10 In Chap. 6 the Lagrangian density in a sound field was shown to be $L = \frac{1}{2}\rho[(\nabla\Psi)^2 - c^{-2}(\partial\Psi/\partial t)^2]$, where Ψ is the velocity potential. This expression is valid in a coordinate system in which the unperturbed fluid is at rest. If the fluid is in motion with a uniform velocity V_0 in the x direction with respect to the laboratory system S, what is the Lagrangian density with respect to S? Use the results obtained in Chap. 6 to derive an expression for the acoustic intensity with respect to S.

11 A monopole source with a source strength $Q = Q_0 \exp(-i\omega t)$ moves along a circular path of radius a in the yz plane with an angular velocity Ω. Calculate the radiated sound field. Show that the spectrum contains frequencies $|\omega \pm n\Omega|$, where $n = 0, 1, 2, \ldots$, and that the corresponding pressure amplitudes are proportional to $J_n(ka \sin\theta)$, where $k = \omega/c$, θ is the angle between the direction of propagation and the x axis, and J_n is the nth-order Bessel function.

12 A rigid cylinder of diameter $d = 1$ cm is mounted between two opposite walls of a square duct perpendicular to the duct axis. The cross section of the duct is 30 by 30 sq cm. When air is passed through the duct, at what average flow speeds will stimulated Kármán-vortex shedding and the related sound emission take place?

13 For isentropic fluid motion, we have $dS/dt = \partial S/\partial t + \mathbf{V}\cdot\nabla S = 0$. If the unperturbed fluid is inhomogeneous, so that $S_0 = S_0(x,z,y)$, and moving with a velocity $\mathbf{V}_0$, which may be a function of the space and time coordinates, show that the linearized wave equation for the pressure is

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\gamma - 1}{c^2}\frac{\partial H}{\partial t} - \nabla\cdot\mathbf{F} + \nabla\cdot(\rho\mathbf{V}_0\,\mathbf{V}_0)\cdot\nabla + \nabla\cdot(\rho\,\mathbf{u}\,\mathbf{V}_0)\cdot\nabla - \frac{\rho_0}{c_v}\frac{\partial}{\partial t}(\mathbf{u}\cdot\nabla S_0 + \mathbf{V}_0\cdot\nabla_0)$$

where H = rate of energy transfer per unit volume of fluid
F = external force per unit volume
$\mathbf{uV}_0$ = tensor with components $u_x V_{0x}$, $U_x V_{0y}$, etc.

Show that the source term can be written in a form that contains the gradient of the sound pressure. Also use the equation of state for an ideal gas $S = C_v \log P + C_p \log \rho + \text{const}$ to express the entropy source term as a function of the gradients of the unperturbed pressure and density. What external force is required to maintain the fluid in this unperturbed state? Consider as an example sound propagation in a moving isothermal atmosphere.

14 Derive Eq. (11.4.8).

15 With reference to Eq. (11.3.24), show that the Green's functions, expressing the sound pressure fields from an axial and a tangential force of frequency ω and unit amplitude at point (r_1,φ_1) in a cylindrical duct of radius r_0, are [see Eq. (9.2.11)]

$$G_z(r\varphi \mid r_1\varphi_1) = \sum_{m,n}\frac{1}{S\Lambda_{mn}} J_m(\kappa_{mn}r_1)J_m(\kappa_{mn}r)e^{im(\varphi-\varphi_1)}\,e^{ik_{mn}z-i\omega t}$$

$$= \sum_{m,n}\frac{m}{S\Lambda_{mn}k_{mn}} J_m(\kappa_{mn}r_1)J_m(\kappa_{mn}r)e^{im(\varphi-\varphi_1)}e^{ik_{mn}z-i\omega t}$$

$$k_{mn}^2 = (\omega/c)^2 - \kappa_{mn}^2$$

Use these functions to determine the sound pressure of Eq. (11.3.24) from the fan-force distributions f_z and f_φ in an inhomogeneous flow, as defined in the discussion preceding Eq. (11.3.24). Discuss the amplitude of the (m,n)th

mode of the resulting sound as a function of the tip speed $V = \Omega r_0$, where Ω is the angular velocity of the fan. In particular, investigate the relative contributions of the axial and the tangential force components when $V \simeq c$. Show that the nth-order harmonic component in the Fourier expansion in φ of the flow perturbation gives rise to a *plane* wave in the duct if n is a multiple of the number of blades B in the fan, and that, under these conditions, the contribution from the tangential (drag) force is zero. Assume that α_s, β_q, f_{z0}, and $f_{\varphi 0}$ are independent of r and calculate the values of the coupling coefficients A_{mn}^{sq} and B_{mn}^{sq}.

CHAPTER

12

PLASMA ACOUSTICS

12.1 WEAKLY IONIZED GAS

In the discussion of sound waves in a neutral gas we have had little occasion to consider effects of external forces acting in the bulk of the gas or fluid. In this chapter the situation is different, since we shall now be concerned with the dynamics of an ionized gas or plasma. External electric and magnetic fields produce forces in the gas and internal fields, and corresponding forces are produced by the gas itself as a result of its motion. These fields and forces lead to many dynamical phenomena: we shall study here only those which may be considered to be acoustic in nature. The dynamical effects of external or internal fields and forces depend to a great extent on the degree of ionization of the gas, i.e., on the ratio between the number densities of the charged particles and the neutrals. For this reason we shall treat separately the weakly ionized and the highly ionized gas.

In the weakly ionized gas the neutral gas plays a dominant part, and in an ordinary sound wave the ions and the electrons are forced to follow the neutrals. From this standpoint the presence of the charged particles is inconsequential. However, the charged particles draw energy from external electric fields, and some of this energy is transferred to the neutral gas, which then can lead to sound generation and amplification in the neutral gas.

The energy transfer from the electrons to the neutrals leads not only to a heating of the neutral gas, but also to ionization. As we shall see, this creation of charged particles, under certain conditions, will travel as a wave through the gas. In the fully ionized gas we shall find, in addition to the ordinary sound wave, a wave mode in which there is charge separation. Such a particular kind of sound wave is often called a *plasma wave*, or *plasma oscillation*. Finally, in the presence of an external magnetic field, the dynamics of the ionized gas is rather complicated; we shall consider in this chapter only some aspects of the problem which involve low-frequency phenomena, such as the magnetoacoustic waves in a conducting fluid.

Coupling to the electromagnetic wave

A weakly ionized gas can be regarded as a mixture of three different gas components, the neutral-gas atoms (or molecules), the ions, and the electrons. Such a gas mixture, often called a plasma, plays an important role in modern electronics. From an acoustic standpoint the basic difference between the plasma and a neutral-gas mixture is the fact that the plasma couples strongly to electric and magnetic fields, whereas for a neutral gas this coupling is very weak, being due to electrostriction only. Actually, a plasma not only responds to an electromagnetic field, but ordinarily depends on an electric field for its existence. The energy of the electrons is drawn from the external electric field that maintains the discharge, and in steady state an electron energy density is reached such that the energy-loss rate equals the rate of energy transfer to the electrons.

The energy losses of the electrons are due to several processes. In a weakly ionized gas, as encountered in an ordinary glow discharge, the main energy loss of the electrons is the energy transfer to the neutral-gas particles through elastic collisions. Other loss mechanisms correspond to diffusion of electrons to the walls of the chamber and to ionization and excitation of the neutrals. All these losses depend on the average speed or temperature of the electrons, and steady-state conditions are reached at a particular temperature of the electrons. For example, in a typical weakly ionized gas, such as in a glow discharge with a neutral-gas density of about 10^{16} cm^{-3} and an electron and ion density of 10^{10} to 10^{11} cm^{-3}, the steady-state electron temperature is of the order of 10^4 to 10^5 °K, whereas the neutral-gas temperature is about 500°K. In other words, in contrast to the situation in a mixture of neutral-gas components, the steady-state temperatures of the gas components in a plasma differ widely, the electron temperature being considerably higher than the ion and neutral-gas temperatures. Because of this temperature difference, there is a continuous energy flow from the electrons to the neutral-gas component, and if this transfer is made to vary with time, sound will be produced in the neutral gas.

The mechanism of the energy transfer from the electrons (or the ions) to the neutrals is roughly as follows: In the time interval between collisions, an electron is accelerated in the direction of the electric field and receives a certain amount of energy corresponding to the distance of travel between collisions. As a result of collisions with the neutral-gas particles, the motion of the electrons is randomized, and most of the energy received from the electric field between collisions goes into random motion. However, there remains a small net drift motion of the electron, and the drift velocity $\mathbf{V}_e$ is proportional to the electric field

$$\mathbf{V}_e = b_e\mathbf{E} \tag{12.1.1}$$

where b_e is called the *electron mobility*. A similar relation holds for the ions, $\mathbf{V}_i = b_i\mathbf{E}$. The average rate of energy absorbed by the electrons per unit

volume can then be expressed as

$$H_e = eN_e b_e E^2 \tag{12.1.2}$$

with a similar expression for the ions (e, of course, is the electronic charge). Since we have $N_e \simeq N_i$, it follows that the energy contributions from the electrons and the ions are related as are their mobilities. Ordinarily, the mobility of the electrons is considerably larger than for the ions, and therefore the energy contribution from the ions can be neglected.

If the electric field or the electron density varies with time, it follows from Eq. (12.1.2) that there is a corresponding time dependence of the energy transfer to the neutrals. There is very little time delay in this process, between the time variation of H and of E^2 or N_e, the characteristic time being of the order of the electron collision time, typically of the order of 10^{-8} sec in a glow discharge. Thus, if the electric field is modulated at acoustic frequencies, we may neglect the collision time and treat (12.1.2) as true also when N_e and E are time-dependent.

In a similar manner we can express the rate of momentum transfer from the electrons and the ions to the neutrals. However, the contributions from the ions and the electrons tend to cancel each other, and the net effect is very small at the frequencies that are considered here.

Since the energy transfer from the electric field may be regarded as instantaneous, and the energy loss from the gas due to heat conduction is a comparatively slow process, with a much longer relaxation time than the acoustic periods of interest here, we may neglect the effect of heat conduction and regard Eq. (12.1.2) as the time-dependent *net* heat transfer to the gas. We have already seen in Eq. (7.1.19) that the corresponding acoustic source term in the equation for the sound pressure is given by $(\gamma - 1)c^{-2}(\partial H/\partial t)$, so that

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\gamma - 1}{c^2}\frac{\partial H}{\partial t} = \frac{\gamma - 1}{c^2}\, e\, \frac{\partial}{\partial t}(N_e b_e E^2) \tag{12.1.3}$$

After having obtained the expression for the source term in the wave equation for the acoustic pressure in the neutral gas, we can now proceed to calculate the sound pressure field in terms of the space-time dependence of the electric field and the electron density if these quantities are known *a priori*. This will be assumed to be the case in the two examples, which follow, but later, in the discussion of wave amplification and spontaneous oscillations, the reaction of the sound wave back on the plasma will be accounted for.

Sound generation by ionization waves

As mentioned briefly in the introduction, and as will be discussed in more detail later, in a plasma traveling waves frequently exist in the electric field and in the electron density. For example, it is well known that an ordinary

glow discharge often contains traveling "striations" that move in the direction from the anode to the cathode with a velocity of approximately 10^4 cm per sec. The frequency of oscillations typically is in the range 10^3 to 5×10^4 sec^{-1}.

Thus, with the static field $\mathbf{E}_0$ and the perturbation $\epsilon_1\mathbf{E}_0$, we can express the total electric field as $E = E_0[1 + \epsilon_1 \cos \omega(t - x/V)]$, where V is the wave speed. Similarly, for the electron density, we have $N_e = N_{e0}[1 + \epsilon_2 \cos \omega(t - x/V)]$, where we have assumed that the two waves are in phase with each other; x is the coordinate along the axis of the tube, positive in the direction of the electric field. For simplicity we shall consider a somewhat idealized geometry of the plasma such that the unperturbed values N_{e0} and E_0 of the electron density and the electric field are both constant along the tube in the interval $-L < x < L$ and zero outside this region, where only the neutral-gas component is present. Also, we shall be interested only in plane sound waves along the axis of the tube; consequently, only the average values of N_{e0} and $\mathbf{E}_0$ across the tube will contribute to the generation of these waves. In the present example we assume that N_{e0} and $\mathbf{E}_0$ stand for these averages.

If the perturbations are sufficiently small, we need include only first-order terms in ϵ_1 and ϵ_2 in the source term in Eq. (12.1.3). Furthermore, we shall neglect the perturbation in the mobility resulting from the perturbation in the electric field. The quantity H in Eq. (12.1.2) can then be written in the form

$$H = H_0 + h_0 \exp\left(\frac{i\omega x}{V} - i\omega t\right)$$
$$\equiv eN_{e0}b_eE_0^2\left[1 + (\epsilon_1 + 2\epsilon_2)\exp\left(\frac{i\omega x}{V} - i\omega t\right)\right] \quad (12.1.4)$$

in the region $-L < x < L$, and zero outside this region. This function is now used in the source term in the wave equation (12.1.3), and to obtain the solution to this equation, we conveniently use the Green's-function techniques, which we have done so frequently throughout the book. The one-dimensional Green's function $G(x \mid x_0)$, being a solution to $(1/c^2)\partial^2 p/\partial t^2 - \partial^2 p/\partial x^2 = \delta(x - x_0)\exp(-i\omega t)$, is

$$G(x \mid x_0) = \frac{i}{2k} e^{ik|x-x_0|}e^{-i\omega t}$$

where $k = \omega/c$. The pressure field in the region $|x| > L$ outside the plasma is then given by

$$p = -i\omega \frac{\gamma - 1}{c^2} h_0 e^{\pm ikx - i\omega t} - \frac{i}{2k}\int_{-L}^{L} e^{i\omega x_0/V} e^{\mp i\omega x_0/c}\, dx_0$$

or

$$p = -(\epsilon_1 + 2\epsilon_2)p_0 \frac{\sin[(\omega L/V) \mp (\omega L/c)]}{(\omega L/V) \mp (\omega L/c)} e^{\pm ikx - i\omega t} \quad (12.1.5)$$

where

$$p_0 = \frac{(\gamma - 1)LH_0}{c} = \frac{(\gamma - 1)eN_{e0}b_eE_0^2L}{c}$$

The minus and plus signs refer to the acoustic waves emitted in the positive and negative x directions, respectively. We note that the amplitude of the wave in the positive x direction, the direction of the source wave, reaches a maximum value p_0 when the speed V of the source wave is the same as the speed of sound c in the neutral gas. In the opposite direction the amplitude of the sound wave is always less than p_0.

It is interesting to rewrite the expression for the characteristic sound pressure p_0 somewhat differently in order to relate it to some characteristic pressure in the plasma. To do this, we express the rate of energy transfer to the neutrals in a different manner, making use of the energy transfer of an electron to a neutral particle in an elastic collision. Since the electron temperature of the electrons is considerably higher than the temperature of the neutrals, we can regard the neutrals as stationary in the collisions with the electrons. Then the average energy transfer in a single collision is $\frac{4m_e}{m_n}\frac{m_e v_e^2}{2}$, where v_e is an average value of the speed of the incident electrons. The number of collisions per second can be expressed as v_e/l_e, where l_e is an average mean-free-path of the electrons. The total rate of energy transfer from the N_e electrons per unit volume, then, is

$$H = \frac{4m_e}{m_n}\frac{m_e v_e^3}{2}\frac{N_e}{l_e} = \frac{4m_e}{m_n}\frac{v_e}{l_e}p_e \tag{12.1.6}$$

where we have introduced $p_e = N_e m_e v_e^2/2$. The characteristic sound pressure in (12.1.5) thus can be expressed as

$$p_0 = 4(\gamma - 1)\frac{m_e}{m_n}\frac{v_e}{c}\frac{L}{l_e}p_e \tag{12.1.7}$$

To get an idea of the order of magnitude of the sound-pressure amplitudes obtained, we consider a He plasma at a neutral-gas pressure $\simeq$1 mm Hg, an electron density $N_{e0} \simeq 10^{10}$ cm^{-3}, and an electron temperature of about 1 eV. Assuming $\epsilon_1 = \epsilon_2 = 0.1$, and with $L = 10$ cm, we find a value of the sound-pressure amplitude $(\epsilon_1 + 2\epsilon_2)p_0$ of the order of about 1 dyne per sq cm.

Inside the plasma the sound field is a superposition of waves traveling in both directions. We leave it as one of the problems to determine this field.

Plasma afterglow

A second example of some importance in plasma acoustics is the generation of sound waves in the plasma afterglow. By plasma afterglow is meant the period of decay of the plasma after the electric field has been turned off. In the afterglow period the electron temperature decays quite quickly to the temperature of the neutral-gas component. For the purposes of the present discussion of sound generation, this decay can be regarded as instantaneous. The electron density, on the other hand, decays much more slowly and, as far as sound generation is concerned, is of minor

importance. However, the sound produced at the instant of turnoff of the electric field, due to the change in the heating of the neutral gas, frequently produces acoustic oscillations in the discharge tube which modulate the decaying electron density at the acoustic frequency. Such a modulation actually has been observed in several experiments.

To get an idea of the strength of these oscillations, we shall consider the sound produced as a result of a sudden turnoff of the electric field E_0 that maintains a discharge. In other words, the time dependence of the heating function H in (12.1.3) is now such that $H = eN_{e0}b_eE_0^2$ for $t < 0$, and $H = 0$ for $t > 0$. We let the plasma be contained in a tube with rectangular cross section, and for simplicity, only the modes perpendicular to the tube axis and to the tube walls at $y = 0$ and $y = D$ will be treated. The coupling to these modes depends on the y dependence of the heating function H. The electron density is approximately of the form $N_{e0} = N_0 \sin(\pi y/D)$, and therefore the y dependence of the heating function also is given by $\sin(\pi y/D)$.

Since the time dependence of H is described by a step function as discussed above, it follows that the wave equation describing the excitation of sound in the afterglow is of the form

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial y^2} = H_0 \sin\left(\frac{\pi y}{D}\right)\delta(t) \tag{12.1.8}$$

where

$$H_0 = \frac{eNb_eE_0^2(\gamma - 1)}{c^2}$$

To obtain the solution, we conveniently make a Fourier transform in time and express the y dependence of the pressure in terms of a Fourier series of modes symmetrical with respect to the axis of the tube (since the source is symmetrical). Thus

$$p(t,y) = \sum_n \int_{-\infty}^{+\infty} A_n(\omega)\cos(k_n y)e^{-i\omega t}\,d\omega \qquad k_n = \frac{2\pi n}{D} \tag{12.1.9}$$

By expanding the right-hand side of Eq. (12.1.8) in a similar way, we obtain

$$A_n = \frac{2}{\pi^2} H_0 \frac{c^2}{4n^2 - 1}\frac{1}{\omega^2 - k_n^2 c^2}$$

Finally, after we have carried out the integration in Eqs. (12.1.9), the pressure field in the plasma afterglow is found to be

$$p(t,y) = \sum_{n=1}^{\infty} \frac{2p_0}{\pi^2 n(4n^2 - 1)} \sin\frac{2\pi nct}{D}\cos\frac{2\pi ny}{D} \tag{12.1.10}$$

where p_0 is the same as in (12.1.5) or (12.1.7), with L replaced by D. We note that the fundamental mode has an amplitude about ten times larger than the amplitude of the second mode, and for large n the amplitude decreases with n approximately as n^{-3}.

