2-1)}} - 1$$

where $\tau = (Vt - x)/r$.

With

$$\frac{1}{c}\frac{dR^\pm}{dt} = \frac{M\cos\theta^\pm}{M\cos\theta^\pm - 1}$$

and considering (11.2.18), the pressure field can be expressed as

$$p = \frac{-q'[t-(R^+/c)]}{4\pi R^+(M\cos\theta^+ - 1)^2} + \frac{q'[t-(R^-/c)]}{4\pi R^-(M\cos\theta^- - 1)^2}$$

$$- \frac{q[t-(R^+/c)]}{4\pi(R^+)^2}\frac{(M-\cos\theta^+)V}{(M\cos\theta^+ - 1)^3} + \frac{q[t-(R^-/c)](M-\cos\theta^-)}{4\pi(R^-)^2(M\cos\theta^- - 1)^3} \quad (11.2.19)$$

If the point of observation is on the x axis, we have, simply, $R^+ = (Vt - x)/(M - 1)$ and $R^- = (Vt - x)/(M + 1)$. Then, if the source has harmonic time dependence, so that $q'(t - R^+/c) = q_0\omega\cos\omega_0[t - (R^+/c)]$, we see that the phase becomes

$$\phi(t) = \omega_0\left[t - \frac{Vt - x}{c(M-1)}\right] = -\frac{\omega_0}{M-1}t + \frac{\omega_0 x}{c(M-1)}$$

This phase corresponds to the signal received from the emission point E^+, in which case the source moves toward the point of observation. We note that the frequency formally is negative, which may be interpreted as a result of the fact that the phase surfaces which were emitted last arrive first. For the sound that arrives from the emission point E^-, the frequency has the same form as in the subsonic case, namely, $\omega_0/(1 + M)$. For any arbitrary position, this frequency is $-\omega_0/(M\cos\theta - 1)$.

An interesting feature of the supersonic field is the interference between the two contributions from the two emission points E^+ and E^-. These waves, which have different frequencies and amplitudes, give rise to a sound-pressure amplitude which oscillates with time. We leave it for one of the problems to study this interference field.

Power spectrum

The calculation of the frequency spectrum of the radiated field from a moving source can be carried out directly by Fourier transform of the pressure field in Eq. (11.2.15). However, it is instructive to start directly from the wave equation,

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\partial}{\partial t}[q(t)\,\delta(x - Vt)\,\delta(y)\,\delta(z)]$$

and take its Fourier transform. Thus we multiply by $e^{i\omega t}$ and integrate over time. Rewriting the delta function as

$$\delta(x - Vt) = \delta\left[V\left(t - \frac{x}{V}\right)\right] = \frac{1}{V}\,\delta\left(t - \frac{x}{V}\right)$$

the Fourier transform of the right-hand side gives

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\frac{\partial}{\partial t}[q(t)\,\delta(x - Vt)]\,dt = -\frac{i\omega}{2\pi V}\,q\left(\frac{x}{V}\right)e^{i\omega(x/V)}$$

Similarly, the Fourier transform of the left-hand side of the equation is $\nabla^2 p(\mathbf{r},\omega) + (\omega/c)^2 p(\mathbf{r},\omega)$, where $p(\mathbf{r},\omega)$ is the Fourier transform of the pressure. We shall specialize further and consider harmonic time dependence of the source so that

$$q(t) = q_0 \cos(\omega_0 t) = \tfrac{1}{2}q_0[\exp(-i\omega_0 t) + \exp(i\omega_0 t)] \qquad (11.2.20)$$

Consider, first, the contribution to the pressure field from $\exp(-i\omega_0 t)$ and denote this contribution by p^+. Then

$$\nabla^2 p^+ + \left(\frac{\omega}{c}\right)^2 p^+ = \frac{i\omega}{4\pi V}\,q_0 e^{i(\omega-\omega_0)(x/V)}\,\delta(y)\,\delta(z) \qquad (11.2.21)$$

Since the radiation field is symmetrical about the line of motion, the sound field will be of the form

$$p^+ = f(r)\exp\frac{i(\omega - \omega_0)x}{V}$$

where $r = \sqrt{y^2 + z^2}$. Inserting this expression into the wave equation, we obtain

$$\nabla_{y,z}^2 f(r) + \left[\left(\frac{\omega}{c}\right)^2 - \frac{(\omega - \omega_0)^2}{V^2}\right]f(r) = \frac{i\omega}{4\pi V}\,q_0\,\delta(y)\,\delta(z) \qquad (11.2.22)$$

The free-space Green's function $g(\mathbf{r},\omega)$ to this Helmholtz equation in two dimensions is a solution to

$$\nabla_{x,y}^2 g(\mathbf{r},\omega) + k^2 g(\mathbf{r},\omega) = -\delta(y)\,\delta(z)$$

and as we have found previously [Eq. (7.3.17)],

$$g(\mathbf{r},\omega) = \frac{i}{4} H_0^{(1)}(kr)$$

In this case $$k^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega - \omega_0}{V}\right)^2$$

Since the field contributions from $e^{-i\omega_0 t}$ and $e^{i\omega_0 t}$ in the source term merely involved changes in the sign of ω_0, it is convenient to introduce the notation k^+ and k^-, where

$$(k^{\pm})^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega \mp \omega_0}{V}\right)^2 \tag{11.2.23}$$

The solution to Eq. (11.2.22), then, is

$$f(r) = \frac{\omega q_0}{16\pi V} H_0^{(1)}(k^+ r)$$

and the pressure field is

$$p^+ = \frac{\omega q_0}{16\pi V} H_0^{(1)}(k^+ r) e^{i(\omega-\omega_0)x/V}$$

Inverting the Fourier transform, we find the space-time dependence of the sound pressure field p^+.

$$p^+ = \frac{q_0}{16\pi V} \int_{-\infty}^{+\infty} \omega H_0^{(1)}(k^+ r) e^{i(\omega-\omega_0)x/V} e^{-i\omega t}\, d\omega \tag{11.2.24}$$

The corresponding radial component of the particle velocity field obtained from $\rho(\partial u_r/\partial t) = -\partial p/\partial r$ is then

$$u_r^+ = -\frac{q_0}{16\pi V(i\rho)} \int_{-\infty}^{+\infty} k^+ H_1^{(1)}(k^+ r) e^{i(\omega-\omega_0)x/V} e^{-i\omega t}\, d\omega \tag{11.2.25}$$

where we have used $dH_0^{(1)}(z)/dz = -H_1^{(1)}(z)$.

The contributions p^- and u_r^- produced by the second term, $\exp(i\omega_0 t)$, in the source term (11.2.20) are obtained simply by changing the sign of ω_0 in Eqs. (11.2.24) and (11.2.25), and the total sound pressure field and the total radial particle velocity are then

$$p = p^+ + p^- \qquad u_r = u_r^+ + u_r^-$$

Before investigating these fields further, we shall study the frequency dependence of the propagation constants $k^{\pm}$ as given by (11.2.23). We have

$$(k^+)^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega - \omega_0}{V}\right)^2 = \frac{M^2 - 1}{V^2}\left(\omega - \frac{\omega_0}{1 - M}\right)\left(\omega - \frac{\omega_0}{1 + M}\right)$$

where $M = V/c$. By considering the signs of the two factors in this equation,

we see that, for $M < 1$, the propagation constant k^+ will be real only in the range

$$\frac{\omega_0}{1+M} < \omega < \frac{\omega_0}{1-M} \tag{11.2.26}$$

Similarly, for $M > 1$, there will be *two* intervals,

$$-\infty < \omega < -\frac{\omega_0}{M-1} \qquad \frac{\omega_0}{M+1} < \omega < \infty \tag{11.2.27}$$

where k^+ is real. The corresponding ranges for k^- are obtained by changing the sign of ω_0.

This behavior is consistent with our previous observations about the Doppler shift. In the subsonic case, $M < 1$, the observed frequency was found to be $\omega_0/(1 - M\cos\theta)$, which has a maximum value $\omega_0/(1 - M)$ when $\theta = 0$ (emission point at $x = -\infty$) and a minimum value $\omega_0/(1 + M)$ when $\theta = \pi$ (emission point at $x = +\infty$). Similarly, in the supersonic case, the frequency is $-\omega_0/(M\cos\theta - 1)$, which gives $-\omega_0/(M - 1)$ when $\theta = 0$ and goes to $-\infty$ when $\cos\theta = 1/M$. Then ω switches to $+\infty$ as θ passes through the value $\theta = \cos^{-1}(1/M)$, and approaches $\omega_0/(1 + M)$ when $\theta \to 0$ $(x \to \infty)$. As discussed earlier, the negative sign of the frequency in one of these regions expresses the fact that the signals that were emitted last from the source arrive first at the point of observation. In the frequency intervals in which $(k^+)^2$ is negative, k^+ is imaginary, and the wave functions $H_0^{(1)}(k^+r)$ and $H_1^{(1)}(k^+r)$ will decay exponentially away from the line of motion of the source. Thus these frequencies will not contribute to the sound field in the far-field region, and the corresponding frequency intervals can be excluded in the far field.

The acoustic power emitted by the moving source will be evaluated by integrating the radial component of the acoustic energy flux over the surface of an infinitely long cylinder enclosing the line of motion of the source. Actually, we shall determine the average power over one period, $2\pi/\omega_0$, of the source. Thus, the radius of the cylinder being r, the acoustic power Π is expressed as

$$\Pi = \int_{-\infty}^{+\infty} 2\pi r\overline{pu_r}\,dx = \int_{-\infty}^{+\infty} 2\pi r\overline{(p^+ + p^-)(u_r^+ + u_r^-)}\,dx \tag{11.2.28}$$

where the bar indicates the time average over one period, $2\pi/\omega_0$. In evaluating this integral the asymptotic form of the Hankel functions will be used,

$$H_0^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi kr}}\, e^{ikr - \frac{1}{4}\pi i}$$

and $H_1^{(1)} \simeq -iH_0^{(1)}(kr)$.

In the far field only the frequency intervals in which k^- and k^+ are real will contribute to the pressure and velocity fields. To indicate this restriction in the range of integration, we introduce the function $S_+(\omega)$, which is unity when k^+ is real and zero elsewhere. Similarly, $S_-(\omega)$ is unity when k^- is real and zero elsewhere. Clearly, $S_-(-\omega) = S_+(\omega)$.

The integrals involved in (11.2.28) are then of the form

$$\int p^+u^- \, dx = C \int_{-\infty}^{+\infty} dx \iint_{-\infty}^{+\infty} d\omega \, d\omega' \, S_+(\omega)S_-(\omega')e^{i(k^+ + k'^-)r} \times \exp\left[(\omega + \omega')\left(\frac{x}{V} - t\right)\right]$$

where $k'^- = k^-(\omega')$, and $C = 4q_0^2/(16\pi V)^2\rho$. Because $S_-(-\omega) = S_+(\omega)$, this integral turns out to be independent of t; thus an average over a period is not zero. A similar integral is obtained for p^-u^+. The contributions from p^+u^+ and p^-u^- will be zero when averaged over one period of the source because they both oscillate with time. The integration over x gives $2\pi V\delta(\omega + \omega')$, and therefore the integration over ω' merely involves replacing ω' by $-\omega$ in the integral. Using $S_-(-\omega) = S_+(\omega)$, the remaining integration over ω, then, is

$$\int p^+u^- \, dx = (2\pi V)C \int_{-\infty}^{+\infty} S_+^2(\omega) \sqrt{\frac{k'^-}{k^+}} \, \omega e^{i(k^+ + k'^-)r} e^{-i\pi/2} \, d\omega$$

On account of $S_+^2(\omega)$, the integration reduces to the interval in which k^+ is real. It follows from (11.2.26) that in the range where k^+ is real (and positive), we have $k'^- = -k^+$ and $\sqrt{k'^-} = i\sqrt{k^+}$. Thus

$$\int p^+u^- \, dx = (2\pi V)C \int_{-\infty}^{+\infty} S_+^2(\omega) \, \omega \, d\omega \tag{11.2.29}$$

It is not difficult to see that $\int p^-u^+ \, dx$ is equal to $\int p^+u^- \, dx$.

Thus, in the subsonic case in which the range of integration is $a = \omega_0/(1 + M) < \omega < \omega_0/(1 - M) = b$, we obtain for the radiated power

$$\Pi = \frac{h_0^2}{16\pi(\rho c)M} \int_a^b \omega \, d\omega = \frac{q_0^2\omega_0^2}{8\pi\rho c} \frac{1}{(1 - M^2)^2} = \frac{\Pi_s}{(1 - M^2)^2} \tag{11.2.30}$$

where

$$\Pi_s = \frac{(q_0\omega_0)^2}{8\pi\rho c} = \frac{(\rho S_0\omega_0)^2}{8\pi\rho c}$$

is the power radiated by a source at rest, which is identical with the result obtained in Eq. (7.1.5) (when we take into account that q_0 and S_0 here are maximum, not rms, values). Consequently, for a point monopole source the relative motion between the source and the fluid causes an increase of the

acoustic power output by a factor $(1 - M^2)^{-2}$. The energy spectrum, defined by $\Pi = \int_0^\infty E_s(\omega)\, d\omega$, is seen to be

$$E_s(\omega) = \frac{{q_0}^2}{16\pi\rho c}\frac{\omega}{M} \qquad \frac{\omega_0}{1+M} < \omega < \frac{\omega_0}{1-M} \qquad M < 1$$

Although the source is simple-harmonic, the sound reaching the observer has a spread of frequency because the source is moving, another way of stating the Doppler effect.

In the supersonic case, $S_+(\omega)$ is different from zero in the two regions given in (11.2.27), and the integral in (11.2.29) becomes

$$\int_{-\infty}^{\beta} \omega\, d\omega + \int_{\alpha}^{\infty} \omega\, d\omega = \frac{{\omega_0}^2}{2}\left[\frac{1}{(M-1)^2} - \frac{1}{(M+1)^2}\right] = \frac{2{\omega_0}^2 M}{(M^2-1)^2}$$

where $\beta = -\omega_0/(M-1)$ and $\alpha = \omega_0/(M+1)$, and the power is

$$\Pi = \Pi_s \frac{1}{(M^2-1)^2} \qquad M > 1$$

Thus we see that, formally, the expressions for the total radiated power in the subsonic and supersonic cases are the same. The energy spectrum differs markedly, however; for $M < 1$, the frequencies are those below $\omega_0/2\pi(1 - M)$, whereas for $M > 1$, they are all those greater than $\omega_0/2\pi(1 + M)$, clear to infinity.

Dipole sources

The ideal point dipole is represented by an oscillating point force acting on a fluid. At sufficiently low frequencies many actual sound sources such as an oscillating piston are approximately equivalent to dipoles. An airplane in "bumpy" flight through turbulent air will act as an acoustic dipole source moving through the atmosphere, and in a similar manner a ship moving over the ocean waves will be a dipole source for low-frequency underwater sound. The sound radiation from propellers and turbines also can be described in terms of moving-dipole-source distributions, as will be discussed later in this section.

An ideal dipole source moving in the x direction with a velocity V is described by a force distribution per unit volume with components of the form

$$F_j = f_j(t)\delta(x - Vt)\delta(y)\delta(z)$$

The corresponding source term in the wave equation is $\partial F_j/\partial x_j$, as we have seen already in Eq. (7.1.23). Thus, if we introduce a vector $\mathbf{A}$ with the components A_j such that

$$p = \operatorname{div} \mathbf{A} \qquad (11.2.31)$$

it follows that the equation for A_j is

$$\nabla^2 A_j - \frac{1}{c^2}\frac{\partial^2 A_j}{\partial t^2} = f_j(t)\delta(x - Vt)\delta(y)\delta(z) \tag{11.2.32}$$

In complete analogy with (11.2.13), we obtain the solution, for $V < c$,

$$A_j = -\frac{f_j[t - (R^+/c)]}{4\pi R_1}$$

where R^+ and R_1 are defined in (11.2.18). The pressure then follows from (11.2.31). For future reference we express the fields from the transverse dipole with the force components $f_j = (0, f_2, 0)$.

$$p_2 = \frac{1}{4\pi R_1}\frac{y}{R_1}\left(\frac{1}{c}f_2' + \frac{1}{R_1}f_2\right) \tag{11.2.33}$$

Here the prime signifies differentiation with respect to the argument. Similarly, the sound pressure field for a longitudinal dipole, in which the force components are $f_j = (f_1, 0, 0)$, is

$$p_1 = \frac{1}{4\pi R_1 c}f_1'\left[\frac{M}{1 - M^2} + \frac{x - Vt}{R_1(1 - M^2)}\right] + \frac{1}{4\pi R_1^2}\frac{x - Vt}{R_1}f_1 \tag{11.2.34}$$

The detailed study of the properties of these fields is left for some of the problems.

We turn here, instead, to the Fourier transform of A_j obtained from Eq. (11.2.32) when the force has harmonic time dependence $f_j(t) = f_{0j}\cos(\omega_0 t)$. Again, by analogy with (11.2.24), we get

$$A_j^+ = \frac{if_{0j}}{16\pi V}\int_{-\infty}^{+\infty} H_0^{(1)}(k^+ r)e^{i(\omega - \omega_0)x/V}e^{-i\omega t}\,d\omega \tag{11.2.35}$$

The sound pressure field (11.2.31) and the corresponding radial velocity component then follow from A_j, and in much the same manner as for the monopole, we find that the power emitted from a longitudinal dipole (force component f_x) is

$$\Pi = \frac{f_{0x}^2\omega_0^2}{16\pi V^3\rho}\int_{\gamma_1}^{\gamma_2}\frac{(\gamma - 1)^2}{\gamma}\,d\gamma = \Pi_d G_1(M) \tag{11.2.36}$$

where

$$\gamma = \frac{\omega}{\omega_0} \qquad \gamma_1 = \frac{1}{1 + M} \qquad \gamma_2 = \frac{1}{1 - M}$$

$$\Pi_d = \frac{\omega_0^2 f_{0x}^2}{24\pi\rho c^3}$$

$$G_1(M) = \frac{3}{2}\frac{1}{M^3}\left[\log\left(\frac{1 + M}{1 - M}\right) + \frac{2M(2M^2 - 1)}{(1 - M^2)^2}\right]$$

We leave it as one of the problems to derive this and some similar expressions for other moving sources.