Acoustic wave amplification

The preceding examples have been concerned with an electric field, imposed from outside the system, generating a sound wave. Next we should study the cases where there is a static electric field, and an acoustic wave is imposed from outside, or else is generated spontaneously from the interaction between acoustic density variations and thermal energy from the electrons. In a plasma the free electrons represent a constant source of additional thermal energy supplied to the gas; when this source is coupled to the acoustic wave, a positive feedback can occur which can transform electronic energy into acoustic energy.

To investigate this problem in great detail would require the analysis of the equations of motion for the neutrals, electrons, and ions in the plasma, accounting for the interaction between them in terms of momentum and energy transfer. Such a general analysis would include not only the acoustic mode of motion, but other wave modes as well. We shall not go into such an analysis here, but merely point out that, as far as the acoustic mode of motion is concerned, the neutrals, ions, and electrons all move approximately in phase with each other and with the same amplitude of oscillation. The relative density fluctuations for all components are also the same, so that a perturbation n_n in the neutral-particle density produces a perturbation n_e in the electron density such that

$$n_e \simeq \frac{N_e}{N_n} n_n \tag{12.1.11}$$

We have already seen in Eq. (12.1.3) that the source term in the wave equation for the sound pressure contains a term proportional to the rate of change of the total electron density. If this rate of change is produced as a result of a perturbation in the acoustic density, as in Eq. (12.1.11), and if the unperturbed electron density N_{e0}, as well as the electric field E_0, is time-independent, the corresponding contribution to the source term in Eq. (12.1.3) is $(\gamma - 1)c^{-2}eb_eE_0^2(\partial N_e/\partial t) = (\gamma - 1)c^{-2}eb_eE_0^2(N_{e0}/N_n)(\partial n_n/\partial t)$. Since the acoustic pressure p is related to the perturbation in the density through the relation $p = c^2m_nn_n$, where m_n is the neutral particle mass, we can express $\partial n_n/\partial t$ in terms of $\partial p/\partial t$, and the wave equation (12.1.3) takes the form

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = (\gamma - 1)\frac{H_0}{\rho_0 c^4}\frac{\partial p}{\partial t} \tag{12.1.12}$$

where we have made use of Eq. (12.1.2).

It should be noted that the sign of the term on the right-hand side is opposite to that for a resistance in the simple-oscillator equation

$$M(d^2x/dt^2) + R(dx/dt) + Kx = 0$$

($-\nabla^2$ takes the place of K). Thus energy is being added, rather than lost.

This energy of course comes from the electrons, kept at high temperature by the constant electric field. If the wavelength $\lambda = 2\pi/k$ is specified, the frequency will have a positive-imaginary part, representing an exponential increase with time; if the frequency $\omega/2\pi$ is specified, the wavenumber k will have a negative exponential part, representing an exponential buildup of the wave as it travels along.

To study the behavior of such a plane wave, transmitted through the plasma at a frequency ω, we introduce $p \sim \exp(ikx - i\omega t)$ into Eq. (12.1.12), and find that the corresponding value for the propagation constant is given by the dispersion relation

$$k = \frac{\omega}{c}\sqrt{1 - \frac{i}{\omega\tau}} = \alpha - i\beta \tag{12.1.13}$$

where

$$\tau = \frac{\rho_0 c^2}{(\gamma - 1)H_0}$$

This time can be described approximately as the time required to transfer from the electrons an amount of energy equal to the thermal energy of the neutral gas. Using the expression (12.1.6), we can express this time in terms of the electron collision time l_e/v_e and obtain

$$\tau = \frac{\gamma}{3(\gamma - 1)}\frac{m_n}{m_e}\frac{N_n}{N_e}\frac{T_n}{T_e}\left(\frac{l_e}{v_e}\right) \tag{12.1.14}$$

where T_n, T_e are the gas and electron temperatures. Under normal glow-discharge conditions, this time is of the order of 0.1 sec, and therefore for the frequencies of interest here, $\omega\tau \gg 1$. Under special conditions, however, τ may be considerably shorter.

Returning to the dispersion relation (12.1.13), we obtain the following expressions for the real and imaginary parts of the propagation constant:

$$\alpha = \frac{\omega/c}{\sqrt{2}}[\sqrt{1 + (\omega\tau)^{-2}} + 1]^{\frac{1}{2}} = \begin{cases} \sqrt{\dfrac{\omega}{2c^2\tau}} & \omega\tau \ll 1 \\ \dfrac{\omega}{c}[1 + \frac{1}{2}(2\omega\tau)^{-2}] & \omega\tau \gg 1 \end{cases} \tag{12.1.15}$$

$$\beta = \frac{\omega/c}{\sqrt{2}}[\sqrt{1 + (\omega\tau)^{-2}} - 1]^{\frac{1}{2}} = \begin{cases} \sqrt{\dfrac{\omega}{2c^2\tau}} = \beta_1 & \omega\tau \ll 1 \\ \dfrac{1}{2c\tau} = \beta_2 & \omega\tau \gg 1 \end{cases}$$

Then, for $\omega \ll 1/\tau$, the growth rate increases with frequency as $\sqrt{\omega}$, and in the limit of high frequencies the growth rate is independent of frequency.

In obtaining these results, we have neglected the attenuation of the acoustic wave resulting from various loss mechanisms. When sufficiently small, the growth and the attenuation rates are additive, so that we can treat them separately. For a monatomic gas the acoustic losses are caused solely by viscosity and heat conduction. Normally, the plasma is contained in a tube, and we can make use of the results obtained in Chap. 6 to obtain the attenuation rate caused by viscous and heat-conduction losses. As we recall, it is convenient to consider separately the losses in the bulk of the gas outside the viscous and thermal boundary layers and the losses in the boundary layers themselves. These loss contributions lead to spatial attenuation rates for the fundamental mode [see Eq. (6.4.14) and page 519].

$$a_1 = \frac{\omega/c}{2d}\left[\sqrt{\frac{2\nu}{\omega}} + (\gamma - 1)\sqrt{\frac{2K}{\omega\rho C_p}}\right] \simeq \frac{1}{d}\sqrt{\frac{2\omega l_n}{c}} \qquad \text{(boundary)}$$

$$a_2 = \frac{1}{2}\left(\frac{\omega}{c}\right)^2\left[\frac{4}{3}\frac{\nu}{c} + (\gamma - 1)\frac{K}{\rho C_p c}\right] \simeq 1.16\left(\frac{\omega}{c}\right)^2 l_n \qquad \text{(bulk)} \tag{12.1.16}$$

where $l_n = \nu/c$ is approximately equal to the mean-free-path of the neutrals, K is the heat-conduction coefficient, and ν is the kinematic viscosity $\nu = \mu/\rho_0$. In obtaining the approximate expressions (12.1.16), we have set $\mu \simeq 0.67 \times K/c_p$, which is approximately valid for a monatomic gas.

When β and the a's are small enough, the growth and attenuation effects are additive and the sound-pressure amplitude will depend on x through the factor $\exp(\beta - a_1 - a_2)x$; amplification will occur if $\beta > a_1 + a_2$. At very low frequencies all the quantities are small; in fact a_2 is much smaller than a_1. Thus in the very low frequency range, it appears that amplification occurs if $4l_n c\tau < d^2$.

However, in the frequency range of major interest, β and the a's are not small and the effects are not additive. To obtain a more generally valid result, we would have to solve the coupled equations of motion for the three fluid components, electrons, ions, and neutrals. When this is done we find that the three fluids do not move exactly in phase, so that a sound wave produces charge separation and consequent electric fields. The situation is analogous to the coupled waves of Eq. (10.1.2). We shall not go into details but shall mention a few results of the more complete analysis.

We find that the effects are additive and the pressure depends on the factor $\exp(\beta - a_1 - a_2)x$ only for frequencies such that

$$\omega \ll \Omega_n \frac{N_n T_n}{N_e T_e}$$

where Ω_n is the mean frequency of collision per second of a neutral particle with charged particles and the N's and T's are the particle densities and

absolute temperatures of neutrals and electrons, respectively. The wave-number k of the acoustic-type wave is obtained from

$$k^2 = \left(\frac{\omega}{c}\right)^2\left[1 - \left(\frac{\Omega_n}{\omega} + \frac{1}{\omega\tau}\right)\frac{\delta^2}{1+\delta^2} + i\left(\frac{1}{\omega\tau_n} + \frac{\Omega_n}{\omega}\frac{\delta^2}{1+\delta^2} - \frac{1}{\omega\tau}\frac{1}{1+\delta^2}\right)\right] \quad (12.1.17)$$

where

$$\delta = \frac{1}{\gamma}\frac{T_e N_e}{T_n N_n}\frac{\omega}{\Omega_n}$$

and

$$\frac{1}{\tau_n} = \left(\frac{\omega}{c}\right)^2\left[\tfrac{4}{3}\nu + (\gamma-1)\frac{K}{\rho C_p}\right] + \frac{\omega}{d}\left[\sqrt{\frac{2\nu}{\omega}} + (\gamma-1)\sqrt{\frac{2K}{\omega\rho C_p}}\right]$$

Quantity τ is defined in Eq. (12.1.14). It is not difficult to show that when $\delta \ll 1$ *and* when both $\omega\tau$ and $\omega\tau_n$ are *larger* than unity, the imaginary part of k reduces to $-\beta + a_1 + a_2$ and the implications of Eqs. (12.1.13) to (12.1.16) have validity.

Spontaneous excitations

In addition to the study of the behavior of an externally produced wave transmitted into the plasma, it is interesting to investigate the possibility of spontaneous excitations of normal modes. For simplicity, we consider the lateral modes between two rigid walls in a rectangular tube. The pressure field is then of the form $\cos(\pi n y/B)\exp(-i\omega_n t)$, the reflecting walls being at $y = 0$ and $y = B$. If we insert this into the wave equation (12.1.12), we obtain

$$\omega_n = \frac{\pi n c}{B}\sqrt{1 - \left(\frac{B}{2\pi n c\tau}\right)^2} + \frac{i}{2\tau} \quad (12.1.18)$$

where, as before, τ stands for the characteristic time defined in Eq. (12.1.14). Under the conditions of interest here, $B/2\pi n c\tau \ll 1$, and we note that the oscillations grow in time at a rate $\exp(t/2\tau)$.

To obtain a criterion for the onset of spontaneous oscillations, we must compare the growth rate with the decay caused by the acoustic losses in the gas. We now conveniently apply the general results obtained in Chap. 6 to the calculations of the energy losses in the lateral modes of oscillation of interest here.

In Eq. (6.4.13), the loss rate per unit volume in a fluid is given by

$$L = D + \frac{K}{T}(\text{grad }\theta)^2 + \text{div}\left(\frac{K}{T}\text{grad }T\right) \quad (12.1.19)$$

where D = viscous dissipation function defined in Eq. (6.4.10)
K = thermal conductivity coefficient
θ = temperature in the sound field

Since we are interested in the average of the loss rate over one period of oscillation, the last term in Eq. (12.1.19) does not contribute. As in Chap. 6, to calculate L in the bulk of the fluid, we use the loss-free field distribution, and the temperature field then can be expressed in terms of the sound pressure (assuming adiabatic conditions), to give

$$\langle L_{\text{bulk}} \rangle = \langle D \rangle + \frac{(\gamma - 1)K}{\rho C_P \rho c^2} \langle \text{grad } p \rangle^2$$

where the brackets indicate time average. For the one-dimensional motion of interest here we have $D = \left(\frac{4\mu}{3}\right)\left(\frac{\partial u_y}{\partial y}\right)^2$. As was shown in Chap. 6, the evaluation of L in the thin boundary layers leads to an expression for the average loss rate per unit area of the surface,

$$\langle L_s \rangle = \sqrt{\frac{\mu\omega}{2\rho}} \frac{\rho |u_s|^2}{2} + (\gamma - 1)\sqrt{\frac{K\omega}{2\rho C_P}} \frac{|p|^2}{2\rho c^2}$$

where $|u_s|$ is the amplitude of the tangential particle velocity just outside the boundary layer (which is approximately the same as the tangential particle velocity at the boundary in the loss-free solution), and $|p|$ is the sound-pressure amplitude at the boundary (loss-free solution).

In the case of the lateral oscillations in a rectangular tube of width B, we have $p = p_0 \cos(\pi n y/B)$, and $u_y = (p_0/\rho c) \sin(\pi n y/B)$. Since the motion is perpendicular to the walls $y = 0$ and $y = B$, only conduction losses will obtain, whereas at the remaining two walls we have both viscous and conduction losses. The total energy in the sound wave per unit length of the tube is $W = (p_0^2/4\rho c^2)A$, where A is the cross-sectional area of the duct. The loss rate is proportional to W, and if we call the constant of proportionality $2/\tau_n$, we have $dW/dt = -(2/\tau_n)W$. The time dependence of the pressure field is then given by $\exp(-t/\tau_n)$. From the expressions for the losses $\langle L_{\text{bulk}} \rangle$ and $\langle L_s \rangle$, we find for a duct with square cross section that the decay time for the nth mode is given by

$$\frac{T_1}{\tau_n} = \frac{8\pi^2}{3}\frac{n^2 l_n}{B} + \sqrt{2\pi}\left[1 + 3(\gamma - 1)\sqrt{\frac{K}{\eta c_P}}\right]\sqrt{\frac{nl}{B}} \simeq 26\frac{n^2 l_n}{B} + 8\sqrt{\frac{nl_n}{B}} \tag{12.1.20}$$

where we have used $K/C_P \approx 1.5\mu$ approximately valid for a monatomic gas. T_1 is the period of the first mode corresponding to $n = 1$. The two terms in (12.1.20) correspond to the bulk and surface losses, respectively. The criterion for the onset of spontaneous oscillations is given by $1/2\tau > 1/\tau_1$, where the growth and attenuation rates $1/2\tau$ and $1/\tau_1$ are given in Eqs. (12.1.18) and (12.1.20).

Ionization waves

Although the ionization wave, or traveling striation, is not an acoustic wave in the ordinary sense of the word, we shall describe it briefly, since we used it earlier in this section as a particular type of sound source in one of the examples of sound generation in a plasma. The ionization wave actually is a self-sustained oscillation of the plasma, which may be described approximately as follows: The plasma is maintained by a constant electric field E_0, which under steady-state conditions produces as many electrons (and ions) through ionization as are lost by diffusion to the walls in the discharge tube.

A perturbation in the electric field in the plasma produces a perturbation in the ionization rate, and hence a perturbation in the charge density, the electron density and the ion density being initially perturbed in the same manner. However, the electrons are more mobile than the ions and seek to distribute themselves according to a Boltzmann distribution in the periodic potential that corresponds to the electric field perturbation. Thus the electrons move away from the regions of large electric field, leaving an excess of ions. This change, in turn, gives rise to an electric field which is in the same direction as E_0 on one side of the ions and in the opposite direction to E_0 on the other side. This perturbation in the electric field is displaced with respect to the initial perturbation in the direction of E_0, which is the direction toward the cathode. We have then closed the loop, and we realize that under certain conditions the process can be self-sustained, so that the perturbation in the electric field travels toward the cathode without decrease in amplitude.

We can put these observations in somewhat more quantitative terms through the following linearized equations: The first expresses the fact that the perturbation in the production rate of the ion density is proportional to the electric field perturbation E and the equilibrium density N. We assume that the characteristic time of change is much longer than the time required for the electrons to become diffused. Thus

$$\frac{\partial n_+}{\partial t} = \gamma E N$$

The second equation is simply the Poisson equation

$$\frac{\partial E}{\partial x} = 4\pi e n_+$$

where e is the unit charge.

A plane wave of the form $\exp(ikx - i\omega t)$ satisfies these equations of motion if the following dispersion relation obtains:

$$k = 4\pi e \gamma N \frac{1}{\omega}$$

This dispersion is unusual inasmuch as the wavelength is proportional to the frequency. As a result, the group velocity $d\omega/dk$ [Eq. (9.1.18)] is equal to the phase velocity but opposite in direction.

$$\frac{d\omega}{dk} = -\frac{\omega}{k} = -4\pi\frac{\gamma eN}{k^2}$$

From a more detailed analysis of this type of wave motion the conditions for wave growth can be established, but it is found that the relation between the group and phase velocity as given above is approximately correct under most conditions.

12.2 HIGHLY IONIZED GAS

In the weakly ionized gas considered in the preceding section, the number density of the charged particles was very small by comparison with the neutral particle density, say, one electron (and ion) for every 10^5 neutral particles. The dynamics of the gas is then much the same as for the neutral gas, and were it not for the energy transfer from the electrons, the charged particles could be regarded merely as "tracer" elements, which make possible the study of the motion of the gas by a study of the behavior of the charged particles. This is so because the charged particles are strongly coupled to the neutrals through collisions and, as was mentioned in the previous section, the relative changes in the density are all the same in the acoustic mode of motion. For this motion, charge separation and the creation of (fluctuating) electric field perturbations in the gas are of little or no consequence. The speed of sound in the gas is, for all practical purposes, the same as in the neutral gas. The contribution from the ions or even from the electrons to the total pressure in the gas is negligible, and the speed of sound is determined by the thermal speed in the neutral gas.

Sound waves

In the highly ionized gas the situation is different. The effect of the neutrals can be neglected, and we are left with the ions and the electrons, which have different thermal speeds and, in general, even different temperatures; the electron temperature usually is much greater than the ion temperature. Thus there is no longer one unique thermal speed that might be approximately identified with the sound speed, as in the case of the neutral gas. However, we can express the speed of sound as $\sqrt{\gamma P/\rho}$, where γ is an appropriate specific-heat ratio. The total pressure P in the plasma is the sum of the ion pressure and the electron pressure $P = P_i + P_e$, and the total density is $\rho = \rho_i + \rho_e = N(m_i + m_e) \simeq Nm_i$, where N is the particle density of the ions (and the electrons) and m_i and m_e are the ion mass and the electron mass. Since under ordinary conditions $P_e \gg P_i$, we expect that the

speed of an ordinary sound wave in the highly ionized gas will be

$$c = \sqrt{\frac{P_e\gamma}{Nm_i}} = \sqrt{\frac{\gamma kT_e}{m_i}} \tag{12.2.1}$$

An analysis of the various wave modes in a plasma shows that there is indeed an acoustic wave mode that travels with a speed given by (12.2.1). We note that in this mode, often called an ion-acoustic wave, the ions provide the inertia, whereas the electrons provide the "restoring" force.

Plasma oscillations

In the acoustic mode of motion of the plasma there is no charge separation, and the wave motion represents a perturbation of the density and the corresponding (adiabatic) change in the temperature. We shall now consider a mode of motion which is longitudinal but in which charge separation and the related electric field are the essential characteristics.

It is very simple to get a qualitative idea about the nature of this mode of motion. Consider an initially unperturbed portion of the plasma with an equal number N of ions and electrons. Suppose the electrons in this region are displaced an amount ξ from their equilibrium position. This will leave an excess $N\xi$ of ions per unit area, and an electric field is established between the electrons and the ions, $E = 4\pi eN\xi$. The corresponding restoring force on one of the displaced electrons is then $4\pi e^2N\xi$, proportional to the displacement. As a result, the electrons will perform an oscillatory motion with a frequency given by

$$\omega_p^2 = \frac{4\pi e^2N}{m_e} \tag{12.2.2}$$

According to Eq. (12.2.2), the frequency of oscillation does not depend on the wavelength of the oscillations, which means that the oscillation has zero group velocity, $d\omega/dk = 0$. In this qualitative analysis, however, the thermal motion of the electrons, clearly, is not accounted for, and the expression for the frequency obtained can only be approximately correct. Furthermore, this simple picture of the motion does not provide for any damping of the oscillation.

To explore the motion further, we shall carry out here a hydrodynamic analysis of plasma oscillations. Let the electron density be $\rho = Nm_e$, with the perturbation $\delta = nm_e$. We assume the ions to be stationary to form a uniform positive background to keep the plasma neutral on the average. The electric field $\mathbf{E}$ in the plasma is then due solely to the perturbation in the electron density, and we have

$$\text{div } \mathbf{E} = -4\pi ne \tag{12.2.3}$$

The equations of mass and momentum balance of the electrons are

$$\frac{\partial \delta}{\partial t} + \rho \operatorname{div} \mathbf{u} = 0 \tag{12.2.4}$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\operatorname{grad} p - Ne\mathbf{E} \tag{12.2.5}$$

where p is the perturbation in the electron pressure. Introducing a characteristic thermal speed v, we may set $p = v^2\delta = v^2 nm$. Eliminating $\mathbf{u}$ and $\mathbf{E}$ and using $\delta = nm$, these equations lead to the following wave equation for δ:

$$\frac{\partial^2 \delta}{\partial t^2} = v^2 \nabla^2 \delta - \frac{4\pi e^2 N}{m} \delta \tag{12.2.6}$$

To find the dispersion relation, we insert the plane wave $\delta = \delta_0 \exp(ikr - i\omega t)$ and obtain

$$\omega^2 = \omega_p^2 + k^2 v^2 \qquad k = \frac{\omega}{v} \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2} \tag{12.2.7}$$

where

$$\omega_p = \sqrt{\frac{4\pi e^2 N}{m}}$$

is 2π times the plasma frequency. We note that k becomes imaginary when $\omega < \omega_p$, and the plasma wave then decays exponentially.

Comparing Eqs. (12.2.7) with the result (12.2.2) obtained from the qualitative observations, we see that by accounting for the thermal motion of the electrons (as we did here by introducing the electron pressure in the equations of motion), we get a dispersion relation that indicates that the oscillation is not stationary but that it propagates. The group velocity is $d\omega/dk = kv^2/\omega$, or $\dfrac{\omega}{k}\dfrac{d\omega}{dk} = v^2$; the product of the phase velocity V_p and the group velocity V_g is $V_p V_g = v^2$. The two velocities can be expressed as

$$V_p = \frac{\omega}{k} = \frac{v}{\sqrt{1 - (\omega/\omega_p)^2}}$$

$$V_g = v\sqrt{1 - (\omega/\omega_p)^2} \tag{12.2.8}$$

which shows that the phase velocity is always larger than v for a propagating wave and that the opposite holds true for the group velocity. We can also express the phase velocity as

$$V_p = \frac{\omega_p}{k} \sqrt{1 + \left(\frac{\lambda_d}{\lambda}\right)^2}$$

where we have introduced $\lambda_d = v(2\pi/\omega_p)$, the characteristic length traveled by an electron in thermal motion in a time equal to a period of plasma

oscillation. This length is called the *Debye length*. For wavelengths considerably longer than the Debye length, the phase velocity is approximately equal to ω_p/k, that is, proportional to the wavelength. For wavelengths of the order of λ_d or smaller, a more detailed analysis of plasma waves will show that the waves are strongly damped. This will be explained under the next heading.

The plasma wave is a longitudinal wave, a kind of acoustic wave, that involves fluid motion in the direction of propagation, as well as a longitudinal electric field. Returning to the equations of motion (12.2.3) to (12.2.5), we can now determine the relationship between the various field quantities. We find, for example,

$$p = \rho V_g u \qquad \delta = \frac{u}{V_p}\rho$$

which in our acoustic terminology means that the plasma-wave "impedance" relating the electron pressure and the particle velocity in the wave is the density times the group velocity. Similarly, we find

$$\frac{E^2}{8\pi} = \left(\frac{\omega_p}{\omega}\right)^2 \frac{\rho u^2}{2} \tag{12.2.9}$$

which says that the electric field energy density in the propagating plasma wave ($\omega > \omega_p$) is always smaller than the kinetic energy density due to the collective motion of the electrons. For $\omega = \omega_p$ they are equal.

After having obtained Eq. (12.2.9), it is interesting to determine the total energy density in the plasma wave. It is the sum of three parts, the kinetic energy of the electrons U_K, the electrostatic field energy U_E, and the "potential" energy U_p due to the compression of the electron gas (resulting in a perturbation of thermal energy). The potential energy in the wave can be expressed as $\kappa p^2/2$, where κ is the compressibility,

$$\kappa = \frac{\delta}{p\rho} = \frac{1}{\rho V_p V_g} = \frac{1}{\rho v^2}$$

Hence the potential energy density is

$$U_P = \frac{\rho^2 V_g^2}{2\rho v^2} u^2 = \left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right] \frac{\rho u^2}{2}$$

The total energy density in the wave then can be written

$$U = \frac{\rho u^2}{2} + \left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right] \frac{\rho u^2}{2} + \frac{E^2}{8\pi} = \rho u^2 = \frac{E^2}{4\pi}\left(\frac{\omega}{\omega_p}\right)^2 \tag{12.2.10}$$

where we have made use of Eq. (12.2.9).

Landau damping

In our hydrodynamical analysis of plasma oscillations we have neglected viscous stresses as well as heat conduction in the electron fluid, and the resulting plasma waves were undamped. However, we can account for the effect of viscosity and heat conduction in the same way as we did in the preceding discussion and in Chap. 6.

Actually, even in the absence of viscosity and heat conduction, a kinetic theory of plasma oscillations leads to a damping of the plasma waves. This damping, however, cannot be explained on the basis of the hydrodynamical model, since it is related to the trapping of electrons in the traveling periodic potential produced by the plasma wave and depends on the form of the velocity distribution function of the electrons. The hydrodynamic analysis deals only with averages of the distribution function. The electrons that will be trapped by the wave are those with a thermal velocity component v_x in the direction of wave propagation about equal to the phase velocity ω/k of the wave. As we shall see, the degree of damping resulting from this trapping depends on the slope of the velocity distribution function in the neighborhood of $v_x = \omega/k$.

To study the trapping mechanism to obtain an expression for the Landau damping, let the unperturbed velocity distribution function of the electrons be $f(v_x,v_y,v_z)$ such that

$$N = \iiint f(v_x,v_y,v_z)\, dv_x\, dv_y\, dv_z$$

If propagation of the plasma wave is chosen to be in the x direction, we shall be interested mainly in the dependence of f on v_x. It is convenient, therefore, to introduce

$$g(v_x) = \frac{1}{N}\iint f(v_y,v_z)\, dv_y\, dv_z \tag{12.2.11}$$

The plasma wave represents a periodic electric field $E = E_0 \sin kx$ and a corresponding potential $V = (E_0/k)\cos kx$ that travels with a velocity $V_p = \omega/k$ in a direction which we choose to be the x direction. Imagine that the wave is formed at $t = 0$. The electrons which at this time have a velocity $(V_p - \Delta v) < v_x < (V_p + \Delta v)$ will be trapped in the potential well of the plasma wave if the velocity increment Δv is such that

$$\frac{m}{2}(\Delta v)^2 = \frac{eE_0}{k} \tag{12.2.12}$$

This value of Δv should be regarded as an average since for any particular electron the actual value of the velocity required to make the electron go over the potential barrier depends upon the location of the electron at $t = 0$ in the potential wave trough. Electrons that have a velocity in the range $(V_p - \Delta v) < v_x < V_p$ will be speeded up and receive energy as a result of the

trapping, whereas the opposite holds true for the electrons in the range $V_p < v_x < V_p + \Delta v$. The corresponding energy will be drawn from or transferred to the plasma wave.

An electron that has been trapped will perform an oscillatory motion in the potential wave trough with a frequency of oscillation which, at least for sufficiently small amplitudes, is

$$\omega_c = \sqrt{\frac{eE_0k}{m}} \tag{12.2.13}$$

A trapped electron with the initial velocity v_x oscillates back and forth with a velocity amplitude $v_x - V_p$, and its average total energy of oscillation is $(m/2)(v_x - V_p)^2$. In addition, the trapped electron has the kinetic energy of translation $(m/2)V_p^2$ with respect to the laboratory system. Therefore the net amount of energy gained by the electrons, and hence lost by the plasma wave, is

$$\Delta W = \frac{m}{2}\int_{V_p-\Delta v}^{V_p+\Delta v} g(v_x)\,dv_x\,[V_p^2 + (v_x - V_p)^2 - v_x^2] \tag{12.2.14}$$

Since we are interested in values of v_x in the neighborhood of V_p, we expand $g(v_x)$.

$$g(v_x) = g(V_p) + (v_x - V_p)g'(V_p) + \cdots$$

where $g'(V_p) = (dg/dv_x)_{V_p}$. If we insert into (12.2.14) this expansion of $g(v_x)$, keeping only the first two terms, we obtain

$$\Delta W = -\frac{2m}{3}V_p g'(V_p)(\Delta v)^3 \tag{12.2.15}$$

The trapping process, and hence the energy exchange between the electrons and the wave, takes place approximately in one period $T_c = 2\pi/\omega_c$ of the oscillations of the electrons in the potential well, and as an average rate of energy transfer we may use $\omega_c \Delta W$. Then, if the internal energy density of the plasma oscillations is denoted by U, we have

$$\frac{\partial U}{\partial t} = -\omega_c\,\Delta W \tag{12.2.16}$$

Now we can express Δv and ω_c in terms of the electric field E, using Eqs. (12.2.12) and (12.2.13), which yields ΔW in terms of E^2. From Eq. (12.2.10), E^2 can be expressed in terms of U to give

$$\frac{\partial U}{\partial t} = -\frac{4\sqrt{2}}{3}\frac{\omega}{k}\frac{\omega_p^2}{k}g'\left(\frac{\omega}{k}\right)\left(\frac{\omega_p}{\omega}\right)^2 U \tag{12.2.17}$$

The corresponding decay of the amplitude of the plasma oscillations is then

exp $(-\beta t)$, where

$$\frac{\beta}{\omega} = \frac{2\sqrt{2}}{3}\left(\frac{\omega_p}{k}\right)^2\left(\frac{\omega_p}{\omega}\right)^2 g'\left(\frac{\omega}{k}\right) \tag{12.2.18}$$

For a Maxwellian velocity distribution of the electrons, we have $g(v_x) = (1/c\sqrt{\pi}) \exp(-v_x^2/c^2)$ and $g'(\omega/k) = -(2\omega/kc^3\sqrt{\pi}) \exp(-\omega^2/k^2c^2)$. The corresponding expression for the decay constant becomes

$$\beta = \frac{4\sqrt{2}}{3\sqrt{\pi}}\omega_p\left(\frac{\omega_p}{kc}\right)^3 \exp\left(-\frac{\omega^2}{k^2c^2}\right) \tag{12.2.19}$$

where $c^2 = 2kT_e/m$, T_e being the electron temperature.

12.3 MAGNETOACOUSTIC WAVES

In the presence of a magnetic field, the plasma becomes nonisotropic, and the wave motion becomes considerably more complicated than in the absence of the field. A systematic study of this problem would carry us far beyond the scope of this chapter, and we shall limit the discussion to low-frequency phenomena in which we can neglect charge-separation effects.

To save 4π's in this section, we shall use a "normalized" set of electromagnetic units; in this section only, $\mathbf{B} = \mathbf{B}_c/\sqrt{4\pi}$; $\mathbf{D} = \mathbf{D}_c/\sqrt{4\pi}$; and $\mathbf{j} = \sqrt{4\pi}\,\mathbf{j}_c$; $\mathbf{B}_c$, $\mathbf{D}_c$, and $\mathbf{j}_c$ being in the usual cgs units.

Magnetic stress tensor

The electrons and ions then move together as in a single conducting fluid. Under these conditions the principal effect of a magnetic field $\mathbf{B}$ on the motion of the conducting fluid is to produce a force per unit volume of the fluid,

$$\mathbf{F} = \frac{1}{c}(\mathbf{j} \times \mathbf{B}) \tag{12.3.1}$$

where $\mathbf{j}$ is the current density. The current density, in turn, is related to the magnetic field through the Maxwell equation

$$c\,\text{curl}\,\mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \simeq \mathbf{j} \tag{12.3.2}$$

In the last step we have neglected the displacement current $\partial\mathbf{D}/\partial t$, which is justified in a fluid with high conductivity, at least for the low-frequency phenomena we wish to study here. To see this, we note that, in a periodic field with a frequency ω, $\partial\mathbf{D}/\partial t$ is of the order of $\omega\epsilon\mathbf{E}$ and $\mathbf{j} = \sigma\mathbf{E}$, where ϵ is the dielectric constant, σ the conductivity, and $\mathbf{E}$ the electric field. Thus the ratio between the magnitudes of the displacement current and the conduction current is then $(\omega\epsilon/\sigma)$, which is much less than unity for the problems to be considered here.

Now if we introduce $\mathbf{j} = c \operatorname{curl} \mathbf{B}$ into (12.3.1) and make use of the well-known relation $\mathbf{B} \times \operatorname{curl} \mathbf{B} = \operatorname{grad} (B^2/2) - (\mathbf{B} \cdot \nabla)\mathbf{B}$, it follows that the ith component of the force in Eq. (12.3.1) can be written in the form, remembering that $\Sigma\, \partial B_j/\partial x_j = 0$,

$$F_i = -\sum_j \frac{\partial}{\partial x_j}\left(\frac{B^2}{2}\,\delta_{ij} - B_i B_j\right)$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. This expression for F_i makes it possible to write the equation of motion in the form

$$\frac{\partial(\rho u_i)}{\partial t} + \sum_j \frac{\partial}{\partial x_j}\left(P\delta_{ij} + \rho u_i u_j + \frac{B^2}{2}\,\delta_{ij} - B_i B_j\right) = 0 \qquad (12.3.3)$$

This way of writing the equation of motion demonstrates that the magnetic field is dynamically equivalent to a mechanical stress tensor $(B^2/2)\delta_{ij} - B_i B_j$. To see the physical meaning of this stress a little better, we express it with reference to a coordinate system, with the x_1 axis in the direction of the magnetic field. The components of the stress are then given by $(B^2/2)\delta_{ij} - B^2\delta_{i1}\delta_{j1}$. In other words, the effect of the magnetic field is equivalent to an isotropic magnetic pressure component $B^2/2$ and a "pull" B^2 per unit area in the direction of the field lines.

The magnetic Reynolds number

Before proceeding to study specific motions, it is interesting to mention an additional general property of motion which has to do with the coupling between the magnetic field and the fluid velocity field. We shall find that in the limit of infinite conductivity, the coupling is so strong that the magnetic field lines are convected with the fluid as if they were "trapped by" or "frozen into" the fluid. To prove this important aspect of the motion, we first seek an expression for the time rate of change of the magnetic field in a coordinate system moving with the fluid. We start from the expression for the current density,

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{\mathbf{U}}{c} \times \mathbf{B}\right) \qquad (12.3.4)$$

together with Maxwell's equations,

$$c \operatorname{curl} \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \qquad \operatorname{div} \mathbf{B} = 0 \qquad (12.3.5)$$

$$c \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \operatorname{div} \mathbf{E} = 0 \qquad (12.3.6)$$

where we have assumed zero net charge density in the fluid.

From these equations follows

$$\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl}\left(\frac{\mathbf{j}}{\sigma} - \frac{\mathbf{U}}{c} \times \mathbf{B}\right) \qquad (12.3.7)$$

and if we neglect the displacement current $\partial \mathbf{E}/\partial t$ we get

$$\frac{\partial \mathbf{B}}{\partial t} - \text{curl}\,(\mathbf{U} \times \mathbf{B}) = -\frac{1}{\sigma} c^2 \,\text{curl}\,(\text{curl}\,\mathbf{B}) = \frac{c^2}{\sigma} \nabla^2 \mathbf{B} \qquad (12.3.8)$$

By making use of curl $(\mathbf{U} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{B}$ and introducing $d/dt = \partial/\partial t + \mathbf{U} \cdot \nabla$, this equation can be expressed as

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B} \cdot \nabla)\mathbf{U} + \frac{c^2}{\sigma} \nabla^2 \mathbf{B} \qquad (12.3.9)$$

Next consider two fluid particles located at the positions $\mathbf{r}$ and $\mathbf{r} + \mathbf{dr}$ on a particular magnetic field line at time t. The separation between the particles is specified by the vector $\mathbf{dr}$, which has the same direction as $\mathbf{B}$, so that we can set $\mathbf{dr} = \epsilon \mathbf{B}$, where ϵ is a small quantity. As the fluid moves, the separation between the particles will also change, in general, so that at time $t + dt$ the new vector is given by

$$\mathbf{dr}' = \mathbf{dr} + (\mathbf{dr} \cdot \nabla)\mathbf{U}\, dt = \epsilon[\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{U}\, dt]$$

However, for the time dependence of the $\mathbf{B}$ vector, we have already seen that, for $\sigma \to \infty$

$$\frac{d\mathbf{B}}{dt} = (\mathbf{B} \cdot \nabla)\mathbf{U} \qquad (12.3.10)$$

so that

$$\mathbf{dr}' = \epsilon(\mathbf{B} + \mathbf{dB}) = \epsilon \mathbf{B}'$$

where $\mathbf{B}'$ is the magnetic field vector at $t + dt$. In other words, $\mathbf{dr}$ and $\mathbf{B}$ remain lined up at all times. We can interpret this result pictorially as a comovement of the magnetic field lines with the fluid, just as if the lines were embedded in the fluid like rubber bands that stretch and bend to conform to the variations in the fluid flow field.

This interpretation, taken together with the result obtained earlier that the electromagnetic stress corresponds to a tension along the field lines equal to B^2 per unit area, suggests the existence of a transverse wave motion in the fluid in much the same way as on a stretched string. To see this, let the unperturbed magnetic field be a uniform field in the z direction of magnitude B. The fluid is initially at rest. Consider now perturbation of the fluid in the form of a local displacement perpendicular to the z axis. This leads to a local transverse displacement of the magnetic field lines. Since there is a force B^2 along the field lines per unit area, and since a fluid string of unit area has a mass per unit length equal to ρ, the density of the fluid, we expect, by analogy with wave motion on a string, that the transverse displacement of the fluid and the magnetic field "string" will propagate with a speed $\sqrt{B^2/\rho} = B/\sqrt{\rho}$. We shall show under the next heading that this observation is consistent with more quantitative detail.