The quantity Π_d is the power emitted from a dipole at rest, in agreement with the result obtained in Eq. (7.1.7), for $f_0 = \omega D$, and allowing for the change from rms to maximum amplitude. The function $G_1(M)$ expresses the velocity dependence of the power. For the moving transverse dipole

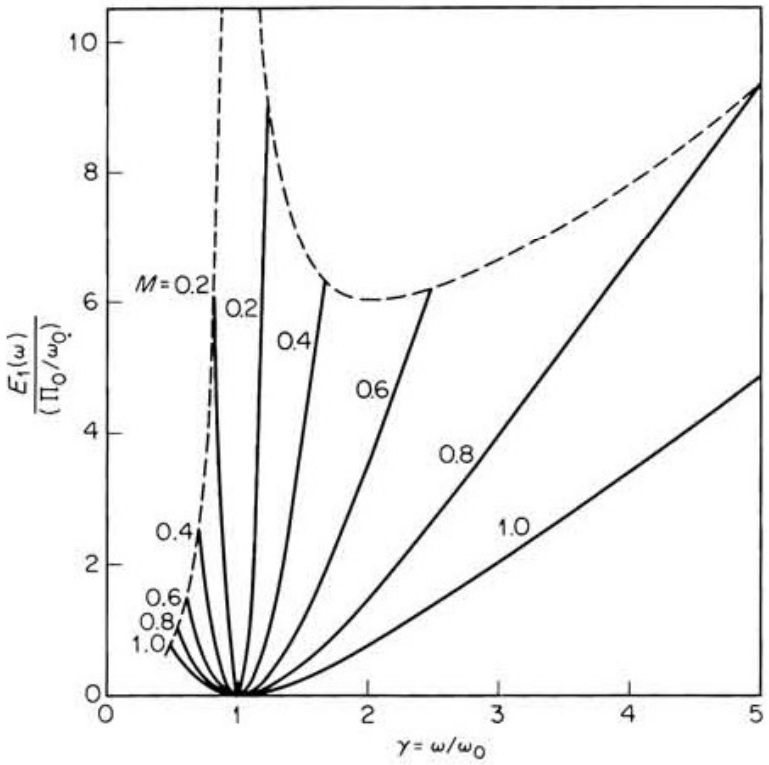


FIGURE 11.15
Power spectra of a longitudinal dipole in uniform subsonic motion for some different values of the velocity. $V/c = M$.

corresponding to the force component f_y (or f_z), the total power emitted is found to be

$$\Pi = \Pi_d G_2(M) \tag{11.2.37}$$

where $$G_2(M) = \frac{3}{4}\left[\frac{2}{M^2(1-M^2)} - \frac{1}{M^3}\log\frac{1+M}{1-M}\right]$$

and Π_d is given in (11.2.36).

The power spectra, defined by $\Pi = \int_0^\infty E(\omega)\,d\omega$, for the longitudinal and transverse moving dipoles, are obtained directly from the integrals for Π; they are found to be

$$E_1(\omega) = \frac{\Pi_d}{\omega_0}\frac{3}{2M^3}\frac{(\gamma-1)^2}{\gamma}$$
$$E_2(\omega) = \frac{\Pi_d}{\omega_0}\frac{3}{4M^3\gamma}[(M-1)\gamma+1][\gamma(M+1)-1] \tag{11.2.38}$$

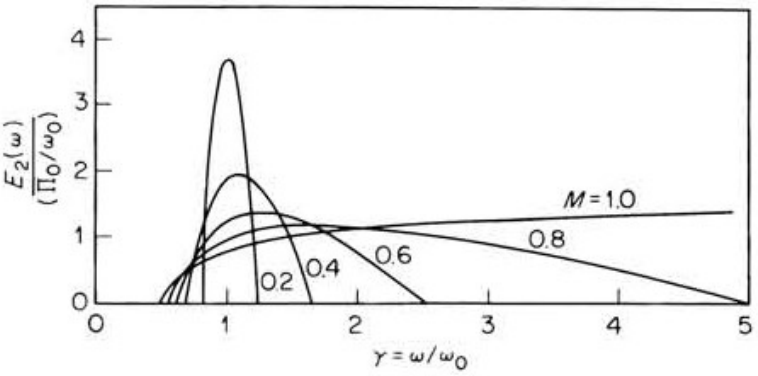


FIGURE 11.16
Power spectra of a transverse dipole in uniform subsonic motion for some different values of the velocity. $V/c = M$.

It may appear that the spectrum shape of the longitudinal moving dipole is independent of the speed. We should keep in mind, however, that the width of the spectrum determined by the extreme values of the Doppler-shifted frequencies does depend on M. These spectra are shown in Figs. 11.15 and 11.16 for some different values of M. For the longitudinal dipole the intensity is zero for $\omega = \omega_0$. This is understandable since, for a dipole lined up in the direction of motion, the direction of zero Doppler shift coincides with the direction of zero intensity in the dipole radiation pattern.

For the "transverse" dipole, on the other hand, the directions of zero intensity are along the line of motion, and on these lines the Doppler-shifted frequencies are $\omega_0/(1 - M)$ and $\omega_0/(1 + M)$, in front of and behind the source, respectively. Consequently, the energy spectrum is expected to be zero at these frequencies, and this is indeed the case.

Sound emission from a piston source in a moving medium

In many applications we encounter problems in which a sound source is at rest in the laboratory system and the fluid is moving. As already pointed out, however, the problem of sound emission from a point source moving in the positive x direction is equivalent to the emission from a stationary source in a fluid moving in the negative x direction. Therefore we should not need to say anything further about this case, except perhaps to point out that it is convenient to express the field in terms of coordinates attached to the source rather than the fluid. In addition, some comments should be made about the calculations of the sound field from an extended source, and we shall choose as an illustration the radiation from an oscillating piston in an infinite plane in contact with a fluid moving past it.

Thus, returning to the expression for the sound field from a moving monopole source in (11.2.13) or (11.2.15), we introduce the coordinates

$$\xi = x - Vt \qquad \eta = y \qquad \zeta = z$$

attached to the moving source. In terms of these coordinates, the solution represents the sound field from a stationary source in a flow field moving in the negative ξ direction. With reference to (11.2.15), in which we set $R_1 = R(1 - M\cos\theta)$ [see (11.2.6)], the sound pressure field from a monopole

at the point (ξ_0,η_0,ζ_0), when the surrounding fluid moves with a velocity V in the negative ξ direction, is given by

$$p = \frac{1}{4\pi R_1}\left[\frac{R}{R_1}q'\left(t - \frac{R}{c}\right) + \frac{\xi V}{R_1^2}q\left(t - \frac{R}{c}\right)\right] \qquad (11.2.39)$$

where

$$R_1^2 = (\xi - \xi_0)^2 + (1 - M^2)[(\eta - \eta_0)^2 + (\zeta - \zeta_0)^2]$$

and

$$(1 - M^2)R = M(\xi - \xi_0) + R_1$$

For a source with harmonic time dependence $q = q_0 \exp(-i\omega t)$, this field can be expressed as

$$p(\mathbf{r};\mathbf{r}_0) = \frac{-i\omega}{4\pi R_1}e^{ikR}\left(\frac{R}{R_1} + i\frac{\xi - \xi_0}{R_1}\frac{M}{kR_1}\right)q_0 e^{-i\omega t} \qquad (11.2.40)$$

where

$$M = \frac{V}{c} \qquad k = \frac{\omega}{c}$$

With $q_0 = 1$ this corresponds to the free-space Green's function for a moving fluid (velocity V in the negative ξ direction).

We shall use this result for the calculation of the sound emission from a rectangular piston set in an otherwise rigid plane, coincident with the $\xi\eta$ plane, with the fluid moving in the negative ξ direction with a speed V. The piston, with the dimensions $2a$ and $2b$, has a velocity $U\exp(-i\omega t)$ in the ζ direction, and therefore a displacement $d_\zeta = (iU/\omega)\exp(-i\omega t)$. The corresponding ζ component of the *fluid* velocity at the plane is given by $[\partial/\partial t - V(\partial/\partial\xi)]\, d_\zeta = \{1 + i(V/\omega)[\delta(\xi - a) - \delta(\xi + a)]\}U\exp(-i\omega t)$. The δ-function contribution from the velocity arises when the fluid flows over the edge of the piston as it moves up and down. Since we have assumed that the flow is in the $-\xi$ direction, these jumps are only at the $\xi = \pm a$ edges, not at the others. With q_0 in (11.2.40) replaced by 2ρ times this fluid velocity, the sound field from the piston is obtained by integrating $p(r/r_0)$ in (11.2.40) over the region $-a < \xi_0 < +a$, $-b < \eta_0 < +b$. The integral includes the δ-function contribution to the velocity from $\eta = -b$ to $\eta = +b$. Considering here only the far field, we use the first term in (11.2.40) and obtain for the pressure field

$$p \to -\frac{2i\rho cUkab}{\pi}\frac{e^{ikR'}}{R_1'}\frac{R'}{R_1'}\frac{\sin X}{X}\frac{\sin Y}{Y}\left(1 + M\frac{X}{ka}\right)$$

$$R_1' = \xi^2 + (1 - M^2)(\eta^2 + \zeta^2)$$

$$(1 - M^2)R' = M\xi + R_1'$$

$$X = \frac{ka[M - (\xi/R_1')]}{1 - M^2}\,;\quad Y = kb\frac{\eta}{R_1'} \qquad (11.2.41)$$

The first term in the factor $1 + M(X/ka)$ represents the contribution from

the surface of the piston, whereas the second term comes from the $\xi = \pm a$ edges of the piston. The surface of the piston represents an extended monopole source distribution, and it is interesting to see from (11.2.41) that, in contrast to the point source, this part of the emitted sound pressure field remains finite when $V = c$. The reason for the divergence for the point source when $V = c$ can be traced to the zero size of the source. At a given point of observation upstream from the source, the pressure-wave contributions from the point source all arrive in phase as $V \to c$, regardless of the emission time. But the wavelength of the sound in the fluid then approaches zero, and if the source has a nonzero size, however small, the difference in travel time for rays emitted from different parts of the source will be sufficient to cause destructive interference. This is what happens to the surface contribution from the piston, and as a result the field remains finite for $M = 1$.

The line sources at the edges of the piston, however, having zero size in the direction of motion of the flow, behave much like a point source, and the corresponding sound-field contribution does indeed diverge at $V = c$, as can be seen from Eqs. (11.2.41).

In practice, the behavior of the flow at the edges of the piston will not be steplike, as implied by the use of δ functions in the acoustic source function. Instead, the flow profile will be blurred because of turbulence, so that this contribution, also, will remain finite at $M = 1$.

11.3 SOUND EMISSION FROM RIGID-BODY FLOW INTERACTION

As already mentioned in Sec. 11.1 in the study of the effect of fluid motion on transducer response, the interaction between a body in motion relative to a fluid leads to pressure fluctuations on the body. These fluctuations result either from turbulence or inhomogeneities in the incident flow or from the turbulence produced by the body itself. To obtain the sound field produced in this manner, we must start by investigating the interaction between the body and the flow. A general analysis of this problem, starting from first principles, is hardly possible, and in this phase of the study we must rely upon experimental data relating the interacting forces with the parameters describing the flow field and the body. Such experimental data are available for a number of simple bodies such as spheres and cylinders and, in particular, for airfoils.

In this section we shall start with some comments about the sound from a rigid body in motion relative to a turbulent fluid. Next we consider the case when the incident flow is uniform and the turbulence is produced by the body itself. As a typical example, the Kármán vortex sound from a cylinder will be discussed. Actually, turbulence is not a necessary requirement for sound emission from a rigid body in a moving fluid. For example, if the motion of the body is supersonic, it is well known that a shock wave is produced, and if there is an array of bodies moving through a fluid, as in

the case of a propeller, a periodic pressure field is generated which, under certain conditions, results in a radiated sound field. The important problem of propeller sound will be considered for the case of both uniform and nonuniform flow. Finally, some observations will be made relating to the problem of the effect of flow on a mechanical oscillator and on the acoustical properties of an orifice in a partition.

Sound from a propeller

As an example of sound radiation in Sec. 7.4, we considered briefly the sound field produced by a source distribution which, in its essentials, could

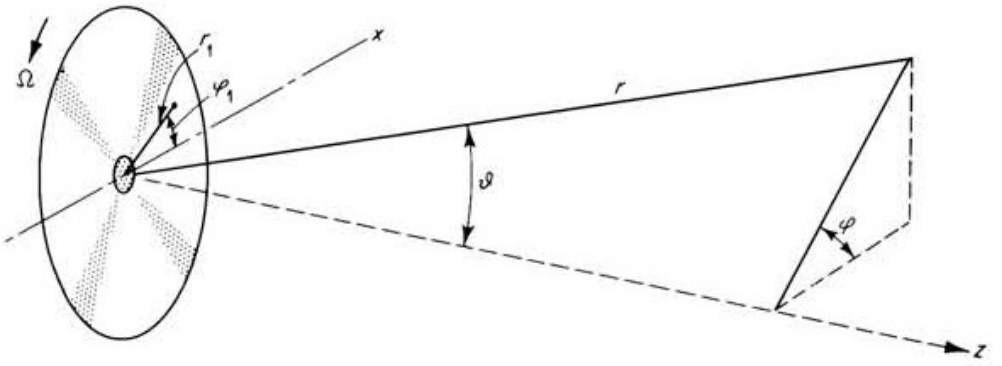


FIGURE 11.17
Sound radiation from a propeller.

serve as a model for a propeller. We shall now justify this model by considering the elementary aerodynamics of a body such as the propeller blade of Fig. 11.17 moving through a fluid, and include the case often encountered in practice when the propeller operates in an inhomogeneous fluid. For example, a ship propeller operates in the inhomogeneous flow in the wake of the ship, a compressor rotor interacts with the wakes of stationary guide vanes in front of the rotor, and an airplane propeller often moves through a turbulent atmosphere. It turns out that flow inhomogeneities give rise to sound radiation which generally far exceeds the radiation in a homogeneous flow. To study this aspect of the sound generation by a propeller, we shall start by investigating the interaction between a propeller blade and the surrounding fluid.

When a propeller blade moves through a homogeneous fluid, there will be a pressure distribution about the blade resulting in a net force distribution over the blade. It is customary to express the force per unit area of the blade in terms of a thrust and a drag component f_t and f_d, illustrated in Fig. 11.18, which are expressed as

$$f_t = c_t \frac{\rho V^2}{2} \qquad f_d = c_d \frac{\rho V^2}{2} \tag{11.3.1}$$

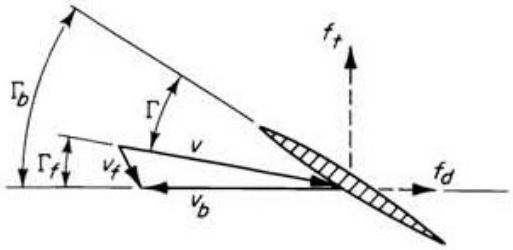


FIGURE 11.18
Airfoil (propeller blade) moving with a velocity V_b through a fluid that moves with a velocity V_f.

The thrust and drag coefficients c_t and c_d depend strongly on the "angle of attack" Γ of the flow velocity (Fig. 11.17) and only weakly on the magnitude of the relative velocity $\mathbf{V}$,

$$\mathbf{V} = \mathbf{V}_f - \mathbf{V}_b$$

where $\mathbf{V}_f$ is the fluid velocity, and $\mathbf{V}_b$ the blade velocity.

The coefficients c_t and c_d must be determined empirically as functions of the position on the propeller blade. If the flow is uniform, the force on a blade is independent of the angular position of a blade as it moves through the fluid. Thus, in a coordinate system rotating with the angular velocity of the propeller, the force distribution over the plane of the propeller is time-independent. However, the effect of the propeller on the surrounding fluid as observed at a fixed point in the laboratory system is represented by a force which is periodic in time with an angular frequency equal to $\omega_1 = B\Omega$, where Ω is the angular velocity of the propeller shaft, and B the number of blades. The plane of the propeller, then, can be regarded as a sound source with a dipole source distribution corresponding to the force (11.3.1) on the portion occupied by the blades and zero between the blades. The coordinates in this plane are shown in Fig. 11.17.

Considering, first, the case of *uniform* flow, the amplitude of the periodic force on the fluid is independent of φ_1, but the phase is proportional to φ_1. Thus, if the z component of the force on the fluid at $\varphi_1 = 0$ is $f_z(t;r_1,0) = \sum_{-\infty}^{+\infty} A_s(r_1)e^{-is\omega_1 t}$, it is, at φ_1,

$$f_z(t;r_1,\varphi_1) = \sum A_s \exp\left[-is\omega_1\left(t - \frac{\varphi_1}{\Omega}\right)\right] = f_z^0 \Sigma \alpha_s e^{isB\varphi_1} e^{-is\omega_1 t} \qquad (11.3.2)$$

$$\omega_1 = B\Omega \qquad \alpha_s = \frac{A_s}{f_z^0}$$

Here the coefficients f_z^0 and α_s generally are functions of the radial coordinate r_1. For the φ_1 component of the force, we obtain

$$f_\varphi(t;r_1,\varphi_1) = f_\varphi^0 \sum_{-\infty}^{+\infty} \alpha_s e^{isB\varphi_1} e^{-is\omega_1 t} \qquad (11.3.3)$$

We have here assumed that the angular dependence of f_φ and f_z is the same but the magnitudes are different. Actually the time average value f_z^0 is considerably larger than f_φ^0 for a well-designed airfoil. Constants f_z^0 and f_φ^0 can

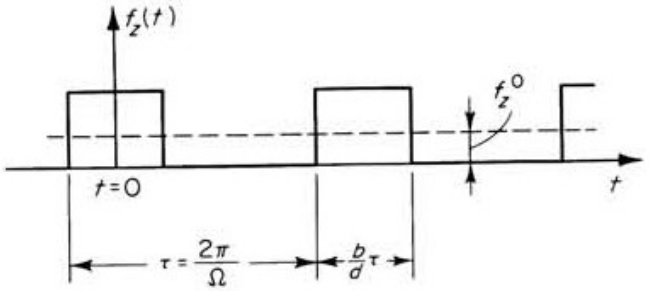


FIGURE 11.19
Example of time dependence of the dipole source function in the plane of the propeller.

be expressed in terms of the total thrust force F_t and the total torque T of the propeller from

$$F_t = \int f_z^0 r_1 \, dr_1 \, d\varphi_1 \qquad \text{and} \qquad T = \int (f_\varphi^0 r_1) r_1 \, dr_1 \, d\varphi_1 \tag{11.3.4}$$

The coefficients α_s are obtained from

$$\alpha_s = \frac{1}{f_z^0} \frac{e^{-isB\varphi_1}}{\tau} \int_{-\tau/2}^{\tau/2} f_z(t,\varphi_1) e^{is\omega_1 t} \, dt \tag{11.3.5}$$

If, as an illustration, we assume that the pressure distribution over a blade is uniform, the time dependence of f is represented by a rectangular function as shown in Fig. 11.19. Then we obtain from Eq. (11.3.5):

$$\alpha_0 = 1 \qquad 2\alpha_s = \frac{\sin(sb\pi/d)}{sb\pi/d} \qquad s > 0 \tag{11.3.6}$$

where b is the width of the blade, and d the separation between two blades.

Nonuniform flow

If the flow is *nonuniform*, so that the velocity V in Eqs. (11.3.1) depends on r_1 and φ_1, the response of a propeller blade is considerably more complicated than in the homogeneous case, but as long as the flow is time-independent, the time dependence of the force distribution will still be periodic, with the fundamental frequency $\omega_1 = B\Omega$. We shall consider only this case of a time-independent (although inhomogeneous) flow field. In contrast to the homogeneous case, the amplitude factors f_z^0 and f_φ^0 are now functions of φ_1. If we express this φ_1 dependence in terms of the Fourier series,

$$f_z^0(\varphi_1) = f_z^0 \sum_{-\infty}^{+\infty} \beta_q e^{iq\varphi_1} \tag{11.3.7}$$

we obtain, replacing f_z^0 by $f_z^0(\varphi_1)$ in (11.3.3),

$$f_z(t;\varphi_1,r_1) = f_z^0 \sum_{sq} \alpha_s \beta_q e^{i(sB+q)\varphi - is\omega_1 t} \tag{11.3.8}$$

This expression includes, through the term $q = 0$, the case of uniform flow considered above. Similarly, the φ_1 component of the force becomes

$$f_\varphi(t;\varphi_1,r_1) = f_\varphi^0 \sum_{sq} \alpha_s \delta_q e^{i(sB+q)\varphi - is\omega_1 t} \tag{11.3.9}$$

The relationship between the coefficients β_q, δ_q, and the flow velocity field will be quite complex in general, but in the case where the width of the propeller blade is sufficiently small, the expression (11.3.1) for the force should be a good approximation, even when V is a function of φ_1. Thus, if the nonuniform part of the flow velocity $\mathbf{V}_f$ is $\mathbf{v}(\varphi_1)$, the nonuniformity in the relative velocity $\mathbf{V}$ is also $\mathbf{v}(\varphi_1)$, since the blade velocity is constant. The corresponding nonuniformity in the force distribution is then obtained from Eq. (11.3.1) as

$$df_t = df_z = \frac{\rho}{2}\,[c_t d(V^2) + V^2\,dc_t] = \frac{\rho}{2}\left[c_t d(V^2) + V^2 \frac{dc_t}{d\Gamma}\,d\Gamma\right]$$

The angle of attack Γ of the flow is $\Gamma = \Gamma_b - \Gamma_f$, where the angles Γ_b and Γ_f are defined in Fig. 11.18.

If the axial and tangential components of the flow velocity are V_{fz} and $V_{f\varphi}$ and the radial component is zero, the relative velocity can be expressed as

$$V = \sqrt{(V_b - V_{f\varphi})^2 + V_{fz}{}^2}$$

and the angle Γ_f is given by

$$\Gamma_f = \tan^{-1}\frac{V_{fz}}{V_b - V_{f\varphi}} \simeq \frac{V_{fz}}{V_b} \qquad \text{if } (V_{f\varphi}, V_{fz}) \ll V_b$$

If the components of the flow perturbation are v_z and v_φ and if $(v_z, v_\varphi) \ll V$, and furthermore, if $V \ll V_b$, we obtain $dV^2 \simeq -2V_b v_\varphi$ and $d\Gamma = -d\Gamma_f \simeq -v_z/V_b$. The corresponding expression above for the perturbation df_z in the z component of the force is then

$$-\frac{df_z}{f_z^0} = \frac{2v_\varphi}{V_b} + \frac{1}{c_t}\frac{dc_t}{d\Gamma}\frac{v_z}{V_b} \tag{11.3.10}$$

In other words, instead of the constant force amplitude in (11.3.2), the expression for the total force is now

$$f_z = f_z^0\left[1 - \frac{2v_\varphi}{V_b} - \frac{1}{c_t}\frac{dc_t}{d\Gamma}\frac{v_z(\varphi)}{V_b}\right] \equiv f_z^0 \sum_q \beta_q e^{iq\varphi_1} \tag{11.3.11a}$$

with a similar expression for the drag force,

$$f_\varphi = f_\varphi^0\left[1 - \frac{2v_\varphi}{V_b} - \frac{1}{c_d}\frac{dc_d}{d\Gamma}\frac{v_z(\varphi)}{V_b}\right] \equiv f_\varphi^0 \sum_q \delta_q e^{iq\varphi_1} \tag{11.3.11b}$$

From this we can express the coefficients β_q and δ_q by a Fourier expansion of the flow field.

It is interesting in this connection to obtain the expressions (11.3.11) in a different and somewhat more direct manner. Since the force $\mathbf{f}$

(components f_i) depends on the relative velocity $\mathbf{V}$ (components V_j) between the propeller blade and the incident flow, it follows that, $\mathbf{V}$ being the only vector from which $\mathbf{f}$ can be constructed, the force components must have the form

$$f_i = \sum_j \alpha_{ij}\left(\frac{\rho V^2}{2}\right)\frac{V_j}{V} \tag{11.3.12}$$

where $\rho V^2/2$ is obtained from dimensional considerations. The dimensionless coupling coefficients α_{ij} may be functions of such dimensionless parameters as the Reynolds and the Mach numbers. These coefficients are assumed to be known from experiments. Splitting up the velocity $\mathbf{V}$ into a uniform part $\mathbf{V}^0$ and a perturbation $\mathbf{v}(\varphi) = (0,v_\varphi,v_z)$, it follows that the force f_z to first order in v becomes

$$f_z = \left[-\alpha_{zz}\frac{v_z}{V^0} + \alpha_{z\varphi}\left(1 - \frac{2v_\varphi}{V^0}\right)\right]\frac{\rho(V^0)^2}{2}$$

Also

$$f_\varphi = \left[-\alpha_{\varphi z}\frac{v_z}{V^0} + \alpha_{\varphi\varphi}\left(1 - \frac{2v_\varphi}{V^0}\right)\right]\frac{\rho(V^0)^2}{2}$$

Comparing these expressions with (11.3.11), we see that the coupling coefficients have the following meaning in terms of the thrust and drag coefficients:

$$\begin{aligned} \alpha_{z\varphi} &= c_t \qquad & \alpha_{\varphi\varphi} &= c_d \\ \alpha_{zz} &= \frac{dc_t}{d\Gamma} \qquad & \alpha_{\varphi z} &= \frac{dc_d}{d\Gamma} \end{aligned} \tag{11.3.13}$$

The radiated sound

After having obtained the acoustic source function, we can now express the corresponding sound field, using the appropriate Green's function as demonstrated several times throughout this book. The acoustic source distribution is defined over the plane of the propeller over a circle of radius a, the radius of the propeller. A point source of unit strength placed at the position r_1, φ_1 in the plane produces the sound field $g_0 = \exp[ik|\mathbf{r} - \mathbf{r}_1|]/4\pi|\mathbf{r} - \mathbf{r}_1|$, where $k = \omega/c$ and $\mathbf{r}$ is the position vector of the point of observation. Expressing the position vectors in the spherical coordinates $\mathbf{r} = (r,\vartheta,\varphi)$ and $\mathbf{r}_1 = (r_1,\pi/2,\varphi_1)$, where $\vartheta = 0$ defines the axis of the propeller, we obtain in the far field $r \gg r_1$, $|\mathbf{r} - \mathbf{r}_1| \simeq r - r_1 \sin\vartheta\cos(\varphi - \varphi_1)$.

The sound field from the unit-strength harmonic point source at r_1 with a frequency ω is then

$$g_0 = \frac{1}{4\pi r}\exp[ikr - ikr_1\sin\vartheta\cos(\varphi - \varphi_1)]e^{-i\omega t} \qquad k = \frac{\omega}{c}$$

If we use the Fourier expansion of Eq. (1.2.9),

$$e^{iz\cos(\varphi-\varphi_1)} = \sum_{-\infty}^{+\infty} i^m J_m(z) e^{im(\varphi-\varphi_1)}$$

where J_m is the Bessel function of the mth order, the expression for g_0 can be written

$$g_0 = \frac{1}{4\pi r} e^{ikr} \sum_{-\infty}^{+\infty} i^m J_m(kr_1 \sin\vartheta) e^{im(\varphi-\varphi_1)} e^{-i\omega t}$$

As we have shown earlier, in Chap. 7, we can generate multipole fields from the monopole field simply by repeated differentiation with respect to the source coordinates. Thus the dipole-field sound field from a unit-strength point force in the z direction ($\vartheta = 0$) is simply $g_{1z} = -\partial g_0/\partial z = -(\partial g_0/\partial r)\cos\vartheta$.

$$g_{1z} = -ik\cos\vartheta \frac{e^{ikr}}{4\pi r} \sum_{m=-\infty}^{\infty} i^m J_m(kr_1 \sin\vartheta) e^{ik(\varphi-\varphi_1)} e^{-i\omega t} \quad (11.3.14a)$$

Similarly, the dipole field from a unit force in the φ_1 direction is

$$g_{1\varphi} = -\frac{1}{r_1}\frac{\partial g_0}{\partial \varphi} = \frac{i}{r_1}\frac{e^{ikr}}{4\pi r} \sum_m m i^m J_m(kr_1 \sin\vartheta) e^{im(\varphi-\varphi_1)} e^{-i\omega t} \quad (11.3.14b)$$

When applying these expressions for the calculation of the sound produced by the sth harmonic of the propeller-blade passage frequency $\omega_1/2\pi$, we have to set $\omega = s\omega_1$ and $k = s\omega_1/c = sk_1$ in the expressions for g_{1z} and $g_{1\varphi}$. The total sound field from the z component of the force distribution is then obtained as

$$p_z(r,\varphi,\vartheta) = \iint f_z(r_1,\varphi_1) g_{1z}(r_1,\varphi_1;r,\vartheta,\varphi) r_1\, dr_1\, d\varphi_1$$

with a similar expression for the field from the φ_1 component of the force. Inserting the expressions for f_z and g_{1z} as given by (11.3.8) and (11.3.14), we find that the integral over φ_1 is zero except when $m = sB + q$, in which case the integral is 2π, and the sound pressure becomes

$$p_\varphi(r,\varphi,\vartheta) = -\frac{ik_1\cos\vartheta}{4\pi r} \sum_{sq} i^{sB+q} e^{isk_1 r} e^{i(sB+q)\varphi - is\omega_1 t} \times \int_0^a s f_z^0 \alpha_s \beta_q J_{sB+q}(u) 2\pi r_1\, dr_1\, d\varphi_1 \qquad u = sk_1 r_1 \sin\vartheta \quad (11.3.15)$$

In general, f_z^0, α_s, and β_q are all functions of r_1. The quantities must be determined from the experimentally known aerodynamic characteristics of the propeller blades, and the integral in (11.3.15) must be evaluated numerically. However, formally, we can express it as

$$(\alpha_s \beta_q J_{sB+q})_{r_1=a_z} \int f_z^0 2\pi r_1\, dr_1$$

where a_z is an average value of the radial coordinate determined from

$$(\alpha_s B_q J_{sB+q})_{r_1=a_z} = (F_t)^{-1}\int_0^a f_z^0 \alpha\beta J_{sB+q} 2\pi r_1\, dr_1 \qquad (11.3.16)$$

where

$$F_t = \int_0^a f_z^0 2\pi r_1\, dr_1$$

is the total thrust force of the propeller. We can then rewrite (11.3.15) to express the sound pressure field from the z component of the force distribution in terms of the mean radius a_z and the total thrust force of the propeller.

$$p_z = (-ik_1 \cos\vartheta)F_t \frac{1}{4\pi r}$$
$$\times \sum s(\alpha_s\beta_q)J_{sB+q}(sk_1 a_z \sin\vartheta)\exp\,[isk_1 r + i(sB+q)(\varphi + \tfrac{1}{2}\pi) - is\omega_1 t]$$

In a completely similar manner we obtain the sound pressure p_φ produced by the φ component of the force

$$p_\varphi = \int f_\varphi g_{1\varphi} 2\pi r_1\, dr_1$$

where f_φ and $g_{1\varphi}$ are given in Eqs. (11.3.11) and (11.3.14). There is one slight difference, inasmuch as the radial integration now contains an extra factor $(r_1)^{-1}$, the radial integral being

$$\int_0^a \alpha_s\delta_q f_\varphi^0 (r_1)^{-1} J_{sB+q} 2\pi r_1\, dr_1$$

Again this integral has to be evaluated numerically, but we can always express it in terms of the total torque $T = \int (f_\varphi^0 r_1)2\pi r_1\, dr_1$ by introducing a mean radius a_φ such that

$$T\left(\frac{1}{a_\varphi}\right)^2 (\alpha_s\delta_q J_{zB+q})_{r_1=a\varphi} = \int_0^a \alpha_s\delta_q f_\varphi^0 (r_1)^{-1} J_{sB+q} 2\pi r_1\, dr_1 \qquad (11.3.17)$$

It should be noted that the mean radii a_z and a_φ, determined from Eqs. (11.3.16) and (11.3.17), in general, are not the same. The sound field produced by the φ component of the force then can be expressed as

$$p_\varphi = \frac{T}{{a_\varphi}^2}\frac{1}{4\pi r}\sum_{s,q}(sB+q)(\alpha_s\delta_q)$$
$$\times J_{sB+q}(sk_1 a_d \sin\vartheta)\exp\,[isk_1 r + i(sB+q)(\varphi + \tfrac{1}{2}\pi) - is\omega_1 t]$$

If as an approximation we assume that the mean radii a_z and a_φ are the same, $(a_z, a_\varphi) = a_m$, the expression for the total sound pressure field $p = p_z + p_\varphi$ can be written in the form

$$p = \frac{1}{4\pi r}\sum_{s,q}\left(sk_1 F_t \cos\vartheta + T\frac{\delta_q}{\beta_q}\frac{sB+q}{{a_m}^2}\right)\alpha_s\beta_q J_{sB+q}(sk_1 a_m \sin\vartheta)$$
$$\times \exp\,[isk_1 r + i(sB+q)(\varphi + \tfrac{1}{2}\pi) - is\omega_1 t] \qquad (11.3.18)$$

Fundamental and harmonics

For numerical calculations and for comparison with experimental results, it is convenient to express the sound pressure field in real form. If we consider, first, the case of *uniform flow*, we have to set $q = 0$ and sum over all values of s. Thus, if n is a positive integer, we have to add terms corresponding to $s = n$ and $s = -n$ and then sum over n. Carrying out this summation and making use of $J_n(z) = J_{-n}(-z)$, we obtain for the pressure field

$$p_{q=0} = \frac{1}{4\pi r}\left(k_1 F_t \cos\vartheta + \frac{BT}{a_m{}^2}\right) \times \sum_{n=1}^{\infty} n(2\alpha_n) J_{nB}(nk_1 a_m \sin\vartheta) \sin\left[nk_1 r + nB(\varphi + \tfrac{1}{2}\pi) - n\omega_1 t\right]$$

This sound field represents a pressure wave which spins with the angular velocity $\omega_1/B = \Omega$ of the propeller. The amplitude of the nth-harmonic component can be written

$$(p_n)_{q=0} = \frac{(2\alpha_n)k_1 F_t}{4\pi r}\left(\cos\vartheta + \frac{F_d}{F_t M}\right) n J_{nB}(nBM \sin\vartheta)$$

where we have introduced the drag force $F_d = T/a_m$ and $M = a_m\Omega/c$, the Mach number of the propeller blade at the mean radius a_m. The value of the Fourier coefficients $2\alpha_n$ is obtained from Eq. (11.3.5), which in the special case when the pressure over the blade is assumed uniform results in Eqs. (11.3.6). For the first few harmonics, the value of $2\alpha_n$ is of the order of unity.

Ordinarily, the propeller blade at radius a_m is not traveling faster than the speed of sound, so that $M \sin\vartheta < 1$. In this case we can use the first term in the power series for the Bessel function [Eq. (5.2.20)] and can use the approximation $(nB)! \simeq \sqrt{2\pi nB}(nB)^{nB} e^{-nB-1}$ to obtain an approximate expression for the amplitude of the pressure wave radiated at frequency $n\Omega B/2\pi$.

$$(p_n)_{q=0} \simeq \frac{a_n\Omega}{2\pi c r}\sqrt{nB}\left(F_t \cos\vartheta + \frac{F_d}{M}\right)(\tfrac{1}{2}eM \sin\vartheta)^{nB} \qquad (11.3.19)$$

The factor $(\sin\vartheta)^{nB}$ peaks sharply at $\vartheta = 90°$, in the plane of the propeller. As long as $\frac{1}{2}eM \simeq 1.4M$ is less than unity, the amplitudes of the higher harmonics fall off fairly rapidly with increasing n, and if this were the sound produced by the usual airplane propeller, aircraft noise would not be the problem it is. The difficulty is that the flow past the propeller blade is *not* uniform, and we must include the terms for $q \neq 0$ in Eq. (11.3.18).