If the conductivity of the fluid is finite, the coupling between the magnetic field and the fluid motion is not complete; the field lines will diffuse, or "slip"

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through the fluid. As a result, the wave motion described above will be damped. To discuss the degree of coupling on somewhat more quantitative terms, let us return to Eq. (12.3.9). If, for simplicity, we consider two-dimensional motion of the fluid in a magnetic field perpendicular to the flow velocity, the equation reduces to

$$\frac{d\mathbf{B}}{dt} = \left(\frac{c^2}{\sigma}\right) \nabla^2 \mathbf{B}$$

which describes the diffusion of the magnetic field in a coordinate system moving with the fluid. We rewrite this equation in dimensionless form, introducing a characteristic length L and a characteristic time T and a corresponding characteristic velocity $U = L/T$ of the fluid. Then, with $\tau = t/T$ and $\xi = x/L$, we get

$$\frac{d\mathbf{B}}{d\tau} = \frac{c^2 T}{\sigma L^2} \nabla_\xi^2 \mathbf{B} = \frac{1}{R_m} \nabla_\xi^2 \mathbf{B}$$

The dimensionless quantity

$$R_m = \frac{UL}{(1/\sigma)c^2} \tag{12.3.11}$$

is called the *magnetic Reynolds number*. It plays the same role in regard to diffusion of the magnetic field as does the ordinary Reynolds number $UL/(\mu/\rho)$ in regard to diffusion of vorticity. The characteristic time of diffusion of the magnetic field is given by

$$T_d = \frac{L^2 \sigma}{c^2}$$

The characteristic time available for the diffusion can be expressed as $T = L/U$, and a necessary condition that the magnetic field be trapped in the fluid is $T \ll T_d$, which means $U \gg c^2/L\sigma$.

This, obviously, is not a sufficient condition for good coupling, however. We must also have a magnetic field present, and for this magnetic field to have a pronounced effect on the motion, it is necessary that the magnetic-field energy density $B^2/2$ be at least of the same order of magnitude as the kinetic energy density $\rho U^2/2$ of the fluid, so that $B^2 > \rho U^2$. If we now use the condition for the velocity $U \gg c^2/L\sigma$ obtained above, we get a condition for the magnetic field,

$$B \gg \frac{c^2\sqrt{\rho}}{L\sigma} \qquad \text{or} \qquad D_m = \frac{BL\sigma}{c^2\sqrt{\rho}} \gg 1 \tag{12.3.12}$$

to assure good coupling between the magnetic field and the velocity. This condition is well satisfied in astrophysical situations, where D_m may range from, say, 10 for the ionosphere to 10^{10} for the solar corona and hot interstellar gases. On the laboratory scale D_m is considerably smaller, say, between 1 and 100 in experiments with liquid metals and hot plasmas.

Transverse waves (Alfvén waves)

The qualitative observations in the preceding discussion about transverse waves traveling along magnetic field lines will now be verified by a study of the linearized equations of motion. We shall choose a particularly simple situation in which the fluid performs one-dimensional motion perpendicular to a uniform external magnetic field. Thus, with an external magnetic field B_0 in the z direction and with a fluid motion in the y direction, we have $\partial/\partial x = \partial/\partial y = 0$. The induced current $\mathbf{j}$ in the fluid then has only an x component, and the corresponding induced perturbation in the magnetic field $\mathbf{b}$ has only a y component. Under these conditions, and assuming an incompressible fluid, Maxwell's equations (12.3.4) to (12.3.6) and the momentum equation (12.3.3) are considerably simplified, and the linearized forms of the equations are

$$\begin{aligned} j_x &= -c\frac{\partial b_y}{\partial z} \qquad & \frac{\partial b_y}{\partial t} &= -c\frac{\partial E_x}{\partial z} \\ j_x &= \sigma E_x + \frac{v_y B_0}{c} \qquad & \rho\frac{\partial b_y}{\partial t} &= -c^{-1} j_x B_0 \end{aligned} \tag{12.3.13}$$

From these equations follows the "wave equation" for b_y:

$$\frac{\partial^2 b_y}{\partial t^2} - \frac{B_0^2}{\rho}\frac{\partial^2 b_y}{\partial z^2} - \frac{c^2}{\sigma}\frac{\partial^3 b_y}{\partial t\,\partial z^2} = 0 \tag{12.3.14}$$

Seeking a wavelike solution of the form $\exp(ikz - i\omega t)$, we have $\partial^2/\partial t^2 = -\omega^2$, $\partial^2/\partial z^2 = -k^2$, and $\partial^3/\partial t\,\partial z^2 = +ik^2\omega$. Equation (12.3.14) then yields

$$k^2 = \frac{\omega^2}{{c_A}^2}\frac{1}{1 - i(c^2\omega/\sigma {c_A}^2)} = \frac{\omega^2}{{c_A}^2}\frac{1}{1 - i\omega\tau} \tag{12.3.15}$$

where

$$c_A = \frac{B}{\sqrt{\rho}} \qquad \tau = \frac{c^2}{{c_A}^2}\frac{1}{\sigma}$$

The time $\tau = \dfrac{c^2}{{c_A}^2}\dfrac{1}{\sigma}$ can be regarded as a "relaxation time" of the magnetic field lines.

If the conductivity is infinite, we get $k = \omega/c_A$, which indicates that the wave speed is $c_A = B/\sqrt{\rho}$, in agreement with the preliminary observations in the previous subsection. If σ is large enough so that $\omega\tau \ll 1$,

$$k \simeq \frac{\omega}{c_A}\left(1 + i\frac{\omega\tau}{2}\right)$$

which means that in this limit the phase velocity is c_A and the wave amplitude is attenuated exponentially, corresponding to the factor $\exp(-\alpha x)$, where

$$\alpha = \frac{c^2}{2\sigma}\frac{\omega^2}{{c_A}^3} \qquad \text{or} \qquad \alpha\lambda = \pi(\omega\tau)$$

The general expressions for the phase velocity and the attenuation constants are

$$c = c_A \frac{\sqrt{2}\sqrt{1+(\omega\tau)^2}}{\sqrt{1+\sqrt{1+(\omega\tau)^2}}} \to \begin{cases} c_A & \omega\tau \ll 1 \\ c_A\sqrt{2\omega\tau} & \omega\tau \gg 1 \end{cases}$$
$$\lambda\alpha = \frac{\sqrt{2}\,\pi\omega\tau}{\sqrt{1+\sqrt{1+(\omega\tau)^2}}} \to \begin{cases} \pi(\omega\tau) & \omega\tau \ll 1 \\ \pi\sqrt{2\omega\tau} & \omega\tau \gg 1 \end{cases} \qquad (12.3.16)$$

where $\lambda = c_A(2\pi/\omega)$.

As an example, let us calculate the energy flow in an Alfvén wave. The wave is generated by a source which produces an oscillatory motion of the fluid, $v_y = v_0 \exp(-i\omega t)$ in the plane $z = 0$. The constant external magnetic field B_0 points in the z direction, and as we have seen before, the transverse motion generated at $z = 0$ will be propagated in the z direction with a speed $c_A = B_0/\sqrt{\rho}$. From Eq. (12.2.13) it follows that the perturbation in the magnetic field is proportional to v_y. In fact, we have $\rho\,\partial v_y/\partial t = B_0\,\partial b_y/\partial z$, and with $v_y = v_0 \exp[i(\omega z/c_A) - i\omega t]$, it follows that

$$B_0 b_y = -\rho c_A v_y \qquad \text{or} \qquad \frac{b_y^2}{2} = \frac{\rho v_y^2}{2} \qquad (12.3.17)$$

This shows that the magnetic and kinetic energy densities in the wave are the same.

The energy flow in the wave can be obtained in several different ways. For example, we can express it as c_A times the total energy density, $I_z = [(\rho v_y^2/2) + (b_y^2/2)]c_A = (\rho v_y^2)c_A$. It is instructive to derive this result in a different manner, using the magnetic stress tensor. As we recall, the magnetic stress tensor $(B^2/2)\delta_{ij} - B_iB_j$ in Eq. (12.3.3) plays the same role as the pressure tensor, and the components of the related energy flow vector $\mathbf{I}$ can be expressed as

$$I_i = \sum_j \left[\left(\frac{B^2}{2}\right)\delta_{ij} - B_iB_j\right]v_j$$

In this example the velocity $\mathbf{v}$ has only the component v_y, and the energy flow in the z direction again is found to be

$$I_z = -B_zB_yv_y = -B_0b_yv_y = \rho c_A v_y^2$$

where we have used Eq. (12.3.17).

The force required in the source plane $z = 0$ to produce the wave must be such that $F_yv_y = \rho c_A v_y^2$, or

$$F_y = \rho c_A v_y$$

The ratio $F_y/v_y = \rho c_A$ is the wave impedance of the Alfvén wave.

Longitudinal waves

Next we wish to study the effect of a magnetic field on the propagation of a sound wave in a conducting fluid. It is clear that there will be no effect if the waves are propagated in the direction of the magnetic lines, and that the largest effect is to be expected when the direction of propagation is perpendicular to the magnetic field. We shall start by studying this particular case. As a preliminary observation we note that on the basis of the previous findings about the coupling between the fluid and the magnetic field, we may expect that the magnetic field will provide an additional "restoring" force for the particle motion in the fluid, and this should result in an increase in the speed of sound.

In the mathematical analysis of the problem let the uniform magnetic field be in the z direction and the sound wave in the x direction. The linearized equations of motion are then

$$\begin{gathered}\frac{\partial \delta}{\partial t} + \rho_0 \frac{\partial v_x}{\partial x} = 0 \\ \rho_0 \frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} - c^{-1} j_y B_z \\ j_y = \sigma(E_y - c^{-1} v_x B_x) \qquad j_y = -c \frac{\partial b_z}{\partial x} \\ p = c_s^2 \delta \qquad \frac{\partial b_z}{\partial t} = -c \frac{\partial E_y}{\partial x}\end{gathered} \tag{12.3.18}$$

A plane wave of the form $\exp(ikx - i\omega t)$ inserted in these equations leads to the following dispersion relation:

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 + c_A^2 \frac{1}{1 + i\omega\tau_1 (c_s k/\omega)^2}$$

where $\tau_1 = c^2/c_s^2\sigma$, $c_s^2 = 1/\kappa\rho_0$, and $c_A^2 = B^2/\rho$. As expected, the magnetic field increases the phase velocity of the sound wave. In fact, in the limiting case of infinite conductivity, so that $\omega\tau_1 = 0$, the phase velocity is $\omega/k = \sqrt{c_s^2 + c_A^2}$. In the other limiting case of zero conductivity, we get $\omega/k = c_s$, as expected. For intermediate values of the conductivity, the propagation constant k is complex for real values of ω, and the sound wave is attenuated through conduction losses. If $\omega\tau_1 \ll 1$, we can insert $c_s^2 + c_A^2$ for $(\omega/k)^2$ in the denominator, and with $\tau_2 = \tau_1[c_s^2/(c_A^2 + c_s^2)] = c^2/(c_A^2 + c_s^2)\sigma$, we obtain

$$k \simeq \frac{\omega}{\sqrt{c_s^2 + c_A^2}} \left(1 + \frac{1}{2} \frac{c_A^2}{c_A^2 + c_s^2} i\omega\tau_2\right)$$

$$\tau_2 = \frac{c^2}{c_A^2 + c_s^2} \frac{1}{\sigma}$$

from which it follows that for $\omega\tau_2 \ll 1$, the attenuation of the wave is proportional to ω^2. It is interesting to note, however, that in both limits, $\omega\tau_2 \to 0$ and $\omega\tau_2 \to \infty$, the attenuation is zero, and consequently attenuation will attain a maximum value for some value of $\omega\tau_2$ (of the order of unity). We leave it as a problem to study the dispersion relation in further detail.

Coupled waves

Under the last two headings we have considered two special cases of motion: first, the transverse (Alfvén) waves traveling along the magnetic field lines, and second, ordinary sound waves traveling perpendicular to the field lines. In the analysis of these waves an arbitrary value of the conductivity was used which enabled us to determine not only the phase velocity of the waves, but also the attenuation.

To carry through a similar general analysis of waves traveling at an arbitrary angle with respect to the magnetic field lines is quite involved, and we shall consider here only the case when the conductivity is infinite.

In analyzing this problem, we first have to deal with the questions concerning the definition of directions, etc. The two prominent directions involved in the problem defined by $\mathbf{k}$ and $\mathbf{B}$ establish a plane, and as we shall see, the nature of the wave depends markedly on whether the fluid velocity is parallel with or perpendicular to this plane. In the latter case we expect the wave motion to be, simply, an Alfvén wave having a speed equal to $B\cos\theta/\sqrt{\rho}$, where θ is the angle between the direction of propagation and magnetic field. As we recall, the perturbation $\mathbf{b}$ of the magnetic field in such a wave is in the same direction as the particle velocity in the wave motion, and if the plane of $\mathbf{k}$ and $\mathbf{B}$ is taken to be the xy plane, the only components of $\mathbf{u}$ and $\mathbf{b}$ then are u_z and b_z. If, on the other hand, the particle velocity is in the plane of $\mathbf{k}$ and $\mathbf{B}$, the wave mode is expected to contain both a transverse and a longitudinal motion, corresponding to the particle velocity components perpendicular to and parallel with $\mathbf{k}$, respectively. The perturbation in the magnetic field again should be perpendicular to $\mathbf{k}$. If the direction of $\mathbf{k}$ is the x axis, this magnetic field perturbation will have a b_y component only. These qualitative observations are easily verified from the linearized equations of motion of a conducting fluid in a magnetic field. In the case of infinite conductivity, and when the wave perturbation is of the form $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$, these equations are

$$\omega\delta = \rho_0(\mathbf{k}\cdot\mathbf{u}) \qquad \omega\mathbf{b} = -[\mathbf{k}\times(\mathbf{u}\times\mathbf{B}_0)] \tag{12.3.19}$$
$$\rho_0\omega\mathbf{u} = \mathbf{k}c^2\delta + [\mathbf{B}_0\times(\mathbf{k}\times\mathbf{b})]$$

where δ, $\mathbf{u}$, and $\mathbf{b}$ are the complex amplitudes of the perturbations in density, velocity, and magnetic field.

It follows from the first of these equations that a density fluctuation is produced only if there is a velocity component in the direction of propagation and the perturbation in the magnetic field is always perpendicular to $\mathbf{k}$.

Furthermore, if the plane defined by **k** and **B** is chosen to be the xy plane, with the x axis along **k**, we find that the velocity component u_z belongs to a transverse wave motion (Alfvén wave) that travels in the x direction with a phase velocity

$$c_1 = \frac{B_0}{\sqrt{\rho_0}} \cos\theta = \frac{B_x}{\sqrt{\rho_0}}$$

where θ is the angle between **k** and $\mathbf{B}_0$, and $B_0/\sqrt{\rho_0} = c_A$ is the characteristic Alfvén speed in the fluid. This mode of motion represents one of three possible characteristic modes of motion that can exist in the fluid. The remaining two modes correspond to wave motion with the fluid velocity components in the xy plane. We can regard these modes as coupled waves involving both transverse and longitudinal motions. One of them is more like an Alfvén wave with a touch of sound in it, and in the other mode the opposite holds true. To find the dispersion relation for these waves, we insert a wave motion with the particle velocity in the **kB** plane. After some algebra we then find that two modes exist with the phase velocities

$$c_{2,3} = \frac{c_s}{2}\left(\sqrt{1 + \alpha^2 + 2\alpha\cos\theta} \pm \sqrt{1 + \alpha^2 - 2\alpha\cos\theta}\right) \quad (12.3.20)$$

where $\alpha = c_A/c_s$, and c_s = sound speed $= \sqrt{1/\kappa\rho_0}$. For small values of the magnetic field, so that $\alpha < 1$, the two modes, apart from second-order terms in α, reduce to an ordinary sound wave with velocity c_s and an Alfvén wave with the velocity $c_A \cos\theta$. We note that if propagation is in the direction of the magnetic field so that $\theta = 0$, the two waves reduce to a pure sound wave and a pure Alfvén wave. For propagation perpendicular to the magnetic field, so that $\theta = \pi/2$, the Alfvén-wave component is absent, and we have only a sound wave with a speed $c_s\sqrt{1 + \alpha^2}$.

Problems

1 A uniform plasma is maintained in a region of length L in a long tube by means of an electric field $\mathbf{E}_0$. The electric field is modulated by a periodic electric field $\mathbf{E}_1 \sin(\omega t)$ with $E_1 \ll E_0$. As a result of this modulation, sound is generated. Derive an expression for the sound field outside the plasma region as well as inside this region. At what frequencies of modulation is the sound pressure field outside the plasma a maximum (minimum)? The tube is so long that reflections can be neglected.

2 The intensity of a microwave beam that has been transmitted through a plasma depends on the electron density of the plasma, and the fractional change in this intensity is approximately equal to the fractional change in electron density. If such a modulation is to be produced by a traveling sound wave, what should be the pressure amplitude of the wave to produce a 1 percent modulation of the transmitted microwave intensity? The plasma parameters are helium gas, neutral-particle density $= 10^{16}\,\text{cm}^{-3}$, electron density $= 10^{11}\,\text{cm}^{-3}$, neutral-gas temperature $= 500°\text{K}$.

3 In Prob. 2 let the electron temperature be 10^{4}°K, the diameter of the discharge tube 3 cm, and the electron-neutral elastic collision cross section 2×10^{-16} sq cm. Under these conditions is acoustic wave amplification possible? If so, determine the frequency range in which amplification is expected.

4 In the study of the amplification and spontaneous excitation of a sound wave we considered only the case of plane waves in a rectangular tube. Consider now the problems of spontaneous excitation of sound waves in a spherical discharge, and derive an expression for the growth rates of radial modes of oscillation. In this case the distribution of electron density and electron temperature is such that Eq. (12.1.12) is of the form

$$\frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 2\beta j_0\left(\frac{\pi r}{R}\right)\frac{\partial p}{\partial t}$$

where R is the radius of the spherical cavity, and β a constant that depends on the electron density and the electron temperature. Set $p = \Sigma P_n(t) j_0(\pi \alpha_n r/R)$, where j_0 is the zeroth-order spherical Bessel function, and $j_0'(\pi\alpha_n) = 0$. Show that the equation for P_n is of the form

$$\frac{d^2 P_n}{dt^2} + \omega_n{}^2 P_n = 2\beta \sum_m f_{n,m} \frac{dP_m}{dt}$$

where $\omega_n = \pi\alpha_n c/R$, and determine $f_{n,m}$ ($f_{1,1} \simeq 0.60$, $f_{1,2} \simeq 0.145$, etc.). Then set $P_n(t) = p_n \exp(-i\omega t)$ and derive, in determinant form, an equation for the eigenfrequencies ω. If, in the determinant, nondiagonal terms containing $f_{m,n}$ ($m \neq n$) are neglected, show that

$$\omega = i\beta f_{n,n} \pm \sqrt{\omega_n{}^2 - \beta^2 f_{n,n}} \simeq i\beta f_{n,n} \pm \omega_n$$

where $\beta f_{n,n}$ is the growth rate.

CHAPTER

13

ACOUSTOOPTICAL INTERACTION

13.1 LIGHT SCATTERING BY A SOUND BEAM

In the preceding chapter we studied some aspects of the acoustics of an ionized gas. We found that an important feature of the plasma is that the electron temperature is considerably higher than the neutral-gas temperature. Consequently, energy flows from the electrons to the neutrals, which may lead to the generation of sound. The problem of the interaction between light and sound can be viewed in much the same manner if we consider the light field as a photon gas. The photons have a considerably higher temperature than the medium carrying the sound wave, and as a result an energy flow into the fluid medium is to be expected.