For nonuniform flow, we introduce the positive numbers n and l and sum the terms corresponding to $s = n, s = -n, q = l, q = -l$. In carrying out this addition, it is convenient to pair up the contributions from ($s = n$,

$q = l$) and ($s = -n$, $q = -l$) and from the terms ($s = n$, $q = -l$) and ($s = -n, q = l$). Making use of $J_n(z) = J_{-n}(-z)$, we then find the following expression for the total sound pressure field:

$$p = \frac{1}{4\pi r}\sum_{n=1}^{\infty}(2nk_1\alpha_n)\left\{\sum_{l=0}^{\infty}\left(\beta_l F_t\cos\vartheta + \delta_l\frac{nB-l}{nBM}F_d\right)J_{nB-l}(nBM\sin\vartheta)\right.$$
$$\times\sin[nk_1(r-ct)+(nB-l)(\varphi+\tfrac{1}{2}\pi)] + \sum_{l=0}^{\infty}\left(\beta_l F_t\cos\vartheta+\delta_l\frac{nB+l}{nBM}F_d\right)$$
$$\left.\times J_{nB+l}(nBM\sin\vartheta)\sin[nk_1(r-ct)+(nB+l)(\varphi+\tfrac{1}{2}\pi)]\right\} \quad (11.3.20)$$

We note that the sound field is composed of two distinct groups of waves, the first representing a pressure field which spins with an angular velocity, greater than Ω,

$$\omega_- = \frac{n\omega_1}{nB-l} = \frac{nB}{nB-l}\Omega \quad (11.3.21)$$

in the positive φ direction. The other wave spins with an angular velocity

$$\omega_+ = \frac{nB}{nB+l}\Omega \quad (11.3.22)$$

which is always smaller than the angular velocity of the propeller. This second kind of wave is negligible, compared with some of the first kind, as long as the propeller Mach number M is less than about 0.7, for the reasons we pointed out in connection with Eq. (11.3.19); Bessel functions of high order ($nB + l \gg 1$) and argument ($nBM\sin\vartheta$) smaller than order are quite small.

The first series over l, however, contains low-order Bessel functions; in fact, the term for $l = nB$ contains the factor $J_0(nBM\sin\vartheta)$, which has its maximum at $\vartheta = 0$ instead of $\vartheta = 90°$. The phase of this wave is independent of φ; its amplitude is

$$\frac{1}{2\pi r}n\alpha_n k_1\beta_{nB}F_t\cos\vartheta\, J_0(nBM\sin\vartheta)$$

which is maximum at $\vartheta = 0$ (along the axis of the propeller) and is zero in the plane of the propeller. The amplitude of the component of the nth-harmonic wave, having a spin velocity $nB\Omega/m$ (where $m = nB - l$), on the other hand, is

$$\frac{\alpha_n nk_1}{2\pi r}\left(\beta_{nB-m}F_t\cos\vartheta+\delta_{nB-m}F_d\frac{m}{nBM}\right)J_m(nBM\sin\vartheta) \quad (11.3.23)$$

which is zero for $\vartheta = 0$, but does not peak sharply at $\vartheta = 90°$, at least when m is small.

Unless M is very close to 1, these terms are larger than the uniform-flow terms of Eq. (11.3.19), even if the nonuniformity factors β and δ are small, particularly in directions away from the equatorial plane of the propeller. Therefore, even a modest flow irregularity gives rise to a radiated sound pressure, which, at least at low Mach numbers, is considerably larger than the sound field from the uniform-flow component.

If an aircraft carrying such a propeller is moving through the air with velocity V_a, these formulas must be modified according to the discussion in Sec. 11.2. The motion modifies the directionality of the emitted sound, as measured by an observer at rest with respect to the fluid. For subsonic flight ($M_a = V_a/c < 1$), the angle ϑ in the foregoing formulas corresponds to the angle θ of Fig. 11.9; in other words, the ϑ for the sound which reaches an observer at O at time t is the angle between the direction of flight and the line of sight R measured at the time $t = R/c$, when this sound originated at the aircraft. All the directional factors will be modified by being multiplied by the factor $1/(1 - M\cos\theta)$, which accentuates the amplitudes ahead of the plane and diminishes those behind. In addition, of course, there are the Doppler changes of frequency and the further modifications of amplitude indicated in Eqs. (11.2.33) and (11.2.34).

Compressor noise

An axial-flow compressor such as in an aircraft jet engine is acoustically equivalent to a propeller which is operating in a cylindrical duct with rigid walls. The flow that enters through the rotating blade assembly of the compressor (the rotor) is generally not uniform, but is distorted by stationary guide vanes placed in front of each rotor. As a result, the flow velocity has a pronounced angular dependence, with a period determined by the number of guide vanes. Since the analysis of the sound produced in this case is quite analogous to the previous free-field radiation from a propeller in inhomogeneous flow, we leave the details of the analysis for the reader, and merely quote the result and make some comments about it.

Let the rotating blade assembly be located in a circular duct of radius r_0 in the plane $z = 0$, where z is the coordinate along the axis of the propeller. The angular frequency of the assembly shaft is Ω, and the fundamental angular frequency of the sound produced is

$$\omega_1 = B\Omega$$

where B is the number of blades. As before, we use Eqs. (11.3.7) and (11.3.9) to obtain the expression for the equivalent force distribution in the plane of the propeller. Thus for the axial force per unit area we have

$$f_z = f_{z0} \sum_{s,q=-\infty}^{\infty} \alpha_s \beta_q e^{i(q+sB)\varphi_1} e^{-is\omega t}$$

The tangential component has the same form, with f_z and f_{z0} replaced by f_φ and $f_{\varphi 0}$. The quantities f_{z0} and $f_{\varphi 0}$ are the time averages of the forces. The Fourier coefficients α_s relate to the force distribution over the propeller in uniform flow [see Eq. (11.3.5)] and the β_q's relate to the expansion of the flow inhomogeneity [see Eq. (11.3.11)].

Using the Green's function for the inside of a cylindrical tube (see Prob. 11.15), we find that the sound pressure produced by the force components f is

$$p = \sum_{s,q=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\pi r_0^2} \left(\pm F_t A_{mn}^{sq} + F_d B_{mn}^{sq} \frac{m}{k_{mn} r_0} \right) \times \frac{J_m(\kappa_{mn} r) e^{ik_{mn} z}}{\left[1 - \left(\frac{m}{\kappa_{mn} r_0} \right)^2 \right] J_m^2(\kappa_{mn} r_0)} e^{im\varphi} e^{is\omega_1 t} \tag{11.3.24}$$

where, as before, $m = q + sB$. We have here introduced the total forces

$$F_t = 2\pi \int_0^{r_0} f_{z0}(r_1) r_1 \, dr_1 \qquad \text{and} \qquad F_d = 2\pi \int_0^{r_0} f_{\varphi 0} r_1 \, dr_1$$

and the coefficients A_{mn}^{sq} and B_{mn}^{sq} that express the coupling of these forces to the (m,n)th duct mode, as

$$A_{mn}^{sq} = \frac{1}{2F_t} \int_0^{r_0} \alpha_s \beta_q f_{z0} J_m(\kappa_{mn} r_1) r_1 \, dr_1$$

and a similar expression for B_{mn}^{sq}. The quantities κ_{mn} are the ratios $\pi\alpha_{mn}/r_0$ of Eq. (9.2.23).

Since the axial force produces an asymmetric field around the propeller plane $z = 0$ and the tangential force produces a symmetric one, we must use the plus sign in Eq. (11.3.24) for $z > 0$ and the negative sign for $z < 0$. (See Prob. 11.15 for a discussion of this equation.)

The propagation constant k_{mn}, of Eq. (9.2.6), is now

$$k_{mn} = \sqrt{\left(\frac{s\omega_1}{c} \right)^2 - \kappa_{mn}^2} \tag{11.3.25}$$

It follows that the (m,n)th mode will propagate if $|s\omega_1/c| > \kappa_{mn}$. For homogeneous flow, corresponding to $q = 0$ and $m = sB$, we have $J'_{sB}(\kappa_{mn} r_0) = 0$. For sufficiently large values of sB the smallest root κ_{m1} is approximately

$$\kappa_{m1} \simeq \frac{sB}{r_0}$$

Therefore the condition for the generation of a propagating mode in homogeneous flow is, approximately,

$$\frac{s\omega_1 r_0}{c} > sB \qquad \text{or} \qquad \frac{\Omega r_0}{c} > 1 \qquad \text{or} \qquad \frac{V}{c} > 1 \qquad (11.3.26)$$

where we have used $\omega_1 = \Omega B$. In other words, the condition is, roughly, that the tip speed $V = \Omega r_0$ of the rotor be larger than c. It comes as no surprise that this condition is much the same as that found in the study of sound generation from the moving corrugated board in Sec. 11.1. In most instances the tip speed of the rotor (with a radius equal to the duct radius) is indeed less than the speed of sound. Then, if the flow were strictly uniform across the duct, so that the load on all the blades was the same, the rotor would produce no propagating sound; all the wave modes would decay exponentially away from the rotor.

It is a well-known fact, however, that a jet-engine compressor *does* produce a very intense high-pitched screech which frequently dominates the noise of airports, so that the conclusion just reached about the absence of propagating modes is not applicable. In practice, however, the flow is never completely homogeneous, and modes other than those produced in homogeneous flow will be produced. Some of these modes represent spinning pressure patterns with angular velocities greater than the rotor velocity, as already indicated in Eq. (11.3.21). (The situation is much the same as when a wheel is photographed by a motion-picture camera. Depending upon the relationship between the angular velocity of the wheels, the number of spokes on the wheel, and the number of film frames per second, the projected motion of the wheel can have an angular velocity considerably different from the actual velocity of the wheel.)

Actually, according to this equation, the lth Fourier component in the angular expansion of the flow velocity gives rise to a spinning pressure pattern with an angular velocity $s\Omega B/(sB - l) = \omega_l$, where Ω is the angular velocity of the shaft, and s is the order of the harmonic of the emitted sound. The approximate condition for propagation of the corresponding acoustic mode produced by this pressure pattern, using $\omega_l r_0 > c$, then becomes

$$\frac{V}{c} > \left|\frac{sB - l}{sB}\right| = \left|1 - \frac{l}{sB}\right|$$

In other words, in order for the mode to propagate, the tip speed V of the blade assembly need not exceed the speed of sound. In fact, the fundamental frequency $\omega = \omega_1$ corresponding to $s = 1$ will be produced at all speeds if $l = B$. Then a plane wave will be produced in the duct.

In the discussions of sound generation by a propeller, we have considered so far only the effect of the momentum transfer to the fluid corresponding to a dipole source distribution. Thus, if the pitch angle of the propeller blades is adjusted to give zero thrust of the propeller, this dipole source

distribution will be eliminated. However, there remain other features of the interaction between the propeller and the surrounding fluid that give rise to sound. The fact that a blade occupies a certain volume means that when a blade moves into a region of the fluid, the volume of the fluid is displaced to make room for the blade. This "thickness effect" of the blade gives rise to an acoustic *monopole* source distribution.

To express this distribution in mathematical terms, we assign a propeller thickness function $S(r_1',\varphi_1')$ over the propeller plane, where the prime indicates coordinates attached to the propeller. This function is zero between the blades and equals the blade thickness at the position of the blades. When expressed in terms of the coordinates of the laboratory system, this function becomes $S(r_1, \varphi_1 - \Omega t)$. The rate of displaced air mass per unit area of the propeller plane at the position (r_1,φ_1) is then $Q(r_1\varphi_1) = \rho(\partial S/\partial t) = -\Omega\rho S' = -\Omega\rho(\partial S/\partial\varphi_1')$. The corresponding acoustic source term in the wave equation for the sound pressure is then

$$\frac{\partial Q}{\partial t} = -\Omega\rho S'' = \Omega^2\rho\,\frac{\partial^2 S}{\partial\varphi_1'^2} \qquad (11.3.27)$$

Once the source function is known, we can proceed to compute the radiated field in a manner completely analogous to the dipole field. We leave this calculation for one of the problems.

If we were to carry the analysis of the sound emission from a propeller even further, we should consider the influence of vibrations of the propeller blades. This effect, however, is ordinarily very small, and we shall not analyze it here. Instead, the problem of interaction of flow with an oscillating body will be discussed a little later, in terms of a simpler example.

In the preceding pages we have shown how the complicated flow field about a propeller can be described acoustically in terms of a monopole and a dipole source distribution that can be calculated from well-known characteristics of the propeller. Although these contributions are the dominant ones, and account quite well for experimental observations, there still remain other more complex contributions to the source function. One of them refers to the vortex shedding and turbulence about the blades. However, for an ordinary well-designed blade profile, the corresponding contribution to the sound field has been found to be small, and we shall not discuss it further in this context. Instead, we shall consider a situation in which the vortex sound actually is the predominant one.

Kármán vortex sound (Aeolian tone)

The interaction force between a rigid body in (subsonic) motion relative to the surrounding fluid is largely determined by the Reynolds number $R = VD/\nu$, where D is a characteristic dimension of the body transverse to the flow, V is the relative speed, and ν is the kinematic viscosity μ/ρ.

For sufficiently small Reynolds numbers, say, $R < 100$, the interaction force is mainly the viscous drag on the body which is proportional to the relative velocity. For example, the force on a sphere is given by the Stokes formula $f = 6\pi a\eta V$, which is familiar to physicists from the Millikan oil-drop experiment. Here η is the shear viscosity coefficient, V the velocity, and a the radius of the sphere. As the speed between the body and the fluid is increased, a value is reached at which the flow starts to separate from the body so that a wake is formed behind the body. The boundary between the wake and the surrounding fluid is a shear layer with a large transverse velocity gradient. This layer is unstable and breaks into turbulent motion farther down in the wake.

For large Reynolds numbers, say, $R > 10^5$, the fluid motion in almost the entire wake is random or turbulent. For lower Reynolds numbers, say, for $300 < R < 10^4$, strong periodic components of the fluid motion in the wake are frequently found. For example, the wake behind a cylinder exhibits a distinct transverse oscillatory motion in which vortices are shed off the cylinder alternately on one side and the other, forming the well-known Kármán vortex street in the wake. The trail of eddies is arranged in a zigzag pattern, and during the formation there is a periodic oscillatory transverse momentum component of the fluid motion in the immediate vicinity of the cylinder. (This phenomenon is not uncommon. We are reminded of the oscillatory or zigzag motion of a piece of paper falling through the air or of a flat stone or a lure moving in water. These motions no doubt are related to the Kármán vortex street.) As a result of the transverse oscillatory momentum transfer to the fluid, a sound field is produced equivalent to that of an oscillatory volume force acting on the fluid. The frequency and magnitude of this equivalent volume force can be determined by measuring the reaction force on the cylinder.

Experimentally, the frequency spectrum of the emitted sound in a free field has a predominant peak about a frequency given by

$$\frac{\omega}{2\pi} = \frac{\alpha V}{D} \tag{11.3.28}$$

where V = incident flow speed
D = diameter of cylinder
α = so-called Strouhal number, approximately constant, $\alpha \simeq 0.2$, over a wide range of Reynolds numbers from 10^2 to 10^5

(Occasionally a slight indication of a second-harmonic component is found.) The width of the peak increases with the Reynolds number until $R \simeq 10^5$, by which value the spectrum becomes fairly uniform without any pronounced peaks. The fluid motion in the wake is then fully turbulent.

The magnitude of the transverse force per unit length of the cylinder is of the form

$$F_t \simeq \beta \frac{\rho V^2}{2} D$$

The value of β, found experimentally by various investigators, lies in the range between $\beta = 0.5$ and $\beta \simeq 2$. There can be several reasons for this scatter. In the first place, the degree of turbulence of the incident flow affects the formation of vortices about the cylinder. In fact, only if the incident flow is comparatively free from turbulence will there be a pronounced oscillatory motion. Second, the vortices shed off a cylinder have been found to be in phase over a comparatively short length of the cylinder, of the order of a few cylinder diameters. The correlation length Δ of the vortices has been found to depend on the total length of the cylinder and on the geometry of the flow system employed. Under such conditions some discrepancies between experimental data are to be expected. As a typical average value of β, we shall use $\beta \simeq 1$ in the following discussion. Correspondingly, a typical value of the coherence or correlation length Δ is about three to four times the diameter. The total acoustic power generated by the cylinder is then obtained approximately as the sum of the power contributions from each length Δ of the cylinder.

The wavelength of the emitted sound from the cylinder, using the frequency in (11.3.28), is

$$\lambda \simeq 5 \frac{c}{V} D$$

Since, ordinarily, $c/V \gg 1$, we have $\lambda \gg D$. Thus the wavelength ordinarily is much longer than the correlation length Δ, and the acoustic power emitted by a length Δ of the cylinder is approximately the same as that produced by an oscillatory point force

$$f_t = \beta \frac{\rho V^2}{2} D \Delta e^{-i\omega t} \tag{11.3.29}$$

To be somewhat more precise, f_t is the force applied to a length Δ of the cylinder by the turbulent layer of fluid surrounding it. If the cylinder does not move, this must equal the transverse force acting on the turbulence-free portion of fluid just outside this layer. If both D and Δ are small compared with λ, this can be equated to a point force f_t acting at the origin on moving fluid, with the cylinder absent. The corresponding sound pressure field can be obtained directly from the sound field of a moving dipole source already derived in (11.2.33). If in this expression we introduce the coordinates $\xi = x - Vt$, η, ζ, attached to the cylinder (cylinder along the ζ axis), we obtain the sound field from a stationary cylinder in a fluid moving in the negative ξ direction. The total emitted power per length Δ is then found by

using the force (11.3.29) in (11.2.36):

$$\Pi_a = \frac{\pi}{12}\alpha^2\beta^2\left(\frac{\rho V^3}{2}\Delta^2\right)M^3G_1^2(M) \tag{11.3.30}$$

Ordinarily, the Mach numbers involved are so small that $G_1(M) \simeq 1$.