The interaction between the electric field and the material medium is due to the response of the atomic electrons to the electric field, which results in a polarization of the material and a resulting index of refraction. This index depends on the density in the material; consequently, the density fluctuations in a sound wave produce a perturbation in the index of refraction. As a result an electromagnetic wave transmitted through the medium will be modulated by the sound wave, and scattering and refraction will occur. Conversely, a spatial variation of the electromagnetic field intensity and the corresponding electromagnetic stress lead to a volume force distribution in the medium (electrostriction), which, under certain conditions, leads to sound generation. This mutual interaction between light and sound may also lead to instabilities and wave amplification, just as in the case of electron-sound interaction discussed in the preceding chapter.

The coupling equation

To study the scattering and diffraction of an electromagnetic wave by sound, we start from the wave equation for the electromagnetic field **E**.

$$\nabla^2\mathbf{E} - \left(\frac{n}{c}\right)^2 \frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

where n is the index of refraction, and c is the speed of light in vacuum. This equation is valid also for a time-dependent index of refraction as long as $\frac{1}{n}\frac{\partial n}{\partial t} T \ll 1$, where T is the period of the oscillation in the **E** field. We write the unperturbed value of the index of refraction as n_0. If the perturbation n_1 caused by the sound wave is small, $n_1/n_0 \ll 1$, we have $n^2 = n_0^2 + 2n_0 n_1$, and the wave equation can be expressed as

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{2(n_1/n_0)}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\delta\epsilon/\epsilon_0}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{13.1.1}$$

In the last term we have introduced the relative variation in the dielectric constant, which is related to the index of refraction as $n \sim \sqrt{\epsilon}$, so that $2\delta n/n = \delta\epsilon/\epsilon_0$.

Similarly, the effect of the electric field on the sound field can be represented as a source term in the acoustic-wave equation, the source term being the divergence of the force distribution produced by the electromagnetic stress. The sound pressure, and therefore the perturbation in the dielectric constant, depends on the electric field, and the wave equations for the electric and acoustic fields are thus coupled, as will be discussed further in Sec. 13.3. However, in the study of the scattering of light by sound, we can ordinarily neglect the influence of the electric field on the sound field, so that $\delta\epsilon$ in Eq. (13.1.1) can be regarded as a known function of space and time, independent of **E**. The wave equation is of the same form as encountered in many of our acoustic-scattering problems.

As an example we shall treat here the problem of light diffraction by a sound beam (usually referred to as the Debye-Sears effect). If the width of the sound beam is not too large, we expect it to act in much the same way as an ordinary optical transmission grating with a grating constant equal to the wavelength λ_s of the sound wave. If, for example, a light wave is incident at right angles to the sound beam, we should expect that the angle for the first-order diffraction line from the "grating" would be given by $\sin\theta = \Lambda/\lambda_s$, where Λ is the wavelength of the incident light and θ is measured from the incident light beam. From the angle of diffraction θ and the wavelength Λ of light, the acoustic wavelength λ_s can be measured. However, the finite width of the ultrasonic grating, its time dependence, and motion make the acoustic grating somewhat more complex than the ordinary optical grating, and we shall devote our initial discussion to this problem.

The beam of sound is assumed to travel in the x direction and to fill the region between the two planes $z = L$ and $z = -L$, as indicated in Fig. 13.1. The incident light beam travels parallel to the xz plane at an angle θ with respect to the z axis. Inside the sound beam the index of refraction varies with the acoustic density fluctuations at the frequency $\omega_s/2\pi$ of the sound. This frequency is considerably lower than the frequency of the light $\Omega/2\pi$.

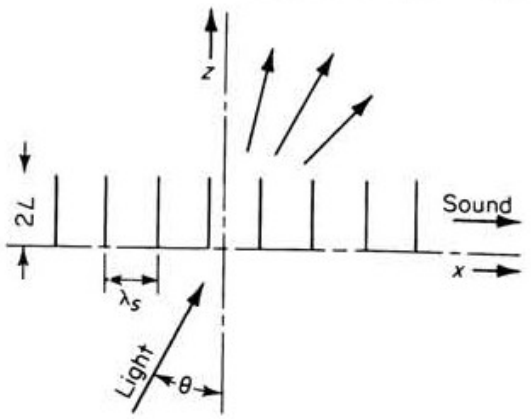


FIGURE 13.1
A sinusoidal wave of sound is a diffraction grating for light.

Actually, this fact justifies the omission of terms involving the rate of change $\partial n/\partial t$ of the index of refraction in the wave equation (13.1.1).

After a Fourier transform in time (and using the result of Prob. 23 of Chap. 1), this equation becomes

$$\nabla^2\mathbf{E}(x,z,\omega) + k^2\mathbf{E}(x,z,\omega) = -c^{-2}\int_{-\infty}^{\infty}(\omega')^2\mathbf{E}(x,z,\omega')\,f(\omega-\omega')\,d\omega' \tag{13.1.2}$$

where $k = \omega/c$, and f is the Fourier transform of $\delta\epsilon/\epsilon_0$. This is a vector wave equation; its general solution would involve dyadic Green's functions. However, for the purposes of this chapter, we need only deal with polarized waves, such that scalar-wave solutions are sufficient.

The Born approximation

The space-time dependence of $\delta\epsilon$ and the corresponding Fourier transform $f(\omega)$ depend on the sound beam that forms the diffraction grating. We shall consider here a single harmonic collimated beam of sound, with the acoustic pressure in the region between $z = -L$ and $z = L$, where the sound pressure may be set equal to $p = p_0\cos(k_s x - \omega_s t)$, where $2\pi/k_s = \lambda_s$ is the acoustic wavelength, and $\omega_s/2\pi$ is the frequency. The perturbation in the index of refraction will have the same form, and we shall set

$$\frac{\delta\epsilon}{\epsilon_0} = \eta\cos(k_s x - \omega_s t) \tag{13.1.3}$$

The Fourier transform f of $\delta\epsilon/\epsilon_0$ in Eq. (13.1.2) then is

$$f(\omega-\omega') = (\eta/2)[\delta(\omega-\omega'-\omega_s)\exp(ik_s x) + \delta(\omega-\omega'+\omega_s)\exp(-ik_s x)]$$

Furthermore, if the incident light is monochromatic with a frequency $\Omega/2\pi$ and with $K = \Omega/c$, we have for the incident light $E = E_0\cos(\mathbf{K}\cdot\mathbf{r} - \Omega t)$, where $\mathbf{K}\cdot\mathbf{r} = Kx\sin\theta + Kz\cos\theta$ (Fig. 13.1).

If we use the transform of this wave in the integral of Eq. (13.1.2), we shall be using a variety of the Born approximation to calculate the diffracted light. The Fourier transform of the incident field is $\mathbf{E}(x,z,\omega') = (\mathbf{E}_0/2)[\delta(\omega' - \Omega)\exp(i\mathbf{K}\cdot\mathbf{r}) + \delta(\omega' + \Omega)\exp(-i\mathbf{K}\cdot\mathbf{r})]$.

If we use these expressions for $f(\omega - \omega')$ and $\mathbf{E}(x,z,\omega')$ in the integral of Eq. (13.1.2), this equation becomes

$$\begin{aligned}\nabla^2\mathbf{E} + k^2\mathbf{E} = -\frac{\Omega^2\eta\mathbf{E}_0}{2}\{&\delta(\omega - \Omega - \omega_s)\exp[i(\mathbf{K}\cdot\mathbf{r} + k_s x)] \\ &+ \delta(\omega - \Omega + \omega_s)\exp[i(\mathbf{K}\cdot\mathbf{r} - k_s x)] \\ &+ \delta(\omega + \Omega - \omega_s)\exp[-i(\mathbf{K}\cdot\mathbf{r} - k_s x)] \\ &+ \delta(\omega + \Omega + \omega_s)\exp[-i(\mathbf{K}\cdot\mathbf{r} + k_s x)]\} \equiv \mathbf{S}(x,z;\omega) \qquad (13.1.4)\end{aligned}$$

This equation expresses the fact that the sound wave modulates the incident light so that electric-field components with frequencies $\Omega \pm \omega_s$ are produced in the interaction region.

The corresponding scattered field, when $\mathbf{E}_0$ is parallel with the y axis, is conveniently calculated with the help of the two-dimensional Green's function of Eq. (7.3.17).

$$G_\omega(x,z \mid x_0,z_0) = \frac{i}{4}H_0^{(1)}(k\sqrt{(x - x_0)^2 + (z - z_0)^2}) \qquad (13.1.5)$$

which is a solution of the equation $\nabla^2 G + k^2 G = -\delta(\mathbf{r} - \mathbf{r}_0)$, representing the field produced by a line source of unit strength at (x_0,z_0). The electric field scattered from the sound wave can then be expressed as

$$\mathbf{E}(x,z,\omega) = \int_{-\infty}^{+\infty} dx_0 \int_{-L}^{+L} dz_0\, \mathbf{S}(x_0,z_0;\omega)G_\omega(x,z \mid x_0,z_0) \qquad (13.1.6)$$

where the source term $\mathbf{S}$ is defined in Eq. (13.1.4).

Setting $\mathbf{K}\cdot\mathbf{r}_0 = x_0 K\sin\theta + z_0 K\cos\theta = K_x x_0 + K_z z_0$ in $\mathbf{S}$, we note that in the integral over x_0 in (13.1.6), the source term contributes exponentials of the form $\exp[\pm i(K_x \pm k_s)x_0]\exp(\pm iK_s z_0)$. Thus we are computing the Fourier transform of the Hankel function $H_0^{(1)}(KR)$, where $R^2 = (x - x_0)^2 + (y - y_0)^2$ and $K = \omega/c$. The trick here is to use Eqs. (7.3.14), (7.3.17), and (1.2.15). We have the two forms for the Green's function in two dimensions,

$$\begin{aligned}\frac{i}{4}H_0^{(1)}(KR) &= \frac{1}{4\pi^2}\iint \frac{\exp[iK_x(x - x_0) + iK_z(z - z_0)]}{K_x^2 + K_z^2 - K^2}\,dK_x\,dK_z \\ &= \frac{i}{4\pi}\int dK_x \frac{\exp[iK_x(x - x_0) + i\sqrt{K^2 - K_x^2}(z - z_0)]}{\sqrt{K^2 - K_x^2}}\end{aligned}$$

and the inverse transform of this is

$$\frac{i}{8\pi}\int_{-\infty}^{\infty} e^{i\mu x_0}H_0^{(1)}(KR)\,dx_0 = \frac{i}{4\pi}\frac{\exp[i\mu x + i\sqrt{K^2 - \mu^2}(z - z_0)]}{\sqrt{K^2 - \mu^2}}$$

For a typical term of **S**, μ would be $\pm(K_x \pm k_s)$; so the integral of the exponential times $H_0^{(1)}$ for this term would be

$$\frac{2}{(\omega/c)\cos\theta_s^{\pm}} \exp\left[\pm i\frac{\omega}{c}x\sin\theta_s^{\pm} + i\frac{\omega}{c}(z-z_0)\cos\theta_0^{\pm}\right] \quad (13.1.7)$$

where $(\omega/c)\sin\theta_s^{\pm} = K_x \pm K_s = K\sin\theta \pm k_s$. This expression represents a scattered plane wave traveling in a direction that makes an angle θ_s with the z axis.

It remains to carry out the integration over z_0 in Eq. (13.1.6).

$$\int_{-L}^{+L} dz_0 \exp\left[i\left(K_z - \frac{\omega}{c}\cos\theta_s\right)z_0\right] = 2L\frac{\sin[K_z - (\omega/c)\cos\theta_s]L}{[K_z - (\omega/c)\cos\theta_s]L} \quad (13.1.8)$$

Collecting the contributions to $E(x,z,\omega)$ from the four exponential terms in S [Eq. (13.1.4)] and inverting the Fourier transform to obtain

$$\mathbf{E}(x,z,t) = \int_{-\infty}^{+\infty} \mathbf{E}(x,z,\omega)e^{-i\omega t}\,d\omega$$

we find that the scattered electric field consists of two plane waves traveling in different directions, one with the frequency $\Omega + \omega_s$, the other with a frequency $\Omega - \omega_s$.

$$\mathbf{E}(x,z,t) = \mathbf{E}_+ \sin[\mathbf{K}_+ \cdot \mathbf{r} - (\Omega + \omega_s)t] + \mathbf{E}_- \sin[\mathbf{K}_- \cdot \mathbf{r} - (\Omega - \omega_s)t] \quad (13.1.9)$$

where

$$\mathbf{E}_\pm = \frac{\Omega^2 L\eta\mathbf{E}_0}{c(\Omega \pm \omega_s)\cos\theta_s^{\pm}} \frac{\sin[(\Omega L/c)(\cos\theta\cos\theta_s^{\pm})]}{(\Omega L/c)(\cos\theta - \cos\theta_s^{\pm})}$$

$$\mathbf{K}_\pm \cdot \mathbf{r} = \frac{\Omega}{c}x\sin\theta_s^{\pm} + \frac{\Omega \pm \omega_s}{c}z\cos\theta_s^{\pm}$$

Here the amplitude of the sound wave is expressed in terms of the parameter η [Eq. (13.1.3)], which is the amplitude of the relative change $\delta\epsilon/\epsilon_0$ of the index of refraction. We can express η in terms of the sound-pressure amplitude p_0 as

$$\eta = \frac{1}{\epsilon_0}\frac{\partial\epsilon}{\partial\rho}\delta\rho = \frac{1}{\epsilon_0}\frac{\partial\epsilon}{\partial\rho}\frac{p_0}{c_s^2} \quad (13.1.10)$$

The diffracted waves

The relation between the angle of incidence θ and the angle θ_1 giving the direction of the diffracted wave is shown in Eq. (13.1.7).

$$\sin\theta_1^{\pm} = \frac{K\sin\theta}{(\Omega \pm \omega_s)/c} \pm \frac{k_s}{(\Omega \pm \omega_s)/c}$$

$$\simeq \sin\theta \pm \frac{k_s}{K} = \sin\theta \pm \frac{\Lambda}{\lambda_s} \quad (13.1.11)$$

where λ_s and Λ are the wavelengths of sound and light. If the incident light beam is perpendicular to the sound beam so that $\theta = 0$, we see that the diffraction lines corresponding to the frequencies $\Omega + \omega_s$ and $\Omega - \omega_s$ are found in the directions given by

$$\sin \theta_1 \simeq \pm \frac{\Lambda}{\lambda_s} \tag{13.1.12}$$

as expected.

To calculate the amplitude of the scattered wave under these conditions, we have, for small values of θ,

$$\cos \theta_s = \sqrt{1 - \sin^2 \theta_s} \simeq \cos \theta + \frac{1}{2} \frac{k_s}{K} \tan \theta \tag{13.1.13}$$

and from (13.1.9),

$$\mathbf{E}_{\pm} \simeq \frac{\Omega L}{c} \eta \mathbf{E}_0 \frac{\sin (\frac{1}{2} k_s L \theta)}{\frac{1}{2} k_s L \theta} \simeq \frac{\Omega L}{c} \eta \mathbf{E}_0 = 2\pi \frac{\eta L}{\lambda} \mathbf{E}_0$$

In other words, the scattered amplitude is proportional to $\eta L = \Delta L$, the perturbation in the optical path length caused by the sound wave. The validity of the Born approximation clearly implies that the corresponding phase shift $2\pi(\Delta L/\lambda)$ be small.

It should be emphasized that the *sinusoidal grating* corresponding to the traveling sound wave produces only a *first-order* diffraction pattern with a single line on either side of the incident-beam direction. If the sound wave is periodic but contains many Fourier components, such as encountered, for example, in a large-amplitude sound wave with nonlinear distortion (as will be discussed in the next chapter), an additional order of diffraction is obtained for each harmonic component of the sound wave. For the nth-harmonic acoustic component we then find a diffraction line at an angle

$$\theta_n = \theta \pm \frac{n\Lambda}{\lambda}$$

and the frequencies of the diffracted light in these directions are $\Omega + n\omega_s$ and $\Omega - n\omega_s$.

Partial-wave analysis

Returning to the expression for the scattered light, we note that the analysis based on the Born approximation is good only as long as the perturbation of the optical path length across the beams is small compared with the wavelength of light (i.e., as long as the width $2L$ in Fig. 13.1 is small). In order to obtain a solution valid over a wider range, we can of course improve upon the Born approximation by using a better approximation than the unperturbed incident field in the source term in Eq. (13.1.2). To illustrate another procedure, however, we return to the basic wave equation (13.1.1) and seek to satisfy it directly by series expansion of the electric field.

In this approach we wish to study how the light beam gradually changes from the initial plane monochromatic wave as it enters the sound beam at $z = -L$ to a superposition of partial waves of different frequencies and directions inside the sound beam. The amplitude of these partial waves will increase gradually with z from zero at the plane of entrance $z = -L$ of the sound beam to some final value at the exit plane $z = L$. To study this gradual deformation of the light beam, we seek a solution to the wave equation (13.1.1) subject to the boundary condition that all partial waves are zero at $z = -L$ except for the initial incident beam.

The acoustic frequency ω_s is always considerably smaller than the optical frequency Ω, and if we express the electric field as

$$E(x,z,t) = \psi(x,z,t) \exp (iKx \sin \theta + iKz \cos \theta - i\Omega t) + \text{complex conjugate} \tag{13.1.14}$$

the amplitude ψ will be a slowly varying function of time. Therefore, if we insert this expression in Eq. (13.1.1) and neglect derivatives of ψ with respect to t, we find that the equation for ψ is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + 2iK\left(\sin\theta \frac{\partial\psi}{\partial x} + \cos\theta \frac{\partial\psi}{\partial z}\right) = -\frac{\delta\epsilon}{\epsilon_0}\left(\frac{\Omega}{c}\right)^2 \psi \tag{13.1.15}$$

If the sound wave travels in the x direction and is periodic in x and t, the amplitude function ψ will have the same periodicity in x and t. The z dependence of ψ is of main interest here, and to obtain the equation describing this z dependence, it is convenient to make a Fourier transform of Eq. (13.1.15) with respect to x and t. Then, if the Fourier transforms of ψ and $\delta\epsilon/\epsilon_0$ are $\psi(z,k_x,\omega)$ and $f(z,k_x,\omega)$, respectively, we obtain from Eq. (13.1.15)

$$2iK\cos\theta \frac{\partial\psi}{\partial z} - k_x^2\psi - 2k_xK\sin\theta\,\psi = -\left(\frac{\Omega}{c}\right)^2 \int_{-\infty}^{+\infty}\int d\omega'\, dk_x'\, \psi(z,k_x',\omega') f(\omega - \omega', k_x - k_x') \tag{13.1.16}$$

Since $\psi(z,k_x,\omega)$ varies slowly with z, we have neglected $\partial^2\psi/\partial z^2$.

As an example we consider, first, the case when we have a single periodic traveling sound wave so that

$$\frac{\delta\epsilon}{\epsilon_0} = \eta \sin (k_s x - \omega_s t) \tag{13.1.17}$$

with the corresponding Fourier transform

$$f(\omega - \omega', k_x - k_x') = \tfrac{1}{2}\eta[\delta(\omega - \omega' - \omega_s)\delta(k_x - k_x' - k_s) - \delta(\omega - \omega' + \omega_s)\delta(k_x - k_x' + k_s)] \tag{13.1.18}$$

Under these conditions the amplitude function $\psi(x,z,t)$ will be periodic in x and t with the period of the sound wave, and the Fourier transform is then a sum of delta functions,

$$\psi(z,k_x',\omega') = \sum_{n=-\infty}^{+\infty} \zeta_n(z)\delta(k_x' - nk_s)\delta(\omega' - n\omega_s) \qquad (13.1.19)$$

Inserting Eqs. (13.1.18) and (13.1.19) into Eq. (13.1.16) and identifying terms of equal frequency, we find the following recursion formula for the wave amplitude ζ_n:

$$2iK\cos\theta\frac{d\zeta_n}{dz} - i\eta K^2(\zeta_{n-1} - \zeta_{n+1}) = [(nk_s)^2 + 2nk_sK\sin\theta]\zeta_n \qquad (13.1.20)$$

$$k_s = \frac{\omega_s}{c_s} = \frac{2\pi}{\lambda_s} \qquad K = \frac{\Omega}{c} = \frac{2\pi}{\Lambda}$$

which can be solved to find the dependence on penetration distance of the various diffracted beams of light.

For example, if the light wave is moving at right angles to the sound wave ($\theta = 0$, the incident light moving in the positive z direction), we can neglect the right-hand side compared with the left-hand side, since $k_s \ll K$. In this case we can take advantage of the second of Eqs. (5.2.21), which may be written

$$\frac{d}{dz}J_n(\alpha z) = \frac{\alpha}{2}[J_{n-1}(\alpha z) - J_{n+1}(\alpha z)]$$

to show that ζ_n should be proportional to the Bessel function $J_n(\eta Kz)$. In fact, if the lower bound of the sound beam is the plane $z = 0$, then the electric vector of the light wave will be, approximately,

$$\begin{aligned} &\mathbf{E}_0 \exp(iKz - i\Omega t) && z < 0 \\ &\mathbf{E}_0 \sum_{n=-\infty}^{\infty} J_n(\eta Kz)\exp[ink_sx + iKz - i(\Omega + n\omega_s)t] && z > 0 \end{aligned} \qquad (13.1.21)$$

Thus, as the light penetrates the sound wave, the incident beam $n = 0$ diminishes in amplitude, slowly at first, then oscillates, proportional to $J_0(\eta Kz)$. The first-order waves ($n = \pm 1$) rise linearly from zero at $z = 0$, reaching maximum value of about $0.6E_0$ at $z \simeq 0.3(\Lambda/\eta)$ (which is many wavelengths since the amplitude η of $\delta\epsilon/\epsilon_0$ is usually very small), and thereafter oscillate. The second-order waves ($n = \pm 2$) rise from zero at $z = 0$ proportional to z^2 and reach their first maximum amplitude (about $0.5E_0$) at $z \simeq 0.5(\Lambda/\eta)$, and so on, each higher order rising more slowly with z initially and reaching its first maximum for greater z.

13.2 BRILLOUIN SCATTERING

The analysis in the preceding section is best applicable to comparatively narrow beams of sound, when the effect of the beam is essentially equivalent to that of a plane grating. As we have seen, these are the conditions for constructive interference in the scattered field at all angles of incidence of the primary light beam. In the other extreme case, when the width of the

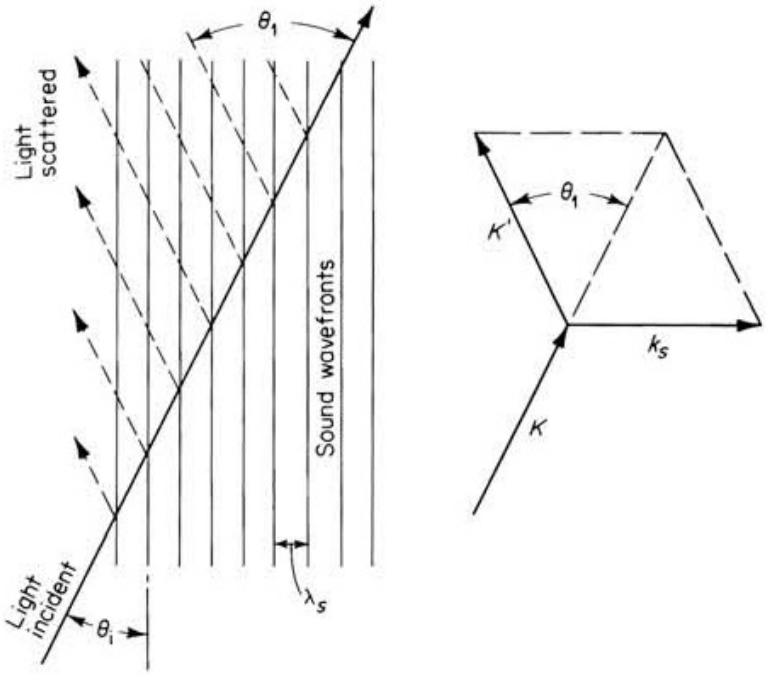


FIGURE 13.2
Brillouin scattering of light from sound waves.

beam is very large so that it can be regarded as a succession of a set of infinite partially reflecting planes, we expect the incident light to be scattered in a manner analogous to Bragg reflection from a crystal. In order for such scattering to take place, however, the width of the sound beam must be considerably larger than $\lambda/\sin\theta$, where λ is the acoustic wavelength and θ is the angle of incidence measured from the direction perpendicular to the sound beam as shown in Fig. 13.2. (In order to get interference, we must have at least two reflected rays for each incident ray.)

Bragg reflection

The condition for constructive interference of the light reflected from successive planes is the familiar Bragg condition for the first-order diffraction line,

$$\sin\theta_i = \frac{\Lambda}{2\lambda_s} \tag{13.2.1}$$

where λ_s, the separation of the planes, is the wavelength of the sound wave, and Λ is the wavelength of the incident light. Thus, for a given wavelength and direction of propagation of the sound, there is only *one* direction of the incident light which will give rise to a (first-order) diffraction line. This is in contrast to the scattering characteristics observed in the case of a narrow beam of sound, discussed in the previous section.

The scattering angle θ_1, that is, the angle between the direction of propagation of the observed scattered light and the direction of the incident light beam, as indicated in Fig. 13.2, is twice the angle of incidence, $\theta_1 = 2\theta_i$. Thus, rewriting Eq. (13.2.1), the angle θ_1 that corresponds to the first-order diffraction line can be obtained from the formula

$$2 \sin \frac{\theta_1}{2} = \frac{\Lambda}{\lambda} = \frac{\omega}{\Omega} \frac{c}{nv_s}$$

or

$$k_s = 2K \sin \frac{\theta_1}{2} \qquad k = \frac{2\pi}{\lambda_s}, \; K = \frac{2\pi}{\Lambda}$$

where n = mean index of refraction of medium
ω = acoustic frequency
Ω = light frequency
v_s = speed of sound

Since the reflecting planes in Fig. 13.2 are moving with the speed of sound, the Bragg scattered light will be Doppler-shifted, and this shift equals $2\Omega(nv_s/c) \sin (\theta_1/2)$. This follows directly from the fact that the relative speed between the fixed point of observation and the (moving) image of the primary light source is $\pm 2v_s \sin (\theta_1/2)$. The plus and minus signs refer to motion of the sound toward and away from the observer. Comparison with Eqs. (13.2.2) below shows that the frequency shift $\Delta\Omega$ equals the frequency of the sound wave, $\Delta\Omega = \omega$. This frequency shift will make the magnitudes of $K' = \Omega'/c$ and $K = \Omega/c$ slightly different, but since $\omega \ll \Omega$, we may use K' instead of K in Eqs. (13.2.2). Then, since the direction of propagation of the sound wave is the same as the direction of $\mathbf{K} - \mathbf{K}'$, it follows from Eqs. (13.2.2) that $\mathbf{k} = \mathbf{K} - \mathbf{K}'$, corresponding to the vector diagram in Fig. 13.2. The kinematic relations that exist between the acoustic wave and the incident and scattered light beams then can be summarized as follows:

$$\mathbf{k} = \mathbf{K} - \mathbf{K}' \qquad \omega = \Omega - \Omega'$$

$$k = 2K \sin \frac{\theta_1}{2} \qquad \frac{\Lambda}{\lambda} = 2 \sin \frac{\theta_1}{2} \tag{13.2.2}$$

$$\Delta\Omega = \Omega - \Omega' = \omega = 2n \frac{v_s}{c} \Omega \sin \frac{\theta_1}{2}$$

the unprimed quantities belonging to the incident wave, the primed quantities to the scattered wave. In other words, from measurements of the frequency shift $\Omega - \Omega' = \omega$ (which determines the acoustic frequency ω) and the corresponding angle of scattering θ_1, we can determine the speed of sound v_s in terms of the speed of light c, the optical frequency Ω, and the index of refraction n of the medium.

If the conditions are such that the scattered wave is in the backward direction, $\theta_1 = \pi$, the frequency shift, and hence the frequency of the target sound wave, is $2n(v_s/c)$. With $v_c \simeq 10^5$ cm per sec, $c \simeq 3 \times 10^{10}$ cm per sec, and $\Omega \simeq 10^{15}$ cps, we see that in order for this type of scattering to take place, the frequency of the acoustic wave involved must be of the order of 10^{10} cps. For scattering in a direction close to the forward direction ($\theta_1 \simeq 0$), the acoustic frequency involved is of course smaller; but from a practical standpoint, measurements cannot be made too close to the forward direction, and the acoustic frequencies usually involved in measurements of this kind lie in the range 10^9 to 10^{10} cps. Thus, because of the high frequencies involved, this method of measurement lends itself to the study of the dispersion relation $v_s = v_s(\omega)$ for high-frequency sound waves, hypersonic waves that are excited in the thermal motion of matter.

Thermal sound waves travel in all directions and have a wide frequency spectrum. Therefore, when a light beam is scattered from these waves, there will be frequency-shifted scattered light at any direction of observation, corresponding to an arbitrary angle θ_1. For each direction, the light beam selects an acoustic target wave with the appropriate wavelength and direction of propagation so as to fulfill the conditions in Eqs. (13.2.2). This phenomenon is usually called *Brillouin scattering*. The scattered light contains two frequencies, $\Omega + \omega$ and $\Omega - \omega$; in a spectroscope they would show as a doublet, the Brillouin doublet. In addition, there is scattered light with unshifted frequency, called the Rayleigh scattered line.

Thermal motion of the medium

In this context the scattered-light components may be regarded as a result of the interaction of the light wave with collective modes of motion of the fluid, in thermal motion. There are several types of such collective modes, and if the fluid is in thermal equilibrium, they are all excited. In our general analysis of the (small-amplitude) dynamics of a real fluid in Chap. 6, we found three types of such collective motions. The first is the ordinary sound wave [the propagational mode of Eqs. (6.4.23)], traveling with the speed v_s and being attenuated at a rate given in Eq. (6.4.14). The second mode of motion is a rapidly decaying shear, or transverse, wave, obeying the second of Eqs. (6.4.17).

The third mode is the thermal wave of Eqs. (6.4.25), also with a comparatively short "lifetime." These two latter modes both contribute to the entropy fluctuations in the fluid, but the viscous wave contributes only to

second order in the field variables, the rate of entropy production being proportional to $\mu(\text{curl } \mathbf{u})^2$. Thus it may be neglected in comparison with the entropy fluctuations resulting from the thermal-wave modes. For these thermal waves the space-time dependence is described by Eq. (6.4.18), $T\rho(\partial\sigma/\partial t) = K\nabla^2 T$, with K the thermal conductivity, and where, at constant pressure, $T(\partial\sigma/\partial t) = C_p(\partial T/\partial t)$. Thus a periodic temperature field with a spatial distribution $\delta T(q,t)\cos(qx)$ will decay in time according to the equation $\partial(\delta T)/\partial t = -(K/\rho C_p)q^2\,\delta T$, which has the solution

$$\delta T = \delta T(0)\exp(-\beta t) \qquad \text{where } \beta = \frac{K}{\rho C_p}q^2 \tag{13.2.3}$$

A corresponding initial perturbation in pressure, $\delta p\cos(k_s x)$, on the other hand, will decay in time as

$$\tfrac{1}{2}\,\delta p\, e^{-\alpha t}[\cos(k_s x + \omega_s t) + \cos(k_s x - \omega_s t)]$$

representing two decaying disturbances that travel with the speed $v_s = \omega_s/k_s$ in opposing directions. A light wave interacting with these traveling, decaying waves will be scattered into two lines with frequencies $\Omega \pm \omega_s$, as was demonstrated previously. Because of the decay of the waves, these lines, the Brillouin lines, will be broadened by an amount of the order of α, the inverse of the lifetime of the propagational waves in the fluid.

In our previous discussion of the decay of acoustic waves (Sec. 4.1), we have considered mainly the spatial decay of a wave driven at a frequency $\omega/2\pi$ and in the corresponding dispersion relation $k = k(\omega)$ [Eq. (6.4.26)] ω has been regarded as real and k as a complex quantity. However, in the present problem of the thermal excitation of acoustic modes in a medium, we are concerned with the decay, in time, of an acoustic wave of given wavelength $2\pi/k$. To obtain this decay we have to return to Eq. (6.4.26) and express ω (now complex) in terms of k (real). Although this inversion of the dispersion relation can be made in principle, it is not always simple, particularly because of the frequency dependence of the bulk viscosity coefficient η. In the cases where a, the spatial decay constant, times the wavelength of the sound is very small compared to unity, the approximate relation $\alpha = v_s a$ is valid for the temporal decay constant α, where v_s is the speed of sound. In many cases of interest for Brillouin scattering this is not the case and a much more complicated formula, relating a and α, holds. We shall not pursue this matter further, but will simply remark that in monatomic fluids, where η is negligible and when the a of Eq. (6.4.14) times λ is quite small, the temporal decay constant has the simple form

$$\alpha = \left[\frac{4}{3}\frac{\mu}{\rho} + \frac{(\gamma-1)K}{\rho C_p}\right]\frac{\omega^2}{2v_s^2} \tag{13.2.4}$$

Since μ/ρ and $K/\rho C_p$ are both of the order of lv_s, where in a gas l is the mean-free-path, we see that $a \simeq \frac{l}{v_s}\frac{\omega^2}{v_s}$. The decay time of the sound wave is thus of the order of $1/\alpha = 1/v_s a \simeq (\lambda_s/l)\omega_s^{-1}$, where λ_s is the wavelength of the sound wave.

For the thermal disturbance, on the other hand, the decay is aperiodic, and there are no traveling waves involved. The light scattered from such a disturbance will not be frequency-shifted, and the corresponding scattered line, the Rayleigh line, will have the same frequency as the incident line, except for a line broadening by an amount of the order of $\beta = (K/\rho C_p)q^2$.

The scattered intensity

The previous considerations, which have been mainly kinematic, do not say anything about the intensity of the scattered light or about the details of the line shapes. To study these aspects of the problem, we return to the wave equation (13.1.1).

Again we use the Born approximation, so that the electric field in the source term is replaced by the incident field $\mathbf{E}_0 \exp i(\mathbf{K}\cdot\mathbf{r} - i\Omega t)$. We express the function $[\delta\epsilon(\mathbf{r},t)/\epsilon_0]$ by its spatial Fourier transform

$$\frac{\delta\epsilon(\mathbf{r},t)}{\epsilon_0} = \int \eta(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{r})\,dv_k \tag{13.2.5}$$

Using the free-field Green's function $(1/4\pi|\mathbf{r}-\mathbf{r}_0|)\,\delta(t_0 - t + |\mathbf{r}-\mathbf{r}_0|/c)$, we find, in a plane perpendicular to $\mathbf{E}_0$, that the scattered field is

$$\mathbf{E}' = \frac{\mathbf{E}_0\Omega^2}{4\pi r c^2}\int dv_k \int \exp\left(i\mathbf{K}\cdot\mathbf{r}_0 - i\Omega\,\frac{t-|\mathbf{r}-\mathbf{r}_0|}{c}\right)\eta(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{r}_0}\,dv_0 \tag{13.2.6}$$

where we have assumed that $1/|\mathbf{r}-\mathbf{r}_0| \simeq 1/r$, which is true if r is large enough.

We introduce $|\mathbf{r}-\mathbf{r}_0| \simeq r - \mathbf{K}'\cdot\mathbf{r}_0/K'$ in the exponential, where $\mathbf{K}'$ is the propagation vector for the scattered light wave and $|\mathbf{K}'| = \Omega/c$. The integration over $\mathbf{r}_0$ gives $V\delta(\mathbf{k} - \mathbf{K}' + \mathbf{K})$, where V is the volume of the source region, and δ the delta function. Consequently, the scattered field is given by

$$\mathbf{E}'(\mathbf{r},\mathbf{K}',t) = \frac{\mathbf{E}_0\Omega^2 V}{4\pi r c^2}\,\eta(\mathbf{k},t)e^{iK'r - i\Omega t} \qquad \mathbf{k} = \mathbf{K} - \mathbf{K}' \tag{13.2.7}$$

To determine the frequency spectrum of the scattered-light intensity, we make use of the fact that the spectrum density of $E^2(\mathbf{r},t)$ is the Fourier transform of the autocorrelation function $\langle E(\mathbf{r},\mathbf{K}',t-\tau)E^*(\mathbf{r},\mathbf{K}',t)\rangle = \psi(\tau)$, where $\langle\ \rangle$ indicates the time average.

Fluctuations of the refractive index

In order to apply this equation to the problem of scattering from the thermally excited collective modes in a substance, as discussed above, we must first relate the fluctuations in the dielectric constant to these modes of motion. From a thermodynamic standpoint this is, in effect, accomplished when we consider ϵ as a function of the two independent thermodynamic variables, pressure and entropy. We have

$$\delta\epsilon = \left(\frac{\partial\epsilon}{\partial P}\right)_S \delta P + \left(\frac{\partial\epsilon}{\partial S}\right)_P \delta S \tag{13.2.8}$$

The fluctuations in ϵ, then, are a result of fluctuations in P and S, and these fluctuations, as we have seen from the previous qualitative considerations, correspond to two of the independent types of collective modes of the fluid, the propagational waves and the thermal waves (the shear waves contribute only to second order in the entropy fluctuations and will not be included in a first-order analysis).