It is of interest to compare this acoustic power with the total flow-energy loss resulting from the steady drag force on the cylinder. Experimentally, this drag force has been found to be

$$f_d \simeq 1.3\frac{\rho V^2}{2}D\Delta$$

on a length Δ of the cylinder, and the corresponding rate of flow-energy loss (conversion into energy of turbulent flow) is

$$W_d \simeq 1.3\frac{\rho V^3}{2}D\Delta$$

The fraction of this energy that is converted into sound is, using Eq. (11.3.30),

$$\frac{W_a}{W_d} \simeq 0.01M^3\frac{\Delta}{D}$$

Effect of cylinder length

For a cylinder of length L, longer than the correlation length Δ of the oscillatory force along the cylinder, a somewhat more involved calculation is needed to obtain the radiated power. We assume a cylinder of diameter D and length L, its axis coinciding with the z axis and center at the origin, the fluid flow being in the x direction. The observer O (at rest with respect to the cylinder) is a distance r from the origin ($r \gg D, L$), the direction of r being given by the spherical angles ϑ (angle between r and the z axis) and φ (angle between the rz plane and the xz plane), as shown in Fig. 11.20. The driving force per unit length F_t is then pointed along the y axis. If M is small, Eqs. (11.2.6) and (11.2.33), combined with Eq. (11.3.30), indicate that, to the first order in M, the pressure wave at O, from an element of length dz of the cylinder at z, is

$$dp \simeq B(z)\frac{\sin\vartheta\sin\varphi}{(1 - M\cos\psi)^2}e^{ikr-ikz\cos\vartheta}$$

where $B(z) = (k\beta D/16\pi)\rho V^2\exp[ig(z) - i\omega t]$, $k = 2\pi\alpha M/D$, $M = V/c$, and $\cos\psi = \sin\vartheta\cos\varphi$. Function $g(y)$ represents the fact that the phase of the force, and thence of p, is randomly distributed along the cylinder, with correlation length Δ. The total pressure p at O is the integral of this over z from $-\frac{1}{2}L$ to $+\frac{1}{2}L$, and the intensity at O is $|p|^2/\rho c$.

$$I = \frac{\sin^2\vartheta\sin^2\varphi}{\rho c(1 - M\sin\vartheta\cos\varphi)^4}\frac{1}{r^2}\int e^{\xi ik\cos\vartheta}\,d\xi\int B(z_1)B^*(z_1 + \xi)\,dz_1$$

where $\xi = z_2 - z_1$, and both integrals are over the length of the cylinder.

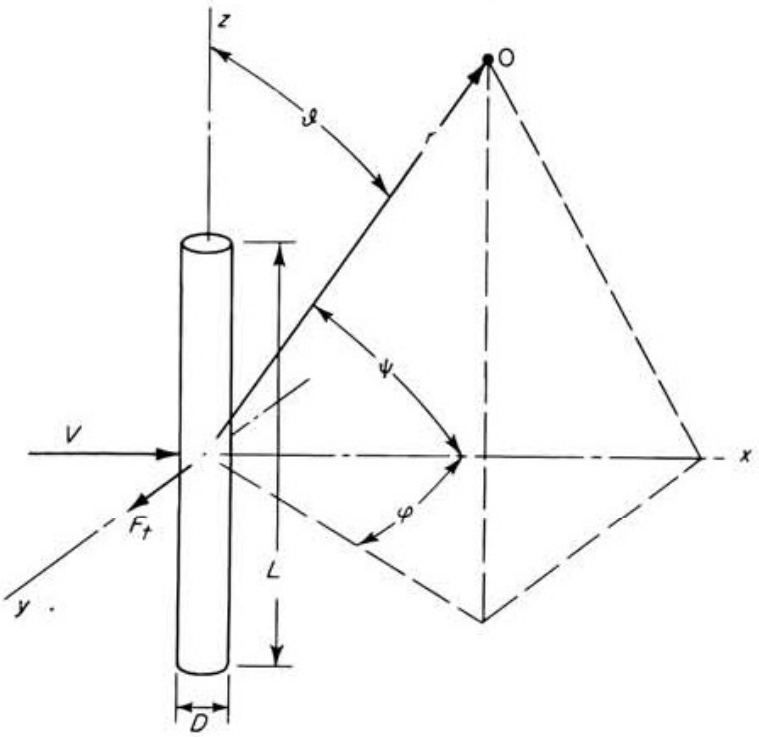


FIGURE 11.20
Cylinder generating an Aeolian tone.

The second integral is, of course, L times the autocorrelation function Υ for B. If the correlation distance along z is Δ, a good approximation for Υ is $|B|^2 \exp(-\xi^2/2\Delta^2)$. Utilization of the familiar formula

$$\int_{-\infty}^{\infty} \exp\left(\frac{-\xi^2}{2\Delta^2} + iu\xi\right) d\xi = \sqrt{2\pi}\,\Delta \exp(-\tfrac{1}{2}u^2\Delta^2)$$

brings us to the desired equation for the intensity at O (assuming $L \gg \Delta$).

$$I \simeq \frac{\sqrt{2\pi}\alpha^2\beta^2}{32r^2}\frac{\rho V^3}{2} M^3 L\Delta \frac{\sin^2\vartheta \sin^2\varphi}{(1 - M\sin\vartheta\cos\varphi)^4} \exp\left[-\frac{1}{2}\left(\frac{2\pi\alpha M\Delta}{D}\right)^2 \cos^2\vartheta\right] \tag{11.3.31}$$

The integral of this over the sphere of radius r is the total power radiated.

$$\Pi \simeq \sqrt{\frac{\pi}{2}}\frac{\pi}{12}\alpha^2\beta^2\left(\frac{\rho V^3}{2} L\Delta\right) M^3 \equiv WLD(k\Delta)$$

When M and $k\Delta$ are small,

$$k\Delta = \frac{2\pi\alpha M\Delta}{D} \qquad W = \sqrt{\frac{\pi}{2}}\alpha\beta^2\frac{\rho V^3}{48} M^2$$

which (aside from the numerical factor $\sqrt{\pi/2} \simeq 1.25$) differs from the result of Eq. (11.3.30), for L not large compared with Δ and λ, only by the factor L/Δ. For a long cylinder the power radiated is proportional to its length (not the square of its length), as well as to the correlation length.

When $2\pi\alpha M\Delta/D = k\Delta$ is less than unity (and M is small), the angle dependence of the intensity is, roughly, the same as for a point dipole pointed in the y direction. This is the usual result, when the fluid flow past the cylinder has enough irregularities so that the correlation length Δ is roughly equal to the cylinder diameter D. On the other hand, if the flow is smooth enough so that Δ is considerably longer than D, $k\Delta$ may be larger than unity, and the exponential function will concentrate the radiation near the equatorial plane ($\vartheta = 90°$). When this is so,

$$\Pi \simeq \frac{\pi}{32}\alpha\beta^2\left(\frac{\rho V^3}{2}LD\right)M^2 = \frac{4}{3\sqrt{2\pi}}WLD \qquad M \text{ small} \qquad k\Delta \text{ large}$$

which is larger than the Π for $k\Delta$ small, but not as large as the ratio between the $k\Delta$'s. In fact, when $k\Delta > 1$, Π becomes independent of $k\Delta$.

The occurrence of an oscillatory wake behind a body is not limited to cylinders only. The basic condition for this phenomenon is the presence of a shear layer regardless of the shape of the body that produces the wake. The characteristic dimension D entering into the formulas (11.3.28) and (11.3.29) more appropriately should be the size of the wake (separation between two opposite shear layers). Thus, for a plate inclined an angle θ with the flow direction, we have $D \simeq L \sin\theta$, where L is the width of the plate.

Stimulated vortex sound; whistles

The fluid oscillations about a cylinder and the corresponding sound field that we have just discussed concerned the conditions in free field, and may be referred to as *spontaneous* acoustic emission from the cylinder. This is in contrast to what might be called *stimulated* generation of vortices and emission of sound when a sound field is present to reinforce the fluid motion about the cylinder. The reinforcing sound may be externally produced, or it may be the sound from the cylinder itself, which has been reflected from surrounding surfaces. Under proper conditions this reflected sound field will stimulate the formation of vortices and thus increase the intensity of the emitted sound.

Such an effect can be realized in a tube with fluid flow if a cylindrical rod is placed in the tube perpendicular to the axis. If the flow speed is adjusted to a value such that the frequency of vortex shedding coincides with one of the transverse resonances in the tube, an intense sound field is produced. The amplitude of the sound field is found to be considerably larger than would be expected on the basis of forced motion of the tube, using the free-field value of the oscillating force as a source function. Therefore the sound emitted from the cylinder, after reflection from the tube walls, is not merely a matter of resonant excitation of a duct by a given dipole source, but rather the result of a reaction of the sound field back on the flow about the cylinder, drawing on the energy of the fluid flow.

Through this feedback mechanism the oscillatory force on the cylinder will depend on the amplitude of the sound, the system can become unstable, and large amplitudes can result. The amplitude presumably will be limited mainly by acoustic losses due to viscous and conduction effects considered in Chap. 6. Such a possibility for transforming the energy of steady flow into acoustic energy is of considerable practical interest, since it has been found to be responsible for the violent oscillations sometimes encountered in industrial test and processing facilities involving flow through tubes and ducts. Not only will enormous sound emission result (sometimes several kilowatts of acoustic power), but mechanical failure of the duct systems frequently has been the result.

A similar instability will occur even in free field if, instead of being rigid, the cylinder is flexible in the transverse direction, with a resonance frequency coincident with the frequency of vortex shedding. In principle, this situation is analogous to what occurred in the famous Tacoma Bridge disaster, in which enormous transverse oscillations occurred as the result of such an instability.

The conditions for stimulated sound emission from the vortex flow around a rod of diameter d set in a duct are easily obtained by setting the Kármán vortex frequency equal to the transverse resonance frequencies of the tube. In the case of a rectangular duct with a transverse dimension D, the transverse resonance frequencies are given by $f_n = nc/2D$. Therefore, using $f \simeq 0.2V/d$ for the vortex frequency as produced by the rod, given in Eq. (11.3.28), the condition for instability in the tube can be expressed as

$$\frac{V}{c} \simeq 2.5n\frac{d}{D} \qquad \text{or} \qquad \frac{V}{d} \simeq 2.5n\frac{c}{D} \tag{11.3.32}$$

An additional condition is that the Reynolds number must lie in a region given approximately by $10^2 < R < 10^5$. It is convenient here to express the Reynolds number as $R = Vd/\nu = \dfrac{V}{c}\dfrac{d}{l}$, where $l = \nu/c$ is of the order of the mean-free-path in the case of a gas, and $\nu = \mu/\rho$ is the kinematic viscosity. Then, introducing the value $V/c = 2.5nd/D$ from Eq. (11.3.32), we can express the additional condition for instability as

$$10^2 < 2.5n\frac{d^2}{Dl} < 10^5 \tag{11.3.33}$$

For a gas at atmospheric pressure, $l \simeq 10^{-5}$ cm, and we get

$$10^{-3} < 2.5n\frac{d^2}{D} < 1 \qquad \text{cm}$$

As an example, a cylindrical rod with a diameter $d = 1$ in. is used in an air duct with $D \simeq 1$ ft; the first resonant instability ($n = 1$) will occur at a flow speed of about 24 ft per sec; and the frequency of oscillation is about 550 cps.

The $n = 2$ mode will be excited at a velocity of about 48 ft per sec with a frequency of about 1,100 cps, etc. The spectrum of the sound in each case contains not only the fundamental frequency, but a number of intense harmonics as well (as many as 10 harmonics have been observed).

As an illustration of the second type of vortex-whistle mechanism, we shall consider the sound emitted from two or more identical cylindrical rods, with their axes perpendicular to the flow, placed one after the other along a line in the direction of the flow. The vortex street produced by one rod interacts with the next rod. This interaction gives rise to a pressure disturbance which affects the vortex shedding by the first rod. If the pressure pulse arrives at the time of formation of a new vortex in a phase such as to stimulate the formation, the vortex is amplified, and through this feedback mechanism, strong acoustic emission results.

To obtain the condition for such a "resonance," it is important to realize that a vortex in the wake of a rod does not travel with the velocity V of the fluid, but somewhat slower. Actually, for a cylindrical rod the vortex drift speed is $V_d \simeq 0.8V$. If the distance between two rods is L, the vortex drift time between the rods is L/V_d. The pressure pulse, produced when a vortex encounters the second rod, will reach the first rod after an additional time L/c, where c is the sound speed. The total time elapsed between the formation of a vortex at the first rod and the arrival of the feedback pressure pulse from the second rod is then $(L/V_d) + (L/c) \simeq L/V_d$, where we have assumed $V_d < c$. The feedback pulse has been found to stimulate vortex formation when this total time delay equals a multiple of the vortex shedding period, $T = 5d/V$, where d is the rod diameter. The condition for resonance is then

$$\frac{L}{V_d} = 5n\frac{d}{V}$$

or with $V_d \simeq 0.8V$,

$$L \simeq 4nd \tag{11.3.34}$$

Another way of interpreting this result is to say that resonance will occur when the distance L between two rods is a multiple of the distance between two consecutive vortices in the wake behind a rod.

In contrast to the first vortex-whistle mechanism, in which reflection of sound from the walls of the duct provided the feedback mechanism, the present mechanism relies upon the interaction of a vortex with a second object. The characteristic length of significance in the feedback processes is the distance between the vortex source and the vortex receiver, and the characteristic velocity is the vortex drift speed. Most vortex whistles can be described in terms of either one of these two types, or sometimes a combination of both. For example, the well-known jet-edge whistle is analogous to the two-rod whistle. Here a thin jet is emitted from a slit with sharp edges. Because of the shear layers produced, the jet is unstable for transverse

perturbations, and the speed of these perturbations traveling down the jet is equivalent to the vortex drift speed in the previous example. The characteristic length involved is the distance between the slit and the edge which is placed in front of the slit. Resonance occurs when this distance is a multiple of the wavelength of the perturbations of the jet. For a given L, the whistle can operate in different modes, in which the distance between slit and edge is 1, 2, 3, etc., resonant wavelengths.

Similar oscillations occur also if, instead of a slit, we have an annular opening and a circular edge in front of the annulus.

Even the flow through a hole in a rigid partition, with a thickness comparable with the hole diameter, gives rise to feedback oscillations. The edge at the entrance of the hole serves as the vortex generator, and the edge at the other end of the hole as the vortex receiver, so that the characteristic length involved is the thickness of the partition. If a cavity resonator is added to this vortex generator, an intense sound can be produced.

The well-known excitation of cavity resonance by flow across the cavity opening involves a combination of the two basic mechanisms described above. The flow that passes the opening of the cavity defines the characteristic time V/d as in the jet-edge whistle. When this time equals the time of travel of an acoustic pulse being reflected from the walls of the cavity, strong resonance occurs; the approximate condition for oscillation in this case is $V/d = L/c$.

Effect of flow on acoustic impedances

To illustrate one effect of flow on an oscillating body, consider a one-dimensional mass-spring oscillator vibrating in a fluid which is moving in the direction of motion of the oscillator. If the velocity of the mass in the oscillatory motion is u, and the incident flow velocity is V, the force on the body caused by the flow is of the form

$$F = AC_d \frac{\rho(V-u)^2}{2} = AC_d \frac{\rho(V^2+u^2)}{2} - AC_d\rho Vu \qquad (11.3.35)$$

where A is the area of the oscillating body, and C_d is the drag coefficient, discussed earlier in connection with the force on a propeller blade. Considering C_d to be independent of the velocity, it follows from Eq. (11.3.35) that the presence of the flow produces a time-dependent force on the oscillator which is equivalent to a friction force proportional to the velocity u of the oscillator. In addition, there is a force proportional to u^2 which contains twice the frequency of the oscillator, but if $u \ll V$, this force is of little importance. Nonlinear contributions to the force of this type will be discussed further in Chap. 13.

The acoustical characteristics of several other mechanical oscillators are influenced in a similar manner by flow. For example, the free oscillations of strings, membranes, and plates have been found to decay more rapidly

when in relative motion with respect to the surrounding fluid. (There are conditions under which the flow can produce self-sustained oscillations of such systems, but this phenomenon will not be considered here.)

We return to the analysis of sound transmission through an aperture, discussed earlier in Chap. 9, to inquire about the effect of flow through an aperture on its acoustical impedance. To study this question, we must first discuss the behavior of steady flow through the aperture. As in the case of flow interaction with any other rigid body, discussed earlier, we have to resort to experimental facts. At very low speeds the fluid motion no doubt is laminar, but for the velocities generally encountered in practice, the flow forms a turbulent jet on the downstream side of the aperture. The critical Reynolds number $R = VD\rho/\mu$ at which flow separation and jet formation occur depends on the sharpness of the edges of the aperture, but generally the value is of the order of $R \simeq 100$.

The pressure in the jet at the hole is somewhat lower than the ambient pressure P_2 on the downstream side of the plate, but the jet pressure quickly adjusts to the ambient pressure as the jet becomes turbulent.

To get an idea of the effect of a periodic perturbation on the orifice flow, we shall consider sufficiently low frequencies so that we may neglect compressibility of the fluid in the aperture region. The flow in the region between the upstream side of the plate and point 2 in Fig. 11.21, just before the flow becomes turbulent, is laminar, and we can apply the momentum equation in the form of Bernoulli's law for an incompressible fluid [Eq. (6.2.12)].

$$-\rho \frac{\partial \psi}{\partial t} + \frac{\rho}{2} V^2 + P = \text{const}$$

where ψ is the velocity potential, and $V = -\text{grad } \psi$. Denote the incident flow speed and pressure by V_1 and P_1, and the velocity of the jet at point 2 by V_2. The pressure at this point approximately equals the ambient pressure P_2 on the downstream side of the plate. Furthermore, if s is the coordinate along a stream line, and V_s the corresponding velocity component (say, along the axis of the orifice), we have $\psi = -\int V_s \, ds$. From Bernoulli's law, and

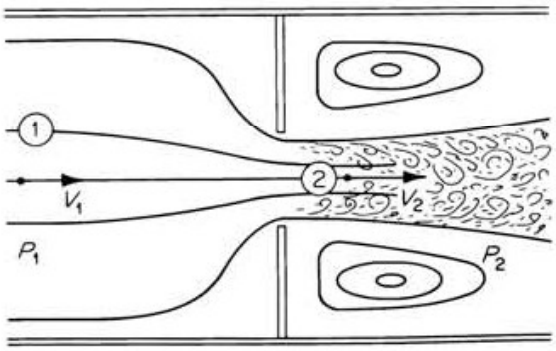


FIGURE 11.21
Flow through an aperture with a superimposed sound field.

using the continuity relation $A_1V_1 = A_2V_2$,

$$\rho\int_1^2 \frac{\partial V_s}{\partial t}\,ds + \frac{\rho}{2}\,V_2^2\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = P_1 - P_2 \qquad (11.3.36)$$

where A_2 and A_1 are the cross-sectional areas at the stations 1 and 2, as indicated in Fig. 11.21.

Next we consider a small perturbation of the flow velocity, such that $V_s = V_s^0 + u_s$, with corresponding perturbations u_2, u_1, Δp in V_2, V_1, and $\Delta P = P_1 - P_2$. This perturbation represents a sound wave (incident) on the plate on the upstream side of the orifice. It follows from (11.3.36) that

$$\rho\int_1^2 \frac{\partial u_s}{\partial t}\,ds + \rho V_2^0 u_2\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \Delta p$$

or

$$\frac{1}{\rho c}\frac{\Delta p}{u_1} = \frac{V_1^0}{c}\left(\frac{A_1}{A_2} - \frac{A_2}{A_1}\right) + \frac{\rho}{\rho c u_1}\int_1^2 \frac{\partial u_s}{\partial t}\,ds \qquad (11.3.37)$$

This expression represents the specific impedance of the orifice. The first term, which represents a resistance, accounts for the fact that the modulation of the jet by an oscillatory flow increases the rate of energy converted into turbulence. The second term is the mass reactance of the aperture, proportional to the M_a of Eq. (9.1.22). If we set $u_s = c(s)u_1$ and consider harmonic time dependence, we can express the integral as $-i\omega\rho l/\rho c$, where $l = \int_1^2 c(s)\,ds$ is a characteristic length that corresponds to the mass end correction of the aperture expressed in a different way in Sec. 9.1 [see, for example, Eqs. (9.1.26) and (9.1.28)]. The resistive term is proportional to the unperturbed flow velocity V_1^0. The reactive term corresponds to an end correction resulting from the convergence of the flow on the upstream side of the aperture. It is expected to be approximately one-half of the mass end correction in the absence of steady flow, because in the latter case the end corrections account for the flow convergence (and divergence) on *both* sides of the aperture. Measurements of orifice impedance in the presence of flow are in good (qualitative) agreement with the result in Eq. (11.3.37).

11.4 GENERATION OF SOUND BY FLUID FLOW

We have already considered some acoustical aspects of fluid motion that concerned the interaction of flow with a rigid body; a nonstationary flow impinging upon a body produces a time-dependent pressure distribution about the body, and as a result, sound is generated. Even in the absence of rigid bodies or boundaries, however, pressure fluctuation will be produced in the main body of a time-dependent flow field. This follows, for example,

from Bernoulli's law, discussed in Chap. 6 [Eq. (6.2.12)]. The pressure fluctuations in the nonstationary fluid-flow region give rise to (small) density fluctuations which propagate as sound to other regions of the fluid that may or may not be in motion. In any event, a nonstationary region of a fluid acts as a sound source, and the question that we wish to study in this section is the determination of the acoustic source strength and the corresponding sound field in terms of the flow field. If the fluid is turbulent, the velocity field and the pressure and density fluctuations are random functions of space and time, and only statistical properties (such as autocorrelation functions) of these quantities can be measured or predicted. Correspondingly, we can describe the sound field only in statistical terms. The results, of course, are similar to those carried out in Chap. 7 in connection with sound radiation from random source distributions.

Qualitative considerations

Before we get involved in mathematical analysis, however, we shall discuss those order-of-magnitude relationships between fluid motion and sound that can be understood from simple qualitative considerations. The mechanism of sound generation in the bulk of a turbulent fluid is different from most of the sources considered so far, inasmuch as the sound is not a direct result of the interaction of the fluid with vibrating boundaries or with external energy sources. In the present case the time dependence of the flow, required for sound emission, is caused by the instability of the fluid flow, so that even if the flow field is produced by a time-independent energy source, flow fluctuations and sound will be produced as a result of the flow instability.

A local fluctuation $\mathbf{v}$ in the fluid velocity will produce a local pressure fluctuation, which (from Bernoulli's law) will be of the order of ρv^2. This local-pressure-fluctuation field acts as a (monopole) source of sound and gives rise to a sound pressure, a distance R away, of the order of $p \simeq (d/R)\rho v^2$, where d is the size of the region over which the pressure fluctuation occurs. The corresponding radiated power $4\pi R^2(|p|^2/\rho c)$ then will be of the order of

$$\Pi_0 \approx \rho v^2 d^3 \frac{v}{c}\frac{v}{d} \tag{11.4.1}$$

Since $d/v = T$ is the characteristic time of fluctuation, this can be interpreted as the fraction v/c of kinetic energy $\rho v^2 d^3$ transformed into sound per unit time. Hence the "acoustic efficiency" of this flow monopole source is of the order of v/c, and the total acoustic output is proportional to the *fourth power* of the velocity fluctuation.

When we consider a larger volume V of the turbulent fluid rather than a (small) volume element d^3, this equation for Π_0, with V substituted for d^3, is valid only if there is a net pressure and corresponding density fluctuation in phase over the entire volume. If the turbulent region is isolated or surrounded by a quiescent region, however, there is no net mass transfer into

the region, so that the net monopole strength will be zero. The pressure fluctuations in the fluid then will be distributed throughout the region, with positive and negative contributions which average to zero.

Next let us consider a pair of such flow monopoles, regions of pressure fluctuations with opposite signs. In Chap. 7 we have seen that two monopole sources of opposite sign represent a dipole source which produces a sound pressure of the order of d/λ times the monopole pressure, where d is the separation of the monopoles and λ the acoustic wavelength. For simplicity we have assumed this separation to be the same as the characteristic "size" of the monopole region used above. It follows, then, that the acoustic power emitted from a "flow dipole" will be of the order $(d/\lambda)^2$ times the power from the monopole. With a characteristic time of flow oscillations $T \simeq d/v$, we have $\lambda = c/T \simeq (c/v)d$, and we obtain for the acoustic power from the flow dipole, using (11.4.1),

$$\Pi_2 \approx \rho v^2 d^3 \left(\frac{v}{c}\right)^3 \frac{v}{d} \tag{11.4.2}$$

We see that, in terms of the interpretation used for the monopole, the acoustic efficiency of the flow dipole is of the order of $(v/c)^3$, and the total emitted acoustic power is proportional to the *sixth power* of the velocity fluctuation.

This conclusion applies also to a finite region of the fluid if the region has a net flow-dipole strength. Since, as we have seen in Chap. 7, a dipole is equivalent to a force on the fluid, a finite dipole strength of the fluid region implies that there is momentum transfer from the region to the surrounding fluid. However, in the absence of external forces or boundaries acting on the fluid region, or if it is surrounded by a quiescent region of fluid, there will be no net momentum transfer and no net dipole strength. Therefore, in a region of velocity correlation in the fluid, two or more flow dipoles will occur, with opposite signs, so that the net dipole strength also is zero.

Thus, in the absence of both a net density fluctuation and a net momentum transfer to the (turbulent) flow region under consideration, the sound emission must be a result of quadrupole and higher-order sources (as was stated in Sec. 7.1). The acoustic pressure Π_4 from a flow quadrupole is of the order of $d/\lambda = d/(c/T) = v/c$ times the dipole pressure, and therefore the radiated power is of the order of $(v/c)^2$ times Π_2 in Eq. (11.4.2). Thus

$$\Pi_4 \approx \rho v^2 d^3 \left(\frac{v}{c}\right)^5 \frac{v}{d} \tag{11.4.3}$$

This expression corresponds to the radiation from a single region of flow correlation (an "eddy") in the fluid of volume d^3. The acoustic efficiency is now proportional to $(v/c)^5$. In contrast to the monopole and dipole formulas, the quadrupole formula *can* apply to a finite region of the fluid, and was applied in obtaining Eqs. (7.1.28) and (7.1.37). The acoustic power from it is merely the sum of the power contributions from the various

uncorrelated eddies in the fluid. The actual expression for this integrated power, although dimensionally of the form (11.4.3), involves additional constants which cannot be found from these simple considerations. For a region of isotropic turbulence we shall see that the constant factor to be used in (11.4.3) is of the order of 10^{-2}.

Thus, for subsonic velocity fluctuations with $M < 1$, the power removed from an eddy through sound radiation is generally very small. For example, with $v/c = 0.1$, the sound emission will remove about 10^{-5} percent of the kinetic energy of an eddy in one period of oscillation T. If sound were the only damping mechanism, the "lifetime" of an eddy would be large indeed. There are more important mechanisms, however, which damp out the eddy motion. For one thing, an eddy will feed energy to smaller eddies, which ultimately, through viscosity, go into heat.

Applying Eq. (11.4.3), we can get an idea about the sound emission from a turbulent jet. The fluctuating velocity component in the jet is known to be proportional to the average flow velocity V of the jet; typically, we have $v \approx 0.1V$. Furthermore, the size of the energy-carrying eddies in the flow is of the order of the diameter D of the jet. The quantity $\rho v^2 d^3(v/d)$ in Eq. (11.4.3), then, is proportional to $\frac{\rho V^2}{2}\frac{\pi D^2}{4}V$, which is the kinetic energy per second being discharged from the jet nozzle. A certain fraction of this power will be radiated as sound, and it follows from (11.4.3) that the sound power is of the form

$$\Pi = (\text{const})\, M^5 \frac{\rho V^3}{2} A \qquad M = \frac{V}{c} \tag{11.4.4}$$

where A is the nozzle area. A large number of experiments have shown that the constant of proportionality in this formula is of the order of 10^{-4} for subsonic circular jets. The efficiency of conversion of jet energy into sound is then about $10^{-4}M^5$, which, for subsonic motion, is quite small. The jet energies involved, however, are often very large, and the acoustic powers emitted can be considerable, often of the order of kilowatts. This should be compared with the acoustic power from a human voice, which is about 10^{-6} watt.

Further analysis

We have already treated the radiation from a randomly fluctuating and turbulent region in Sec. 7.1, and we refer to that section as one approach to the mathematical analysis of our problem of sound generation by turbulence. We shall supplement that analysis by presenting here a somewhat different approach, plus some additional observations regarding this complex problem. There is no significant loss of generality if viscosity and heat conduction are left out of the discussion, and we shall do so. In the absence of an external force, viscosity, and heat conduction, the general equations of fluid motion

(6.1.13) and (6.2.11), expressing mass and momentum balance in the fluid, can be formulated as

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial(\rho V_i)}{\partial x_i} = 0$$
$$\frac{\partial(\rho V_i)}{\partial t} + \sum_j \frac{\partial(\rho V_i V_j)}{\partial x_j} = -\frac{\partial P}{\partial x_i} \tag{11.4.5}$$

where V_i ($i = 1, 2, 3$) stands for the components of the total fluid velocity, and ρ and P are the total fluid density and pressure. Under isentropic conditions the equation of state of the fluid can be expressed as $P = P(\rho)$, the pressure being a function of density only, and the compressibility of the fluid is given by

$$\kappa = \frac{1}{\rho}\frac{d\rho}{dP} \tag{11.4.6}$$

By differentiating the first of Eqs. (11.4.4) with respect to t and the second with respect to x_i, we obtain, using (11.4.6),

$$\rho\kappa \frac{\partial^2 P}{\partial t^2} - \sum_{ij} \frac{\partial^2(\rho V_i V_j)}{\partial x_i \partial x_j} = \sum \frac{\partial^2 P}{\partial x_i^2} = \nabla^2 P \tag{11.4.7}$$

where, as we recall from Eq. (6.2.4), $c^2 = 1/\rho\kappa$. In the absence of an external force field, viscosity, and heat conduction, this equation is a version of the general "wave equation" (7.1.21).

Both the compressibility κ and the density ρ, even in the absence of sound in the fluid, may be functions of position [Eq. (7.1.22)]. However, we have neglected a possible time dependence of $\rho\kappa = 1/c^2$. Before we proceed further, let us consider some general properties of Eq. (11.4.7). First we observe that if the compressibility is zero, the pressure field is fixed by the velocity field via the equation $\nabla^2 P = -\Sigma(\partial^2 \rho V_i V_j / \partial x_i \partial x_j)$; the consequent pressure then is of the order of ρV^2. Under these conditions the ratio between the first and second term is of the order of $(L/T)^2/c^2$, where we have introduced a characteristic time T of pressure fluctuation in the fluid. In a turbulent fluid this characteristic time is of the order of $T \simeq L/V$, where L is the characteristic length L (correlation length), so that $(L/T)^2/c^2 \simeq (V/c)^2$. Under subsonic conditions, so that $V/c \ll 1$, compressibility has only a small effect, and the pressure field therefore is essentially the same as in an incompressible fluid. This pressure field we shall refer to as the *hydrodynamic*, or *aerodynamic*, pressure field. It is characterized by the property that the pressure fluctuation is proportional to the *square* of the velocity fluctuation. This is in contrast to a sound field, in which the sound pressure is proportional to the *first power* of the velocity fluctuation.

In a moving compressible fluid we generally have both hydrodynamic and acoustic pressure fluctuations; their relative strength ordinarily depends on the location of the point of observation. For example, inside a violent

region of motion such as in a turbulent jet, the hydrodynamic pressure fluctuations predominate, whereas sufficiently far outside this region, the pressure in the sound field, emitted from the violent region of motion, is larger than the local hydrodynamic pressure fluctuations. In this far-field region the hydrodynamic motion of the fluid generally is very small, and the characteristic period of oscillation is long, compared with the periods in the sound field which has been generated by the rapid fluctuations in the violent region. If the typical angular frequency of the sound field is ω, the first term in Eq. (11.4.7) is of the order of $\omega^2 p/c^2 \simeq \omega^2\delta$, where δ is the density fluctuation $\delta = p/c^2$. Similarly, the second term can be expressed as $\rho\Omega^2$, where we have introduced the characteristic frequency of the fluid motion $\Omega = V/L$. Consequently, the condition for predominance of the sound pressure field can be expressed as $\dfrac{\delta}{\rho}\left(\dfrac{\omega}{\Omega}\right)^2 \gg 1$.

But even if this condition is fulfilled, we cannot completely ignore the hydrodynamic motion of the fluid, because there will be some coupling between the fluid motion and the acoustic field, resulting in refraction and scattering of the sound field throughout the entire fluid (as discussed in Sec. 8.1). However, in many cases, as for a jet, there is a comparatively sharp boundary between the region of fluid motion and the outside "quiescent" region, and the refraction effects will be localized at the boundary. The refraction is then approximately the same as for a discontinuous change in fluid velocity, already considered in Sec. 11.1.

To account for all the inhomogeneities often encountered in, for example, turbulent jets, we let κ and ρ be functions of position. If the space average values are κ_0 and ρ_0, Eq. (11.4.7) can be written in the form

$$\rho_0\kappa_0\frac{\partial^2 P}{\partial t^2} - \nabla^2 P = \rho_0(\kappa_0 - \kappa)\frac{\partial^2 P}{\partial t^2} + \left(\frac{\rho_0}{\rho} - 1\right)\nabla^2 P + \frac{\rho_0}{\rho}\frac{\partial^2 \rho V_i V_j}{\partial x_i \partial x_j} \equiv S(\mathbf{r},t) \tag{11.4.8}$$

We leave the derivation for one of the problems [see also Eq. (8.1.11)].

Interaction between turbulence and sound

The foregoing discussion serves as a convenient starting point for most problems of sound generation and transmission in a loss-free inhomogeneous fluid, when external volume forces and heat sources are absent. Although it may be difficult to establish a clear separation between the "hydrodynamic" and "acoustic" contributions to the various field variables, the handling of this question in most cases of interest is fairly obvious. Suppose, for example, we are interested in the transmission of an acoustic signal from an external source through an inhomogeneous moving fluid, the state of which is time-independent. The sound wave then is simply represented by the time-dependent part of the field, and if the signal is sufficiently small, it can be treated as a perturbation to the fluid motion, as was done in Sec. 8.1.

The terms on the right-hand side in Eq. (11.4.8) then account for the scattering and refraction of sound from inhomogeneities in compressibility and density and from the motion of the fluid. The latter effect is seen if we set $V_i = U_i + u_i$, where $\mathbf{U}$ is the unperturbed velocity, and $\mathbf{u}$ the perturbation. The cross terms of the form $U_i u_j$ then provide the coupling between the sound wave and the fluid motion.

Another situation, in which the method of separating hydrodynamic and acoustic fields is fairly clear, refers to the generation of sound by a well-defined region of fluid motion, such as a turbulent jet outside which the fluid is practically at rest. To obtain the sound field in the outside region, we integrate the source term multiplied by the appropriate Green's function over the source region. In this region the hydrodynamic portions of the source term may be assumed predominant and known *a priori*. The acoustic contribution to the source term in the source region accounts for the scattering and refraction of the emitted sound that it undergoes before reaching the outside region. However, we shall ignore these effects in this discussion, and consider the source terms in Eq. (11.4.8) as known hydrodynamic variables in the source region, at least in a statistical sense.

Then, using an appropriate Green's function, we can express the radiated sound field directly in terms of integrals, as we have done several times before in this book. For example, in Eq. (7.1.23), we worked with the Fourier-transformed wave equation, and in the absence of boundaries, the appropriate Green's function was g_ω, given in Eq. (7.1.6). We shall now handle the problem a little differently and use the wave equation in its original form, without Fourier transformation [see also the derivation of Eq. (8.1.24)]. Then, in the absence of boundaries, the appropriate Green's function is as given in Eq. (7.1.18),

$$g(\mathbf{r},t \mid \mathbf{r}_0,t_0) = \frac{\delta\{t_0 - [t - (R/c)]\}}{4\pi R}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$. If, for simplicity, we neglect inhomogeneity of ρ and κ in (11.4.8), the solution to this equation then can be written

$$p(\mathbf{r},t) = \int dt_0 \int dv_0 \left(\sum \frac{\partial^2 \rho V_i V_j}{\partial x_i \partial x_j} \right) \frac{\delta\{t_0 - [t - (R/c)]\}}{4\pi R} \qquad (11.4.9)$$

where the volume element dv_0 refers to the source region. Through integration by parts, the far-field contribution in (11.4.9) can be expressed as

$$p(\mathbf{r},t) = \frac{1}{4\pi R c^2} \frac{\partial^2}{\partial t^2} \int (\rho V_r^2)\, dv_0 \qquad (11.4.10)$$

which is analogous to Eq. (7.1.28). Here $V_r = \Sigma x_i V_i / r$ is the component of the fluid velocity $\mathbf{V}$ in the direction $\mathbf{r}$ from the source region to the point of observation, and the parentheses following the integral indicate that the

quantity inside is to be evaluated at the time $t - |\mathbf{r} - \mathbf{r}_0|/c$. It is interesting that only the velocity component V_r contributes to the radiated sound field at $\mathbf{r}$. This is related to the angular distribution of the sound field from the elementary quadrupoles in the region. Recall, for example, that the radiated sound field from a dipole is zero in a direction perpendicular to the direction of the dipole, and similar results apply to the quadrupoles.