After these observations regarding the fluctuation in ϵ, we can proceed to derive an expression for the scattered electric field and the frequency spectrum of the scattered intensity. We obtain, using Eqs. (13.2.7),

$$I(\Omega',r) = |E'(r,\Omega')|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle E'(\mathbf{r}, t+\tau)E^*(\mathbf{r},t)\rangle e^{i\Omega'\tau}\, d\tau$$

$$= E_0^2\left(\frac{\Omega}{c}\right)^4 \frac{1}{(4\pi r)^2}\frac{1}{2\pi}\int \langle \eta(\mathbf{k}, t+\tau)\eta^*(\mathbf{k},t)\rangle e^{-i\Omega\tau}\, e^{i\Omega'\tau}\, d\tau \tag{13.2.9}$$

Since $\eta = \delta\epsilon/\epsilon_0$, we obtain from Eq. (13.2.4), realizing that the fluctuations in P and S are uncorrelated (they represent different collective modes of motion),

$$\Upsilon(\tau) = \langle \eta(\mathbf{k}, t+\tau)\eta^*(\mathbf{k},t)\rangle = A\,\Upsilon_P(\tau) + B\,\Upsilon_S(\tau) \tag{13.2.10}$$

where

$$A = \frac{1}{\epsilon_0}\left(\frac{\partial\epsilon}{\partial P}\right)_S \qquad B = \frac{1}{\epsilon_0}\left(\frac{\partial\epsilon}{\partial S}\right)_P$$

and

$$\Upsilon_P(\tau) = \langle \delta P(\mathbf{k}, t+\tau)\, \delta P^*(\mathbf{k},t)\rangle$$

$$\Upsilon_S(\tau) = \langle \delta S(\mathbf{k}, t+\tau)\, \delta S^*(\mathbf{k},t)\rangle$$

are the autocorrelation functions for the pressure and entropy fluctuations. Here $\delta P(\mathbf{k},t)$ is the amplitude of the spatial component of pressure, having the wavelength $2\pi/k$.

$$\delta P(\mathbf{k},t) = \left(\frac{1}{2\pi}\right)^3 \iiint e^{-i\mathbf{k}\cdot\mathbf{r}}\, \delta P(\mathbf{r},t)\, dv_r \tag{13.2.11}$$

with a similar equation for the amplitude $\delta S(\mathbf{k},t)$. Note the parallel with the discussion following Eq. (8.1.26), for the scattering of sound from turbulence.

To obtain more specific formulas for the spectrum of the scattered light, we must be more explicit about the autocorrelation functions Υ_P and Υ_S for the pressure and entropy fluctuations. As we have seen, in the preceding discussion of the basic properties of these modes of motion, a pressure mode of a certain wavelength $2\pi/k$, which happens to be excited at a certain time, will decay like a damped harmonic oscillator with a temporal decay rate α, given on page 820. Similarly, a thermal wave will decay exponentially without oscillations with a decay rate β, given in Eq. (13.2.3).

In the course of time these modes are excited at random intervals, and the resulting time dependence of δP and δS is analogous to the time dependence of the damped harmonic oscillator driven by a random force, as discussed in Sec. 1.6. There we determined not only the average spectrum density of the oscillator displacement, but also the corresponding autocorrelation function, which we expressed as the root-mean-square displacement of the oscillator multiplied by a function of the time delay variable τ, as shown in Eq. (2.3.15). In a completely analogous manner we can express the autocorrelation functions Υ_P and Υ_S for these randomly excited acoustic and thermal oscillators (the thermal mode corresponds to a damped oscillator with resonance frequency equal to zero) in terms of the mean-square wave amplitudes. Thus, with reference to Eq. (2.3.15), we find

$$\begin{aligned} \Upsilon_P &= \langle|\delta P(\mathbf{k})|^2\rangle\, e^{-\alpha|\tau|}\cos(\omega_s\tau) \\ \Upsilon_S &= \langle|\delta S(\mathbf{k})|^2\rangle\, e^{-\beta|\tau|} \end{aligned} \tag{13.2.12}$$

The scattered lines

The corresponding spectrum density of the light intensity $I' \sim |E'|^2$ scattered from the acoustic oscillations in the medium is then obtained from Eq. (13.2.9).

$$\begin{aligned} I'_P(\mathbf{k},\Omega') &= I_0 D\langle|\delta P(\mathbf{k})|^2\rangle \int_{-\infty}^{\infty} e^{-\alpha|\tau|}\cos(\omega_s\tau)e^{i(\Omega-\Omega')\tau}\,d\tau \\ &= I_0 D\langle|\delta P(\mathbf{k})|^2\rangle \left\{\frac{\alpha}{\alpha^2+[\Omega'-(\Omega-\omega_s)]^2}\right. \\ &\qquad \left. +\frac{\alpha}{\alpha^2+[\Omega'-(\Omega+\omega_s)]^2}\right\}^2 \end{aligned} \tag{13.2.13}$$

where $D = \left(\frac{\Omega}{c}\right)^4\left(\frac{1}{4\pi r}\right)^2\frac{V}{\pi}\left(\frac{\partial P}{\partial\epsilon}\right)_S^2$, and I_0 is the intensity of the incident light. These represent the two *Brillouin lines*, centered at the frequencies $\Omega' = \Omega - \omega_s$ and $\Omega' = \Omega + \omega_s$, broadened by an amount determined by the "lifetime" $1/\alpha$ of the frequency ω_s. The line shape is what is commonly called *Lorentzian*. Thus, from measurements of $\Omega' - \Omega = \omega_s$, the scattering angle θ_1 [Eq. (13.2.1)], and the line width, we can determine not only the speed of sound at the frequency ω_s, but also the decay rate. In other words,

the complete (complex) dispersion relation $\omega = \omega(k)$ can be measured in this way for sound waves in the kilomegacycle region, which are those involved in the Brillouin scattering.

In a completely similar manner we obtain the intensity contribution I_S from the entropy fluctuations of wavenumber k. In fact, the result is obtained directly from Eq. (13.2.13) by putting $\omega_s = 0$ and replacing α by the decay constant β given in Eq. (13.2.3).

$$I'_S(\mathbf{k},\Omega') = I_0\left(\frac{\Omega}{c}\right)^4 \frac{1}{(4\pi r)^2}\frac{\pi}{1}\left(\frac{\partial\epsilon}{\partial S}\right)_P^2 V\langle|\delta S(\mathbf{k})|^2\rangle \frac{\beta}{\beta^2 + (\Omega' - \Omega)^2} \quad (13.2.14)$$

This represents the *Rayleigh line*, centered at the incident-light frequency Ω and broadened by an amount corresponding to the lifetime $1/\beta$ of the entropy waves. As we note from Eq. (13.2.3), this lifetime decreases with k^2.

It remains to relate the mean-square amplitudes $\langle|\delta p(\mathbf{k})|^2\rangle$ and $\langle|\delta S(\mathbf{k})|^2\rangle$ in terms of the intensity of thermal agitation, as measured by the temperature of the medium. For sufficiently long wavelengths the various modes are excited with equal strength, and we may set $\langle|\delta P(\mathbf{k})|^2\rangle \simeq \langle|\delta P(0)|^2\rangle$. Then, from the inverse Fourier transform in Eq. (13.2.11), we get $\langle|\delta P(\mathbf{k},t)|^2\rangle \simeq (2\pi)^{-3}\langle|\delta P(\mathbf{r},t)|^2\rangle V^2$, where $\langle|\delta p|^2\rangle$ is the mean-square pressure fluctuation, and V the volume element under consideration [see also Eq. (1.3.17), for an alternative derivation]. Similarly, we have $\langle|\delta S(\mathbf{k},t)|^2\rangle \simeq \langle|\delta S(0,t)|^2\rangle \simeq (2\pi)^{-3}\langle|\delta S(\mathbf{r},t)|^2\rangle V^2$. Then, from well-known thermodynamic relations, we have $\langle|\delta P|^2\rangle = (k_b T)(V/\kappa_S) = k_b T V \rho v_s^2$, $\langle|\delta S|^2\rangle = k_b C_p \rho V^2$, where k_b is the Boltzmann constant, T is the temperature, $\kappa_S = \dfrac{1}{\rho}\left(\dfrac{\partial P}{\partial \rho}\right)_S = (\rho v_s^2)^{-1}$ is the adiabatic compressibility, and C_p is the specific heat at constant pressure.

We have now determined all the quantities in the expression for $I_P(\Omega',k)$ and $I_S(\Omega',k)$ to enable the numerical calculation of the spectral-intensity distribution in terms of the incident intensity and the thermodynamic properties of the medium. Actually, in such calculations it is convenient to express the power scattered into the solid angle $d\Omega$ at the point of observation a distance r from the region of scattering. This scattered power is $I'r^2\,d\Omega$; from it we can define a scattering cross section $d\Sigma$ given by the equation

$$\frac{d\Sigma}{d\Omega} = \frac{r^2 I'}{I_0} = \left(\frac{\partial\Sigma}{\partial\Omega}\right)_P + \left(\frac{\partial\Sigma}{\partial\Omega}\right)_S \quad (13.2.15)$$

and it follows directly from the sum of Eqs. (13.2.13) and (13.2.14).

We could study several other properties of the scattered light on the basis of the relations we have derived, but we shall stop here, and take up some of these questions in the problems.

13.3 STIMULATED BRILLOUIN SCATTERING

In the discussion of scattering in the two preceding sections, the perturbation in the dielectric constant was assumed to be a known function of space and time, independent of the electric field. However (as was mentioned in the introduction), the electric field does produce small mechanical effects on the medium.

Electrostrictive pressure

Although it is well known from classical electromagnetic theory, we shall demonstrate the presence of electrostrictive pressure in the following manner: The change of the electrostatic energy $\epsilon E^2/8\pi$ per unit volume element V, when a substance is compressed so that the density changes by $d\rho$, is $\dfrac{\partial\epsilon}{\partial\rho}\dfrac{E^2}{8\pi}\,d\rho$. To produce this change, a pressure P_E is required such that the work done by the pressure, $P_E(d\rho/\rho)$ per unit volume, equals the change in the electric energy. Thus

$$P_E = \rho\frac{\partial\epsilon}{\partial\rho}\frac{E^2}{8\pi} = \delta\epsilon\frac{E^2}{8\pi}$$

where $\rho(\partial\epsilon/\partial\rho)$ is of the order of unity for many liquids and solids. As far as the dynamics of the medium is concerned, this pressure has the same effect as the ordinary pressure P. Under more general conditions the effect of an electric field on a material medium is known to be the same as a mechanical stress tensor, $(\epsilon/4\pi)[E_iE_j - (E^2/2)\delta_{ij}]$.

Thus, in a more careful study of the interaction of light and sound waves, we should allow for the feedback from the light to sound and express it as a source term in the acoustic-wave equation. In the same way as the acoustic field enters in the source term of the electric-field equation (13.1.1) through the variation in ϵ, we also have a source term in the acoustic-wave equation that contains the electric field. From our previous discussion it follows that these two equations are

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{\delta\epsilon/\epsilon_0}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{1}{c^2}\frac{1}{\epsilon_0}\left(\frac{\partial\epsilon}{\partial P}\right)_S p\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{13.3.1}$$

for the electric field, and

$$\nabla^2 p - \frac{1}{v_s^2}\frac{\partial^2 p}{\partial t^2} = -\nabla^2\left(\frac{\epsilon}{8\pi}E^2\right) \tag{13.3.2}$$

for the sound field. We have neglected losses in both equations. To study the coupling between the electric field $\mathbf{E}$ and the sound pressure p in connection with Brillouin scattering, we make use of the previous analysis and consider the electric field to consist of three components: a primary wave $\mathbf{E}_0$ with frequency Ω, and two scattered waves $\mathbf{E}_+$ and $\mathbf{E}_-$, with frequencies $\omega_+ = \Omega + \omega_s$ and $\omega_- = \Omega - \omega_s$.

Studying the general character of the source terms in these coupled

equations, we note that the source term in the equation for $\mathbf{E}$, because of the product of the primary field $\mathbf{E}_0$ and the sound field p, contains a term with frequency $\omega_- = \Omega - \omega_s$ (the term with the frequency $\omega_+ = \Omega + \omega_s$ is of no particular interest now). If p and $\mathbf{E}_0$ are traveling waves, this source term is also a traveling wave, which, under the conditions of Brillouin scattering, has a propagation constant $\mathbf{K}_- = \mathbf{K} - \mathbf{k}_s$, where $\mathbf{K}$ and $\mathbf{k}_s$ are the propagation constants for the primary light wave and the sound wave, respectively, as shown in Fig. 13.2.

In fact, if the sound wave and the primary electric wave are $p = p_0 \cos(\omega_s t - \mathbf{k}_s \cdot \mathbf{r})$ and $\mathbf{E} = \mathbf{E}_0 \cos(\Omega t - \mathbf{K} \cdot \mathbf{r})$, respectively, the wave equation for the electric field produced by the term with the frequency $\Omega - \omega_s = \Omega_-$ will be

$$\nabla^2 \mathbf{E}_- - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_-}{\partial t^2} = \left[-\frac{1}{2c^2}\frac{1}{\epsilon_0}\left(\frac{\partial \epsilon}{\partial P}\right)_S p_0 \Omega^2 \mathbf{E}_0 \right] \cos[(\Omega - \omega_s)t - (\mathbf{K} - \mathbf{k}_s)\cdot \mathbf{r}] \tag{13.3.3}$$

The particular solution to this equation, which satisfies the boundary condition $\mathbf{E}_- = 0$ at $\mathbf{K}' \cdot \mathbf{r} = 0$, is

$$\mathbf{E}_- = \frac{1}{4}\frac{1}{\epsilon_0}\left(\frac{\partial \epsilon}{\partial P}\right)_S p_0 \mathbf{E}_0 (\mathbf{K}' \cdot \mathbf{r}) \sin(\Omega_- t - \mathbf{K}_- \cdot \mathbf{r})$$

where, under conditions of Brillouin scattering, we have $\mathbf{K}' = \mathbf{K} - \mathbf{k}_s$. In other words, we see that the amplitude of the electric field $\mathbf{E}_-$ grows linearly with distance if p_0 and $\mathbf{E}_0$ are independent of $\mathbf{r}$.

Let us now see what effect this growing electric field has on the sound field. We note that the contribution to the source term in Eq. (13.3.2), arising from $(\epsilon_0/8\pi)\nabla^2(\mathbf{E}_0 \cdot \mathbf{E}_-)$, has a wave component with the frequency ω_s and propagation constant $\mathbf{k}_s$. Consequently, the corresponding sound pressure field is obtained from the acoustic-wave equation (13.3.2), with a source term that is a wave traveling with the speed of sound in the direction of $\mathbf{k}_s$. Just as in Eq. (13.3.3), we find that the generated sound also will grow with the distance of travel.

Thus the coupled electroacoustic wave system is unstable. In other words, the electric field $\mathbf{E}_-$ feeds energy into the sound field through Brillouin scattering, and the sound field in turn feeds energy into $\mathbf{E}_-$ (and $\mathbf{E}_0$). Through this feedback phenomenon a small initial sound field will build up, and with it the electric field component $\mathbf{E}_-$. However, this growth does not apply to the higher frequency component $\mathbf{E}_+$.

In this discussion we have neglected both acoustic and electric losses. If these were accounted for, we could then show that there exists a critical threshold for the primary electric field strength, which must be exceeded for the stimulated Brillouin scattering to occur. This threshold has been exceeded experimentally in high-intensity (pulsed) laser beams, and stimulated Brillouin scattered light has been produced both in solids and liquids.

Problems

1 In the discussion of Debye-Sears scattering the sound-pressure amplitude in the sound beam was assumed to be uniform across the beam. (*a*) Generalize the results in Eqs. (13.1.9) and (13.1.13) to include a variation of the sound-pressure amplitude across the beam so that $p_0 = p_0(z)$ and $\eta = \eta(z)$ [Eq. (13.1.10)]. (*b*) If the light beam is incident at right angles to the sound beam, discuss the possibility of determining the sound-pressure-amplitude distribution $p_0(z)$ from measurements of the scattered light. (*c*) Suppose the amplitude of the sound wave is uniform across the beam but large enough so that distortion of waveshape takes place (as discussed in Chap. 14). Then the sound wave will be periodic, but not simple-harmonic. What information about the waveshape can be obtained from studies of the scattered light?

2 In Brillouin scattering of a laser beam (frequency 5×10^{15} cps) from a liquid the scattered light in the direction perpendicular to the incident beam is observed. In this direction the frequency separation of the two Brillouin lines is found to be 6×10^9 cps, and the width of a line (at an intensity equal to half the maximum intensity) is 10^8 cps. If the index of refraction of the liquid is $n = 1.4$, determine the frequency, phase velocity, and spatial attenuation of the thermally excited target sound waves involved.

3 Determine the total scattered power in the Brillouin and Rayleigh lines by integrating I'_ρ and I'_s in Eqs. (13.2.13) and (13.2.14) over the frequency Ω'. Obtain an expression for the ratio between these powers.

4 In our discussion of Brillouin scattering only first-order diffraction (corresponding to an optical-path difference of one wavelength between the light rays scattered from neighboring Bragg planes) takes place. Under what conditions (if at all) can higher-order diffraction lines occur?

5 Brillouin scattering can be regarded as an elastic collision between phonons with momentum $\hbar\mathbf{k}_s$ and energy $\hbar\omega_s$ and photons with momentum $\hbar\mathbf{K}$ and energy $\hbar\Omega$. Show that the scattering formulas (13.2.2) follow from conservation of momentum and energy in such a collision.

6 A circular light beam of radius r_0 is intensity-modulated so that the intensity is of the form $I = I_0 [1 + \epsilon \sin(\omega t)]$, where $\epsilon \ll 1$. Derive an expression for the sound field produced by such a beam in a liquid as a result of electrostriction. Also discuss the case of a pulsed light beam, where the intensity is $I = I_0$ for $-\tau < t < \tau$ and $I = 0$ for $t > \tau$.

CHAPTER

14

NONLINEAR OSCILLATIONS AND WAVES

14.1 THE SIMPLE OSCILLATOR

Throughout the preceding chapters we have dealt with linear, or linearized, equations of motion. The unknown quantities in these equations, the displacements, velocities, pressures, densities, and temperatures, all enter the equations to the first power; higher-power terms were either neglected or, as in the case of turbulence, were considered as known source terms. As was mentioned many times, the reasons for this restriction are quite persuasive. In the first place, the usual acoustical motions are small enough so that the nonlinear terms are considerably smaller than the linear ones, and their neglect does not alter the nature of the solution in any important respect.

An equally weighty reason, as far as this book is concerned, is the fact that the techniques for solving linear equations are simpler and much more completely worked out than are the techniques for solving nonlinear equations. Even though there are many acoustical phenomena in which nonlinear effects cannot be neglected, a book dealing with the theory of sound and explaining the methods for computing acoustical behavior must devote most of its space to linear equations, simply because there is so much more that can be said about them. The whole gamut of general techniques, so useful in linear theory, such as the Green's function, Fourier transforms, indeed the whole procedure of adding solutions to produce yet other solutions, is for the most part useless in dealing with nonlinear systems. We cannot employ the useful complex factor $e^{-i\omega t}$, since real and imaginary parts do not separate in a nonlinear equation.

In this final chapter we shall present a representative sample of the methods of solution of those few nonlinear problems to which analytic solutions have been obtained to date. The first section will discuss the motion of a simple oscillator, a point mass moving in one direction, acted on by nonlinear forces, to bring out some of the general characteristics which also apply to more complex systems. The rest of the chapter will present a few examples of nonlinear wave motion. An increasing number of nonlinear

problems are being solved numerically, for specific cases, by the use of electronic computers. Such numerical techniques, for either linear or nonlinear equations, applicable for individual cases but lacking the generality of analytic solutions, are not discussed in this book.

Examples of nonlinear forces

Even the usual examples of the simple oscillator, a mass on the end of a coiled spring, exhibits nonlinear aspects if the amplitude of motion is large. If the spring is stretched too far, it becomes a straight piece of wire, with much greater stiffness than that of the coiled spring. Likewise, if the coiled spring is compressed sufficiently, the "pitch" of the spiral eventually will become zero, and the coil spring will become like a solid tube, with a stiffness which again considerably exceeds that of the original spring.