If rigid boundaries are present so that $\partial P/\partial n = 0$, where $\mathbf{n}$ is the unit vector perpendicular to the boundary, it is advantageous to choose a Green's function G such that $\partial G/\partial n = 0$, as we have seen already in connection with (7.1.17). If we use this Green's function, there will be no contribution from integrals over the boundary. If, then, for example, we deal with a known turbulent flow field outside a boundary, we need not be concerned about the interaction between the flow and the boundary and the corresponding pressure distribution along the boundary. On the other hand, in some cases it may be of interest to relate the sound field to the pressure fluctuations over the boundary, and another choice of Green's function may then be appropriate. In this context we recall the discussion of sound generation from the Kármán vortex street in the wake behind a cylinder, in Sec. 11.3. In that case, the sound, in fact, is generated by the volume source distribution in the oscillatory flow in the vicinity of the rigid cylinder, but we replaced this source distribution by its net dipole strength, determined indirectly in terms of the reaction force on the cylinder. The validity of this equivalence can be formally proved, but we shall not pursue it here.

If, in Eq. (11.4.9), instead of expressing P in terms of $\mathbf{V}$ as we did earlier, we now express $\mathbf{V}$ in terms of P, using Eq. (11.4.7), the source term becomes a monopole distribution $\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} - \nabla^2 P$. Then, if we use the free-space Green's function of Eq. (11.4.10) and integrate by parts, the volume integral contributions to the far field from $\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}$ and $\nabla^2 P$ cancel each other, except for a term $\int \text{div}\,(g\ \text{grad}\ P)\,dv$, which can be transformed to a surface integral over the surface that encloses the source region. Then Eqs. (11.4.4) and (11.4.5) show that the sound pressure in the far field can be expressed as

$$p(\mathbf{r},t) = \frac{1}{4\pi r}\frac{d}{dt}\int (\rho V_n)\,dv_0 \qquad (11.4.11)$$

where V_n is the flow velocity component normal to the surface of integration. As before, the parentheses following the integral indicate that the quantity inside is to be evaluated at $t - |\mathbf{r} - \mathbf{r}_0|/c$. If the surface of integration is taken outside the turbulent region, the velocity component that enters is the acoustic velocity $\mathbf{u}$, and the formula then is merely an expression of Huygens' principle.

Description of turbulence

In a turbulent fluid the motion is random, and the velocity field, and hence the radiated sound pressure field, are stochastic functions; the detailed time dependence cannot be expressed in terms of simple functions. The procedure in calculating the corresponding emitted acoustic power was discussed in Chap. 7 for a particular form (Gaussian) of the correlation function of the velocity fluctuations. We shall extend the calculation here to consider the so-called (Kolmogorov) inertial subrange of isotropic turbulence, and obtain an expression, not only for the total intensity, but also the frequency spectrum. These quantities are conveniently obtained from the autocorrelation function for the sound pressure at the point of observation. This function is defined as

$$\Upsilon(\tau) = \langle p(\mathbf{r},t)p(\mathbf{r},t-\tau)\rangle \tag{11.4.12}$$

where the brackets indicate time average [see Eq. (1.3.13)]. From it we obtain the total acoustic intensity spectrum $E(\omega)$ from the Fourier transform

$$E(\omega) = 2\pi\rho c \int_{-\infty}^{+\infty} \Upsilon(\tau) e^{i\omega\tau}\, d\tau \tag{11.4.13}$$

and the total acoustic intensity is given by

$$I = \frac{\Upsilon(0)}{\rho c} = \frac{\langle p^2\rangle}{\rho c} = \int_{-\infty}^{+\infty} E(\omega)\, d\omega \tag{11.4.14}$$

If we express the sound pressure in terms of the fluid velocity field and introduce the notation

$$T_r = \rho V_r^2$$

the correlation function (11.4.12) can be expressed as [see Eq. (11.4.10)]

$$\Upsilon(\tau) = \left(\frac{1}{4\pi r c^2}\right)^2 \iiint K_t(\mathbf{r}_0,\mathbf{r}_1,\tau)\, dv_0\, dv_1 \tag{11.4.15}$$

where

$$K_t(\mathbf{r}_0,\mathbf{r}_1,\tau) = \left\langle \frac{\partial^2 T_r(\mathbf{r}_0,t_0)}{\partial t^2} \frac{\partial^2 T_r(r_1,t_1)}{\partial t^2} \right\rangle \tag{11.4.16}$$

and

$$t_0 = t - \frac{|\mathbf{r}-\mathbf{r}_0|}{c} \qquad t_1 = t - \tau - \frac{|\mathbf{r}-\mathbf{r}_1|}{c}$$

In other words, the acoustic emission from a turbulent fluid is uniquely specified by the function K, which in essence represents the correlation in the fluctuations of the kinetic energy at different locations. Equations (11.4.13) to (11.4.16) can be regarded as the formal solution to the problem of sound emission from fluid flow.

First let us consider a stationary region of turbulence with zero drift velocity. Under such conditions the characteristic period T of fluctuation v of the velocity is $T = L/v$, where L is the characteristic eddy size. In the

presence of drift velocity $V_0 \gg v$, the corresponding relation would be $T \simeq L/V_0$, but if v is proportional to V, we may still set $T \simeq L/v$. To make a general estimate of the integrand K_t in Eq. (11.4.16), we set $\partial/\partial t \sim v/L$ and find, for the order of magnitude of K_t,

$$K_t \simeq \frac{\rho^2 v^8}{L^4}$$

We also observe, when carrying out the integration over r_1 in (11.4.15), that we shall get a contribution only for r_1 values within a distance of the correlation length L from r_0. Then, assuming that the total volume of the turbulent region is much larger than L^3, the integration over r_1 gives a value of the order of L^3, and the expression for the total emitted acoustic power from the turbulent region then can be expressed as

$$\Pi \simeq 4\pi r^2 I = \frac{4\pi r^2}{\rho c}\,\Upsilon(0) \sim \frac{\rho^2 v^8 L^2}{4\pi\rho c^5}\int \left(\frac{K_t}{\rho^2 v^8/L^4}\right)_{\tau=0} \frac{dv_0}{L^3} = C\left(\frac{v}{c}\right)^5 \frac{\rho v^3 L^2}{2} \tag{11.4.17}$$

which is consistent with (11.4.3) [compare with Eq. (7.1.38)]. The integral is dimensionless, and its value depends on the details of the flow in the turbulent region, but can always be expressed as the total volume of the region multiplied by some average value of the integrand. Therefore the constant factor in the last expression for Π can be regarded as proportional to the volume of the turbulent region, neglecting surface effects. It is interesting to compare this expression for the total radiated acoustic power with the mechanical power required to keep the turbulent region in motion. In the case of statistically stationary fluid motion, there must be a source which supplies mechanical power to the fluid, and this power, through the turbulent motion of the fluid, eventually is dissipated by viscosity and other loss mechanisms and by acoustic radiation. The mechanical power from the source first goes into kinetic energy of the large energy-carrying eddies, which are of the characteristic size of the driving mechanism. Here viscosity plays a minor role, and if the power transferred per unit mass of the fluid is ϵ, it follows from dimensional considerations that the velocity v is of the order of $(\epsilon L)^{\frac{1}{3}}$. Inserting this expression for the velocity in (11.4.17), and realizing that in this expression the constant C is proportional to the volume of the turbulent region divided by L^3, we see that the acoustic power Π emitted per unit mass of the fluid can be expressed as

$$\Pi \simeq (\text{const}) M^5 \epsilon \tag{11.4.18}$$

This shows that the efficiency of conversion of fluid-flow energy into sound via turbulent fluctuations is proportional to the fifth power of the Mach-number fluctuations in the flow. In the particular case of homogeneous isotropic turbulence, the dimensionless constant in Eq. (11.4.18) can be shown to be approximately equal to 38, but we shall not derive this result here.

Frequency distribution

These considerations referred only to the total emitted acoustic power. It is interesting to inquire also about the spectral distribution of this power. If we limit the discussion to statistically stationary turbulence, the time dependence of the function K involves only the time difference $t_0 - t_1$.

$$t_0 - t_1 = \tau - \left(\frac{|\mathbf{r} - \mathbf{r}_0|}{c} - \frac{|\mathbf{r} - \mathbf{r}_1|}{c}\right) \simeq \tau - \frac{\mathbf{Y} \cdot \mathbf{r}}{rc}$$

We have introduced here $\mathbf{Y} = \mathbf{r}_0 - \mathbf{r}_1$. If the fluid is homogeneous, the correlation function depends only on this difference, and not on the actual location of the points $\mathbf{r}_0$ and $\mathbf{r}_1$. In an inhomogeneous fluid this is not so, and it is then convenient to introduce, in addition to $\mathbf{Y}$, the variable $\mathbf{X} = \mathbf{r}_0 + \mathbf{r}_1$. Furthermore, if we introduce the time $\tau_0 = \mathbf{Y} \cdot \mathbf{r}/rc$ for sound to travel a distance Y, the expression for the correlation function takes the form

$$\Upsilon(\tau) = \left(\frac{1}{4\pi rc^2}\right)^2 \iiint d^3X \iiint K_t(\mathbf{Y}, \tau - \tau_0, \mathbf{X})\, d^3Y \qquad (11.4.19)$$

The Fourier transform of the correlation function yields the intensity spectrum

$$E(\omega) = \frac{1}{16\pi^2 r^2 \rho c^5} \iiint d^3X \iiint \phi(\mathbf{Y},\omega,\tau_0,\mathbf{X})\, d^3Y \qquad (11.4.20)$$

where

$$\phi(\mathbf{Y},\omega,\tau_0,\mathbf{X}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K_t(\mathbf{Y}, \tau - \tau_0, \mathbf{X}) e^{i\omega\tau}\, d\tau$$

For homogeneous turbulence the function K is independent of $\mathbf{X} = \mathbf{r}_0 + \mathbf{r}_1$. The integral over d^3Y is then simply the volume of the turbulent region. Consequently, the intensity spectrum radiated per unit volume of the source region has the form

$$e(\omega) = \frac{1}{32\pi^3 r^2 \rho c^5} \iiint d^3Y \int_{-\infty}^{+\infty} K_t(\mathbf{Y}, \tau - \tau_0) e^{i\omega\tau}\, d\tau \qquad (11.4.21)$$

We recall that τ_0 represents the time of travel of sound a distance Y, or since K_t is substantially zero for Y larger than the typical eddy size L, we can say that τ_0 is approximately the time of travel across an eddy. Then, if the frequency of interest is such that the period of the sound wave is smaller than τ_0 (or the wavelength smaller than L), there will be interference between the sound emitted from different parts of an eddy. There is also the possibility that during the time τ_0 the state of an eddy has changed; i.e., the lifetime of the eddy is small compared with τ_0. Under these circumstances it is important to retain τ_0 in the equation. We can easily see, however, that if the fluctuating Mach number is considerably smaller than unity, the lifetime of an eddy is large compared with τ_0 (the characteristic time of the eddy is $T = L/v$). Then, if we restrict ourselves to the case of low Mach numbers, and also to wavelengths large compared with the typical eddy size, we are justified in setting $\tau_0 = 0$ in the equation.

With this simplification we turn to the question dealing with the form of the function K_t. If we have a statistically stationary turbulent region, it is clear that the particular form of the driving mechanism that maintains the turbulent flow must influence the structure of the turbulence, at least for eddies of the order of the size of the driving mechanism. These eddies receive their energy directly from the driver, and this energy is transmitted to smaller eddies through nonlinear coupling, and eventually the energy is degraded to the "microscale" eddies which give up their energy to heat. However, there is a range of eddy sizes, the so-called inertial subrange, for which the statistical properties of the flow are supposed to be independent both of the driving mechanism and of viscosity. The size of the eddies in this inertial, or similarity, range is smaller than the size L of the largest eddies, but is larger than the size l of the microscale eddies, which are influenced more by viscosity than by inertial effects. From dimensional considerations it follows that the size of the microeddies is of the order of $l \approx (\nu^2/\epsilon)^{\frac{1}{4}}$, where ν is the kinematic viscosity, $\nu = \mu/\rho$.

Thus, in the inertial subrange, the magnitude of Y will lie in the range $L > Y > l$. If in this range the statistical properties of the turbulence are independent of the geometry of the driving mechanism and of viscosity, the only quantities that should enter into the description of the motion are the eddy size R, the power ϵ transferred to the fluid per unit mass, and the density ρ of the fluid. (The power ϵ is transferred from larger eddies to smaller ones until it is transformed into heat.) Using these quantities, we can construct from dimensional arguments expressions for the quantities entering our acoustical problem. Thus, for the characteristic velocity, we obtain $v \sim (\epsilon Y)^{\frac{1}{3}}$, and for the characteristic time of oscillation of an eddy of size Y, we get $T = Y^{\frac{2}{3}}\epsilon^{-\frac{1}{3}}$. From dimensional considerations it follows, then, that the function K_t will be of the form $K_t \sim \rho^2(\epsilon Y)^{\frac{8}{3}}\phi$, where ϕ is a dimensionless function of the argument $\tau/T = \tau/Y^{\frac{2}{3}}\epsilon^{-\frac{1}{3}}$. Thus, if we use

$$K_t \sim \rho^2(\epsilon Y)^{\frac{8}{3}}\phi\left(\frac{\tau}{Y^{\frac{2}{3}}\epsilon^{-\frac{1}{3}}}\right)$$

in (11.4.21) to obtain the acoustic power spectrum $4\pi r^2 e_i(\omega)$ of the sound radiated from the inertial subrange, we get, with $dv_Y = 4\pi Y^2\,dY$,

$$4\pi r^2 e_i(\omega) = \frac{4\pi}{8\pi^2\rho c^5}\int_l^{L_i} Y^2\,dY\int_{-\infty}^{+\infty} K_i(Y,\tau)\phi(x)\,dx \qquad (11.4.22)$$

where L_i is the largest eddy size in the inertial range. The lower limit on this integral goes to zero as the kinematic viscosity goes to zero. With this choice of lower limit, and using the dimensionless variables $x = \tau/Y^{\frac{2}{3}}\epsilon^{-\frac{1}{3}}$ and $y = \omega Y^{\frac{2}{3}}\epsilon^{-\frac{1}{3}}$, we obtain

$$4\pi r^2 e_i(\omega) = \tfrac{4}{3}\pi\rho c^2 M^{\frac{21}{2}}\left(\frac{c}{\omega L}\right)^{\frac{1}{2}}\int_0^{y_m} y^{\frac{5}{2}}\,dy\int_{-\infty}^{+\infty} e^{iyx}\phi(x)\,dx$$

where $y_m = \omega L_i/cM$. To obtain the acoustic contribution from the rest of the eddies with $L_i < Y < L$, we must add to this integral a portion in which K_t does not have the similarity form but depends on the geometry of the driving mechanism. However, as $\omega \to \infty$, the upper limit in the integral goes to infinity, and the additional contribution from the larger eddies becomes negligible. Therefore, for $\omega \gg cM/L$, the acoustic spectrum should be "universal," in the sense that it does not depend on the geometry of the driving mechanism. The acoustic power spectrum in this range is

$$4\pi r^2 e(\omega) \simeq (\text{const})\rho c^2 M^{\frac{21}{2}} \left(\frac{c}{\omega L}\right)^{\frac{7}{2}} \qquad \omega \gg \frac{cM}{L} \tag{11.4.23}$$

the constant involving the value of the double integral, with the integral over y extended to ∞.

This result indicates that the power-spectrum density increases as $M^{\frac{21}{2}}$ and decreases with frequency as $\omega^{-\frac{7}{2}}$ in the high-frequency range, whereas the total power, including the entire spectrum, is proportional to M^8.

A simplified line source

After these general considerations it is instructive to work through a simple specific example. Let us consider the sound emission from an idealized model flow field that may be used as a very simple model of a jet. When viewed from far away, a jet may be regarded approximately as an acoustic line source of finite length L, with a source strength that varies with position, and we shall use this model in the present example. The acoustic source strength per unit length of this source, as we have seen, is determined by the velocity fluctuation in the direction along the line of observation. For simplicity, we assume that the time dependence of v_r is a result of convection of a "frozen" pattern of flow irregularity. This means that in a coordinate system moving with the mean speed V_0 of the flow, the time dependence of the flow irregularity (eddies) is only that corresponding to the (sudden) creation at $x = 0$ and absorption or destruction of the eddies at $x = L$. The speed of convection is the mean velocity of the jet flow. To make the calculations simple, the spatial variation of v_r is assumed to be harmonic in the region $0 < x < L$, and zero elsewhere. The time dependence of v_r at a point x is then given by

$$v_r = \begin{cases} v_r = 0 & x < 0;\ x > L \\ v_0 \sin \dfrac{2\pi}{\Lambda}(x - V_0 t) \equiv v_0 \sin (Kx - \Omega t) & 0 < x < L \end{cases} \tag{11.4.24}$$

where $K = 2\pi/\Lambda$ is the wavenumber corresponding to the "wavelength" Λ of the flow irregularity, and $\Omega/2\pi = V_0/\Lambda$ is the frequency of flow fluctuation as seen by an observer in the laboratory system. With $T_r = \rho v_r^2$, we

have

$$\frac{\partial^2 T_r}{\partial t^2} = \frac{\rho v_0^2}{2}(2\Omega)^2 \cos\,[2(Kx - \Omega t)]$$

To evaluate the function K_t that occurs in (11.4.16), we have to evaluate $\partial^2 T_r/\partial t^2$ at (x_0,t_0) and (x_1,t_1), where

$$t_0 = t - \frac{|\mathbf{r} - \mathbf{r}_0|}{c} \simeq t - \frac{r}{c} + \frac{x_0 \cos\theta}{c}$$

$$t_1 = t - \tau - \frac{|\mathbf{r} - \mathbf{r}_1|}{c} \simeq t - \tau - \frac{r}{c} + \frac{x_1 \cos\theta}{c}$$

Here θ is the angle between $\mathbf{r}$ and the x axis. Using these expressions in Eq. (11.4.16), we obtain

$$K_t(x_0,x_1,\tau) = \left\langle \left(\frac{\partial^2 T_r}{\partial t^2}\right)_{r_0,t_0} \left(\frac{\partial^2 T_r}{\partial t^2}\right)_{r_1,t_1} \right\rangle$$

$$= \tfrac{1}{2} v_0^4 \Omega^4 \cos\,\{2[K(x_0 - x_1)(1 - M_0 \cos\theta) - \Omega\tau]\}$$

where $M_0 = V_0/c$. Next we have to integrate K_t over the range $0 < x_0 < L$, $0 < x_1 < L$. If the cross-sectional area of the jet is A, we have $d^3r_0 = A\,dx_0$ and $d^3r_1 = A\,dx_1$, and after integration over x_0 and x_1, we find

$$\Upsilon(\tau) = A^2 \int_0^L\!\!\int K_t\,dx_0\,dx_1 = \left(\frac{AL}{4\pi r c^2}\right)^2 \frac{v^4\Omega^4}{2} \cos\,(2\Omega\tau) \frac{\sin^2(\alpha KL)}{(\alpha KL)^2} \qquad (11.4.25)$$

$$\alpha \doteq 1 - \frac{V_0}{c}\cos\theta$$

According to Eq. (11.4.14), the total emitted intensity radiated in the direction θ is

$$I(\theta) = \frac{\Upsilon(0)}{\rho c} = \frac{\pi^2}{r^2}\frac{A}{\Lambda^2}\left(\frac{L}{\Lambda}\right)^2 \frac{A\rho v_0^4 V_0^4}{c^5} \frac{\sin^2(\alpha KL)}{(\alpha KL)^2} \qquad (11.4.26)$$

The frequency spectrum of the emitted sound in this simple model is proportional to $\delta(\omega - 2\Omega)$, so that only the one frequency 2Ω is present. With $\Omega = 2\pi V_0/\Lambda$, we note that the intensity of the radiated sound is proportional to $v_0^4(\Omega)\Omega^4$, where we have indicated that the velocity amplitude v_0 generally will depend on Ω or Λ. The corresponding total radiated power is

$$\Pi = \int 2\pi r^2 I(\theta) \sin\theta\,d\theta$$

$$= 2\pi^3 \frac{A}{\Lambda^2}\left(\frac{L}{\Lambda}\right)^2 \frac{\rho v_0^4 V_0^4 A}{c^5} \frac{1}{KLM_0} \int_{(1-M)KL}^{(1+M)KL} \frac{\sin^2 y}{y^2}\,dy \qquad (11.4.27)$$

where we have assumed v_0 independent of θ. In this simple model the velocity fluctuations are correlated over the entire length L of the jet, but under more general conditions, the correlation length may be only some fraction of L. In that case we should obtain an expression of the form (11.4.26) for each length of the jet equal to the correlation length. The factor $\sin^2(\alpha KL)/(\alpha KL)^2$, which expresses the interference that takes place between sound emitted from different points along the correlation distance, has a maximum value of unity at an angle given by $\cos\theta = 1/M_0$. In other words, a true maximum in the angular distribution resulting from this factor occurs only if $M_0 > 1$. We also note that the maximum gets sharper as $KL = 2\pi L/\Lambda$ gets larger, i.e., as the frequency V_0/Λ of the emitted sound increases.

Turning to the total emitted power, we see that if $M_0 < 1$, the interference factor $\int(\sin^2 y/y^2)\,dy$ rapidly goes to zero as $L \to \infty$, and the power radiated per unit length of the jet then goes to zero. This behavior depends on the simplified model used here, in which there is no time dependence of the flow in the jet in a coordinate system moving with the mean velocity of the jet. On the other hand, if $M_0 > 1$, we see that, as $L/\Lambda \to \infty$, the interference factor $\int_0^\infty (\sin^2 y/y^2)\,dy$ equals $\pi/2$ for $M_0 > 1$, and the power then becomes *proportional to L*. If, on the other hand, the jet length is short, so that $L/\Lambda \ll 1$, the interference integral is proportional to L, and according to Eq. (11.4.20), the power then is *proportional to* L^2.

To simulate the conditions in a turbulent jet, we have to set $v_0 \sim V_0$ (typically, $v_0 < 0.1V_0$), and we see that, apart from the interference factor $\int(\sin^2 y/y^2)\,dy$, the acoustic power is proportional to V_0^8, consistent with the qualitative result in Eq. (11.4.4). As we have seen in this analysis, there is no radiation (per unit length of the jet) for a subsonic jet of infinite length; this is related to the fact that, in a coordinate system moving with the speed of the jet, the flow is time-independent. For a *finite* jet, however, there will be a radiated sound even at subsonic speed; this depends on the fact that, in a coordinate system moving with the speed of the jet, the flow is no longer time-independent, since there is a creation of a flow irregularity at $x = 0$ and a destruction thereof at $x = L$. The corresponding (single) "lifetime" of an eddy then can be considered to be $\tau \simeq L/V_0$. In practice, there is a distribution of lifetimes among the various eddies in the flow, and for each such lifetime there will be a corresponding length of travel $L_i = V_0\tau_i$ before the destruction of the eddy. Under these circumstances the flow always will be time-dependent, even in the coordinate system moving with the jet, since there is a continuous creation, decay, and destruction of eddies over the length of the jet.

The simple model of sound emission from a jet considered here can easily be extended to incorporate some of these features so that it may be applied to an arbitrary convected flow irregularity containing a continuous

distribution of wavelengths and decay times. To put these observations in quantitative terms, we should introduce an appropriate correlation function for the jet flow and proceed to evaluate the correlation function for the sound pressure field as given in Eq. (11.4.15). Or we could follow the analysis in Sec. 7.1 dealing with the sound radiation from a randomly fluctuating fluid region. In attempting such an analysis, however, considerable difficulties are encountered because the jet flow is neither homogeneous nor isotropic. Furthermore, the mean flow velocity is not the same throughout the jet, and the large velocity gradients transverse to the jet axis couple with the turbulent fluctuations to give rise to large fluctuations in the momentum flux tensor. In addition, the mean velocity gradients cause refraction of the sound as it is transmitted from the jet to the surrounding quiescent atmosphere. The free-space Green's function used in Sec. 7.1 then is no longer appropriate. Qualitatively, the effect of refraction out from the jet is much the same as described in connection with Fig. 11.7.

Effect of translational motion of turbulence

A detailed analysis of the noise from a jet, accounting for the various effects mentioned above, will not be carried out here. Actually, for such an analysis to be meaningful, a better knowledge of the relevant flow correlations is required than is presently available. However, in view of the fact that, in Sec. 7.1, we have already treated the sound emission from a region of stationary turbulence, it is instructive in this context to determine the influence of translational motion of the turbulent region on the sound emission. We exclude the effect of refraction. The basic effect of the flow is to alter the autocorrelation function and the spectrum density of the excitation functions; they must be expressed in terms of coordinates at rest with respect to the observer. The transformation of coordinates, if the mean velocity of the turbulent region is U, in the direction of the z axis, is $x' = x$, $y' = y$, $z' = z - Ut$, and $t' = t$. The space and time correlation becomes mixed, and if, for example, $\Upsilon_s(\mathbf{d},\tau)$ and $|S(\mathbf{K},\omega)|^2$ of Eqs. (7.1.32) and (7.1.33) are the correlation function and spectrum density of the source function in coordinates moving with the jet at velocity U in the z direction, then the corresponding functions in coordinates at rest with respect to the observer and the surrounding air are

$$\begin{aligned}\Upsilon_s(d - U\tau, \tau) &= (\Delta_s^2) \exp\left[-\frac{1}{2}\frac{(d - U\tau)^2}{w_s^2} - \frac{1}{2}(\omega_s\tau)^2\right] \\ S_s(\mathbf{K}, \omega - \mathbf{U}\cdot\mathbf{K}) &= \frac{VT}{(2\pi)^6}\frac{w_s^2}{\omega_s}(\Delta_s^2)\exp\left[-\tfrac{1}{2}K^2 w_s^2 - \frac{(\omega - U\cdot\mathbf{K})^2}{2\omega_s^2}\right]\end{aligned} \tag{11.4.28}$$

In this system of coordinates the correlation between the same portion of fluid ($d' = 0$) at different times must be between the portion at $\mathbf{r}$ at $t = 0$ and the portion at $\mathbf{r} + \mathbf{U}\tau$ at time $t = \tau$. Correlation between times an

interval τ apart, for $d = 0$, corresponds to comparison between two parts of the fluid a distance $U\tau$ apart in the moving fluid. The Fourier transform of this, the spectrum density $|S_s|^2$, has its frequency transformed to $\omega - \mathbf{U} \cdot \mathbf{K}$ because of the motion. Because of the motion, the frequency distribution of the radiation is shifted by an amount $\mathbf{U} \cdot \mathbf{K}$, in accordance with the Doppler shift in frequency of the moving source.

Equations (7.1.35) and (7.1.36) for the intensity at frequency $\omega/2\pi$ and the total intensity, at the distant observation point in quiet air outside the turbulent region, become

$$I_{s\omega} = \rho c \langle \Delta_s^2 \rangle \frac{\omega^2 VT}{16\pi^2 c^2 r^2} \frac{w_s^2}{\omega_s} \exp\left\{\frac{-\omega^2}{2\omega_s^2}\left[k_s^2 w_s^2 + \left(1 - \frac{U}{c}\cos\vartheta\right)^2\right]\right\}$$

$$I_s = \frac{\rho c V}{\sqrt{32\pi r^2}} \langle \Delta_s^2 \rangle \frac{k_s^2 w_s^3}{\{k_s^2 w_s^2 + [1 - (U/c)\cos\vartheta]^2\}^{\frac{3}{2}}} \tag{11.4.29}$$

where ϑ is the angle between the direction of the velocity $\mathbf{U}$ of the jet flow and the vector $\mathbf{r}$ from the jet to the observation point. If the space variation of the random source distribution is more "fine-grained" than its time variation (i.e., if $w_s < c\tau_s = 4\pi c/\omega_s$), then $k_s w_s < 1$, and the radiated intensity is strongly dependent on the angle ϑ. The frequency distribution $I_{s\omega}$ has its maximum at

$$\omega_{\max} = \sqrt{2}\omega_s \left[\left(1 - \frac{U}{c}\cos\vartheta\right)^2 + k_s^2 w_s^2\right]^{-\frac{1}{2}}$$

$$\simeq \sqrt{2}\,\frac{\omega_s}{1 - (U/c)\cos\vartheta} \qquad \text{if } k_s w_s \ll 1$$

When $U < c$, frequency $\omega_{\max}/2\pi$ is greatest at $\vartheta = 0$, downstream of the turbulent region, and is least at $\vartheta = \pi$, upstream of the jet. The total intensity also decreases as ϑ is changed from 0 to π. Thus, as the observer moves in a circle of large radius r about the turbulent region, from a position away from the motion $\vartheta = \pi$ to a point in the direction of the motion $\vartheta = 0$, the mean frequency of the observed sound increases by an amount which is greater the nearer the jet velocity U is to the speed of sound c, and the total intensity also increases considerably.

If the time variation of the source function $s(\mathbf{r}',t)$ is rapid and the size of the turbulent cells is large, then $k_s w_s > 1$, and the radiation is not much dependent on the angle ϑ. In the limit the frequency distribution has its maximum at $\omega_{\max} \simeq \sqrt{2}(c/w_s)$, and the total intensity at the observation point is

$$I_s \simeq \frac{\rho c^2 V}{\sqrt{32\pi}\omega_s r^2} \qquad k_s w_s \gg 1$$

independent of ϑ.

The same remarks can be made regarding the sound produced by the fluctuations of the stress-momentum tensor of Eq. (7.1.38). The factors $1 + k_s^2 w_s^2$ and $1 + k_e^2 w_e^2$ in Eqs. (7.1.36) and (7.1.38) are changed, respectively, to

$$\left[\left(1 - \frac{U}{c}\cos\vartheta\right)^2 + k_s^2 w_s^2\right] \quad \text{and} \quad \left[\left(1 - \frac{U}{c}\cos\vartheta\right)^2 + k_e^2 w_e^2\right]$$

The comments given above regarding the dependence of mean frequency and total intensity of the radiation on the angle ϑ between the mean velocity U and the radius vector to the observation point also apply to the combined radiation. The intensity I_e from the turbulent motion is somewhat more sharply directional than is the intensity I_s from the fluctuating monopoles.

When $U > c$, the maximum of the intensity, as well as the frequency $\omega_{\max}$, comes, not at $\vartheta = 0$, but at $\vartheta = \cos^{-1}(c/U)$.

Problems

1 Derive the expressions (11.2.36) and (11.2.37) for the power emitted from a moving longitudinal and transverse dipole. Also determine the corresponding power spectra given in (11.2.38) in the text. Discuss the relative magnitudes of the power emitted in the two cases.

2 An airplane travels through a turbulent atmosphere with a speed equal to half the sound speed and, as a result, experiences a fluctuation lift force with a characteristic period of the order of $T \simeq 1$ sec and with an amplitude of the order of the weight w of the airplane. As far as the influence on the surrounding atmosphere is concerned, regard the airplane as a traveling transverse dipole source, so that the pressure field produced can be determined from Eq. (11.2.33).

(*a*) Show that the integral of the near-field term in the pressure field over the ground surface (assumed plane) equals the weight of the airplane.

(*b*) What is the maximum amplitude of the sound pressure field at the ground surface if the airplane flies at a height of 5,000 ft and if $w = 10$ tons?

(*c*) What is the radiated acoustic power?

3 Derive the expressions (11.2.33) and (11.2.34) for the sound pressure field radiated from a moving transverse and longitudinal dipole source. Introduce in these expressions the distance R between the point of observation and the point of emission [Eqs. (11.2.6) and (11.2.12)] and the corresponding direction cosines, $\cos\theta_1$, $\cos\theta_2$, and $\cos\theta_3$, and show that the expressions for the pressure fields (11.2.33) and (11.2.34) can be written in the form

$$p_2 = \frac{1}{4\pi R}\frac{\cos\theta_2}{(1 - \beta\cos\theta_1)^2}\left[\frac{1}{c}f_2' + \frac{1}{R(1 - \beta\cos\theta_1)}f_2\right]$$

$$p_1 = \frac{f_1'}{4\pi c R}\frac{\cos\theta_1}{1 - \beta\cos\theta_1} + \frac{f_1}{4\pi R^2}\frac{\cos\theta_1 - \beta}{(1 - \beta\cos\theta)^3} \qquad \beta = \frac{V}{c}$$

4 A stationary acoustic source distribution in a duct has a uniform source strength $Q = Q_0 \exp(-i\omega t)$ per unit volume over a length L of the duct and is zero outside this region. The air in the duct is moving with a uniform velocity V along the duct. With reference to the discussion related to Eq. (11.1.25), calculate the sound field in the duct both outside and inside the source region.

5 A plane boundary is specified acoustically by a specific acoustic impedance ζ. The outside fluid moves parallel with the boundary with a velocity, V as shown in Fig. 11.3. A plane wave is incident on the boundary at an angle ϕ. Show that the pressure reflection coefficient is given by Eq. (11.2.21) and that the power absorption coefficient is

$$\alpha = \frac{4\mu(1 + M\cos\phi)\sin\phi}{[\xi + (1 + M\cos\phi)\sin\phi]^2 + \sigma^2}$$

where

$$\zeta = \frac{1}{\xi - i\sigma}$$

and $M = V/c$, c being the speed of sound in the fluid.

6 A thin plane partition of some kind, a membrane or a thin plate, separates two regions of a fluid in relative motion as indicated in Fig. 11.3. Generalize Eq. (11.1.20) for the reflection coefficient of a plane sound wave at the interface to account for the presence of a partition. The partition is specified acoustically by means of an impedance z, defined as the ratio between the pressure difference between the two sides of the partition and the velocity of the partition. Show that the reflection coefficient in the case of a stretched membrane may be obtained from (11.1.20) by setting

$$\zeta = -\frac{i\omega\varepsilon}{\rho_1 c_1}\left[1 - \frac{{v_t}^2}{{c_1}^2}\frac{\cos^2\phi_1}{(1 + M_1\cos\phi_1)^2}\right] + \frac{\rho_2 c_2}{\rho_1 c_1}\frac{1}{(1 + M_2\cos\phi_2)\sin\phi_2}$$

where ϵ is the mass per unit area of the membrane, and $v_t = \sqrt{T/\epsilon}$, T = tension of membrane. For a thin plate with phase velocity of flexural waves v_p, replace the expression inside the brackets by $1 - (v_p/c_1)^4\cos^4\phi_1/(1 + M_1\cos\phi_1)^4$.

7 As mentioned in the discussion of the sound field from a supersonically moving monopole source [Eq. (11.2.19)], the sound field inside the sound cone can be regarded as a superposition of the field contributions from two emission points, E_- and E_+. In the case of harmonic time dependence of the sound source, this leads to the interesting effect of "standing-wave" formation inside the cone from a moving source. Thus, if the source strength is specified by $q = q_0\exp(-i\omega t)$, show that the sound-pressure amplitude on the axis of the sound cone a distance x from the vertex of the cone (i.e., from the sound source) is given by

$$p = \frac{\omega(2q_0)}{4\pi x(1 - \beta^2)}\left\{\sin^2\frac{\omega x}{c(1 - \beta^2)} + \beta^2\cos\frac{\omega x}{c(1 - \beta^2)}\right\}^{\frac{1}{2}}$$

8 Show that the d'Alembertian operator $\nabla^2 - (1/c^2)(\partial^2/\partial t^2)$ in the acoustic wave equation is invariant under the "Lorentz" transformation

$$x' = \gamma(x - Vt) \qquad y' = y \qquad \gamma = \frac{1}{\sqrt{1 - (V/c)^2}}$$

$$t' = \gamma\left(t - \frac{Vx}{c^2}\right) \qquad z' = z$$

9 Show that a Galilean transformation to a frame of reference moving with velocity V brings the ordinary acoustic wave equation into the form

$$\frac{1}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}\right)^2 p - \nabla^2 p = 0$$

Show that this equation is consistent with Eq. (7.1.21). Is it valid when V depends on position?