We need not, however, go to such extremes in order to illustrate nonlinearity. In fact, some oscillators are nonlinear even for very small displacements, as we shall see shortly. Consider, for example, the oscillator shown schematically in Fig. 14.1. Here both ends of a coil spring are held fixed a constant distance L apart. A mass is attached to the center of the spring and is set in oscillation transverse to the length of the spring. If the relaxed length of the spring is L_0 and the spring constant is K, the potential energy of the spring is $\frac{1}{2}K[2\sqrt{(L/2)^2 + x^2} - L_0]^2$, and the restoring force on the mass is

$$F = -4Kx\left(1 - \frac{L_0}{\sqrt{L^2 + 4x^2}}\right)$$

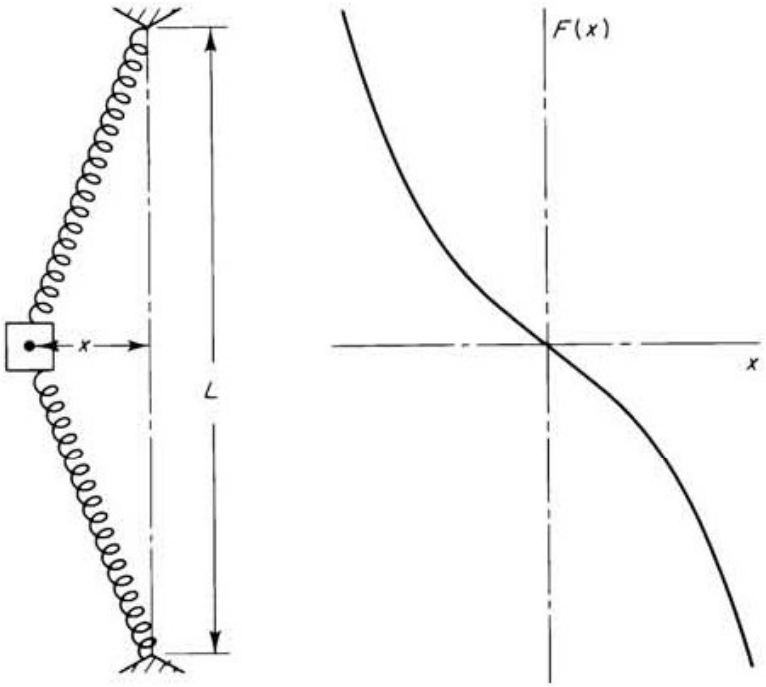


FIGURE 14.1
Example of a nonlinear oscillator with a "hard" restoring force.

For sufficiently small values of x, this can be expanded into the series

$$F = -4K(L - L_0)\frac{x}{L} - 8KL_0\left[\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^5 + \cdots\right] \qquad x < L$$

which contains all the odd powers of x.

For x very much smaller than L, we can usually neglect the higher terms and consider the force to be linear in x. But note that if the relaxed length L_0 of the spring is equal to the distance L between the supports, the linear term vanishes, and even for small values of x, the restoring force, $F \simeq -(8K/L^2)x^3$, is proportional to the cube of the displacement.

The oscillator in Fig. 14.1 is said to have a "hard" restoring force, which means that the slope of the F versus x curve increases with x. There are many other oscillators with this general property, as illustrated by some of the problems. An extreme but simple example is the "particle-in-a-box" oscillator, where the particle bounces elastically back and forth between parallel walls a distance L apart. The force on the particle in the region between the walls is zero; it rises suddenly to a very large value at the walls.

Another familiar example of a nonlinear oscillator is the simple pendulum. In terms of the angle of deflection, the equation of motion of the pendulum is $ML(d^2\theta/dt^2) + MgL\sin\theta = 0$, where M is the mass and L is the length of the pendulum. Thus, for sufficiently small angles of deflection, the restoring torque is $MgL\theta - MgL\theta^3/6$. In this case the nonlinearity has a "softening" effect on the restoring torque, in contrast to the effect of the nonlinearity in previous examples.

Deviations from linear oscillator

As a result of the nonlinearity, the oscillator characteristics will differ from the well-known properties of the linear oscillator. For example, the period of oscillation no longer is independent of the amplitude. In the special case of the particle-in-a-box oscillator, for example, the period clearly is proportional to the distance between the parallel walls, and hence proportional to the amplitude of oscillation. Also, it is inversely proportional to the velocity amplitude of the motion. In other cases it may not be so simple to find an explicit expression for the period, but we shall find that a "hard" nonlinearity generally serves to decrease the period with increasing amplitude and that the opposite is true for the "soft" nonlinearity.

So far we have implied a conservative oscillator in which the force depends only upon the position coordinates. However, there are many examples of oscillators in which velocity-dependent forces, such as friction, play an essential role. Since a friction force changes sign with the velocity, it follows that it will contain only odd powers of the velocity. We shall consider, for example, a force of the form $av + bv^3$. Although in most cases both a and b are positive, there are oscillators which correspond to a negative value of a. Under these conditions self-sustained oscillations will

result; the energy in these oscillations often is produced by a time-independent force, such as in the bowing of a violin string.

When we go from the free to the forced oscillations of a nonlinear oscillator, we find further unfamiliar features in terms of our experience with linear oscillators. Physically, the most important of these features is, perhaps, the fact that the principle of linear superposition of motions produced by two or more driving forces no longer applies. For example, two harmonic forces with frequencies ω_1 and ω_2 will produce a displacement which contains not only these frequencies, but also "combination tones," with frequencies $n\omega_1 \pm m\omega_2$, and in some cases also "subharmonic" frequencies.

The mathematical analysis of a one-dimensional nonlinear oscillator thus involves the solution of an ordinary differential equation which contains nonlinear terms in the position coordinates and/or its derivatives. These can be solved numerically, but there is no general theory for differential equations of this kind; exact solutions are known only in very special cases, which rarely are of interest in problems of practical interest. It should be mentioned in this connection, however, that approximate equations of motion of an oscillator sometimes can be solved exactly. An important example is a conservative oscillator in which the restoring force can be described approximately as $\alpha x + \beta x^3$, typical of the oscillator of Fig. 14.1, to the third order of approximation. This can be solved exactly in terms of elliptic functions which have been tabulated. When more general types of nonlinearities are considered, we have to resort to graphical or other approximate procedures in the analysis of the motion; the purpose of this section is to survey some of these methods and apply them to a few special examples. The survey is not intended to be exhaustive; we refer the reader to the specialized treatises mentioned in the bibliography for further details.

Conservative forces

Let us consider first the problem of free oscillations of a nonlinear one-dimensional oscillator in which the restoring force is conservative, dependent only upon the position coordinates of the oscillating mass. The equation of motion is then [Eqs. (1.3.1) to (1.3.4)]

$$M\frac{d^2x}{dt^2} = Mv\frac{dv}{dx} = F(x) \tag{14.1.1}$$

If we introduce the potential energy

$$V(x) = -\int F(x)\,dx$$

we obtain the energy relation

$$M\frac{v^2}{2} + V(x) = E \equiv \text{const} \tag{14.1.2}$$

which, with $v = dx/dt$, can also be written

$$t - t_0 = \int_{x_0}^{x} \frac{dx}{\sqrt{2(E - V)/M}} \tag{14.1.3}$$

Formally, this relation represents the solution to the equation of motion, and even if we cannot express the integral explicitly, in closed form, we can determine x as a function of time and the period of oscillation T_p of Eq. (1.3.4) to any degree of accuracy by numerical analysis.

In the particular case of a simple pendulum of length L, the position coordinate along the path of motion is $x = L\theta$, where θ is the angle of deflection. The potential energy is $MgL(1 - \cos\theta)$, and if the maximum angle of deflection in the oscillation is θ_0, we have $E = MgL(1 - \cos\theta_0)$. Then, setting $t_1 = 0$ when $x_1 = 0$, we obtain for the pendulum

$$t = \sqrt{\frac{L}{2g}} \int_0^{\theta} \frac{\pm\, d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{1}{2}\sqrt{\frac{L}{g}} \int_0^{\theta} \frac{\pm d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}$$

If we introduce the new variable φ, given by $\sin(\theta/2) = k \sin\varphi$, where $k = \sin(\theta_0/2)$, this equation becomes

$$t = \sqrt{\frac{L}{g}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \equiv \sqrt{\frac{L}{g}}\, F(k,\varphi) \tag{14.1.4}$$

By expanding $(1 - k^2 \sin^2\varphi)^{-\frac{1}{2}}$ in a power series and integrating term by term, we obtain t as a function of φ. The function $F(k,\varphi)$, known as an elliptic integral, is available in tabulated form.

$$F(k,\varphi) = \varphi + \tfrac{1}{2}k^2 I_1(\varphi) + \frac{1 \cdot 3}{2 \cdot 4} k^4 I_2(\varphi) + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} k^6 I_3(\varphi) + \cdots$$

where

$$I_n(\varphi) = \int_0^{\varphi} \sin^{2n}\varphi \, d\varphi$$

Using the first term only, we obtain the harmonic motion $\theta = \theta_0 \sin(\sqrt{g/L}t)$, familiar from the linear analysis. The period of oscillation can be expressed as

$$T = 4\sqrt{\frac{L}{g}}\, F\left(\frac{k, \pi}{2}\right) = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{2} + \frac{3}{8}k^4 + \cdots\right) \tag{14.1.5}$$

For an amplitude of oscillation corresponding to $\theta_0 = 90°$, we have $k = 1/\sqrt{2}$, and the period of oscillation is $T \simeq 1.4T_0$, where T_0 is the small-amplitude period, $2\pi\sqrt{L/g}$.

Thus the motion of a simple pendulum is an example of a nonlinear oscillation that can be described exactly in terms of elliptic functions. These functions can also be used for an exact description of a conservative oscillation in which the restoring force is of the form $\alpha x + \beta x^3$, as will be discussed further in one of the problems.

The phase plane

Even if the integral of Eq. (14.1.3) cannot be evaluated explicitly in terms of known functions, it is always possible to describe the behavior of the oscillator qualitatively by plotting the momentum p or the velocity v as a function of the position coordinate x. This representation of the motion in the vx plane, the *phase plane*, follows directly from conservation of energy. In the case of a harmonic oscillator, we have

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}Kx^2 = E$$

Thus, for each given energy of oscillation E, the relation between v and x is represented by an ellipse in the phase plane. It is often convenient to normalize the coordinates v and x. For example, if we introduce v/v_0 and x/x_0, where v_0 and x_0 are the velocity and displacement amplitudes in the harmonic motion, we obtain

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{v}{v_0}\right)^2 = 1 \qquad x_0^2 = \frac{2E}{K} \qquad v_0^2 = \frac{2E}{M} \tag{14.1.6}$$

Thus, in terms of the coordinates x/x_0 and v/v_0, the phase-space trajectory representing the motion is a circle of unit radius.

A deviation of the restoring force from linearity will produce a distortion of the circle. For example, in the extreme case of the particle-in-a-box oscillator, the circle is replaced by a square path. In this particular oscillator the force can be considered to be of the form $(x/x_0)^n$, where n is a very large odd number. For more moderate nonlinearities the deviation of the phase-space trajectory from the circle depends on the energy of oscillation, as will be demonstrated in the following discussion.

As pointed out in connection with Fig. 1.2, if the potential energy of the oscillator increases monotonically from a stable equilibrium point, as in the linear oscillator, the motion is always periodic and the phase-plane trajectories are all closed paths about the equilibrium point. If, on the other hand, the potential energy has a maximum corresponding to an unstable equilibrium point, we shall find that phase-plane trajectories intersect at that point. To illustrate these characteristics and the use of the phase plane in general, we consider the mass-spring oscillator mentioned in connection with Fig. 14.1.

If, as before, the relaxed length of the spring in Fig. 14.1 is L_0 and the distance between the supports is L, we obtain, for the potential energy corresponding to a transverse displacement x,

$$V(x) = 2Kx^2 + KLL_0\left(1 - \sqrt{1 + \frac{4x^2}{L^2}}\right) + \tfrac{1}{2}K(L - L_0)^2$$

or in dimensionless form, and omitting the useless constant,

$$\frac{V(x)}{2KL^2} = \eta^2 + \frac{L_0}{2L}(1 - \sqrt{1 + 4\eta^2})$$

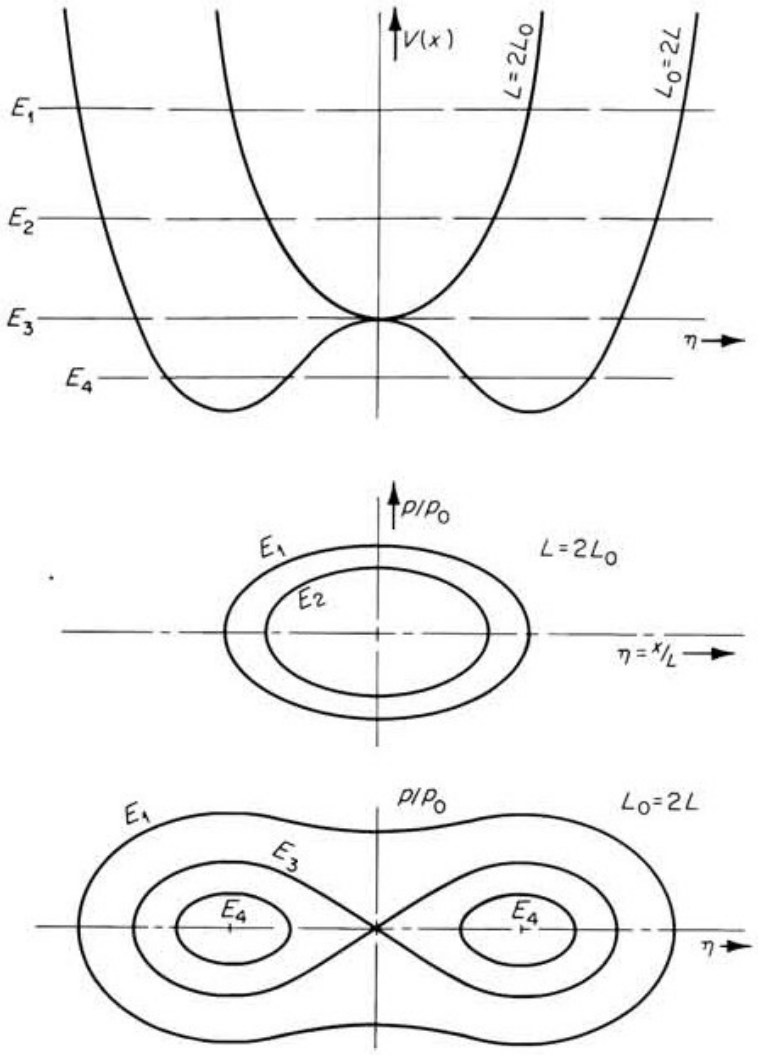


FIGURE 14.2
Potential-energy and phase-plane trajectories for the oscillator of Fig. 14.1.

where $\eta = x/L$. We have already seen that the behavior of the oscillator depends, essentially, upon the ratio (L_0/L) between the relaxed length of the spring and the distance between the fixed end points of the spring. If $L_0 < L$, the equilibrium position $x = 0$ is stable, and the potential-energy curve has a minimum. On the other hand, if $L_0 > L$, the spring is compressed when $x = 0$ and the potential-energy curve has a maximum as shown in Fig. 14.2, where the special cases $L_0/L = 0.5$ and $L_0 = 2$ are considered. Note that in the latter case there are two stable equilibrium points, at approximately $x = \pm L$, about which oscillatory motion can occur for sufficiently small energies.

In Fig. 14.2 are shown the integral curves in phase space for some different energies. For $L_0/L = \frac{1}{2}$, the energy curves are all ellipses centered about $p = 0$, $x = 0$. For $L_0/L = 2$, however, the curves are essentially different when $E > 0$ and when $E < 0$. In the former case the integral curves are closed paths around the origin. For $E = 0$, on the other hand, the integral curve runs through the origin, and the slope of the curve at the origin depends on the direction of motion; the particle comes momentarily

to rest as it passes through the origin. Finally, for $E < 0$, the integral curve splits up into two separate closed branches centered about the two stable equilibrium points mentioned earlier.

In the calculation of the period of oscillation, we shall consider only the case when $L > L_0$. For sufficiently small values of the amplitude of oscillation, the restoring force becomes

$$F(x) \simeq \frac{4Kx(L - L_0)}{L} + \frac{8Kx^3}{L^2} = kx\left[1 + \left(\frac{x}{d}\right)^2\right]$$

where $k = 4K(L - L_0)/L$ is positive, and $d = L\sqrt{2L_0/(L - L_0)}$ is the displacement at which the linear and nonlinear parts of the restoring force are equal. The potential energy is then (again omitting the constant term)

$$V(x) = \frac{kx^2}{2}\left[1 + \frac{1}{2}\left(\frac{x_0}{d}\right)^2\right] \tag{14.1.7}$$

If the amplitude of oscillation is x_0, the total mechanical energy of oscillation can be expressed as

$$E = \frac{kx_0^2}{2}\left[1 + \frac{1}{2}\left(\frac{x_0}{d}\right)^2\right]$$

and the period of oscillation, according to Eq. (14.1.3), becomes

$$T_p = 4\sqrt{\frac{M}{2}}\int_x^{x_0}(E - V)^{-\frac{1}{2}}\,dx = \frac{8d}{\sqrt{2}}\sqrt{\frac{M}{k}}\int_x^{x_0}\frac{dx}{\sqrt{(x_0^2 - x^2)(x_0^2 + 2d + x^2)}}$$

If we introduce $x = x_0 \cos\psi$, this integral reduces to

$$T_p = T_0\left[\frac{2}{\pi}\int_0^{2/\pi}\frac{d\psi}{\sqrt{1 + (x_0^2/2d^2)(1 + \cos^2\psi)}}\right]$$

where $T_0 = 2\pi\sqrt{M/k}$. Expanding the square root, we obtain

$$T = T_0\left[1 - \frac{3}{8}\left(\frac{x_0}{d}\right)^2 + \cdots\right] \tag{14.1.8}$$

and T_0 is shown to be the limiting period, when the amplitude of motion x_0 is much smaller than d, so that the motion is simple-harmonic. We note that the period decreases with increasing amplitude, a characteristic feature of an oscillator with a "hard" spring. For an oscillator with a "soft" nonlinearity, as demonstrated already in the case of a pendulum, the period increases with increasing amplitude.

Nonconservative forces

In many cases it is necessary to consider, in addition to the conservative force $F(x)$, a nonconservative force $G(x,v)$, which depends on the velocity v.

In the damped linear oscillator in Chap. 2 we had, for example, $G(x,v) = -Rv$. The force $G(x,v)$ clearly will change the total mechanical energy H of the oscillator at a rate given by

$$\frac{dH}{dt} = G(x,v)v$$

where

$$H = \frac{Mv^2}{2} + \int (-F)\,dx$$

This result follows, of course, from Eq. (1.3.2).

If we again use the linear damped oscillator as an example, we have $Gv = -Rv^2$; that is, energy is removed from the oscillation at a rate equal to Rv^2. As a result, the amplitude of oscillation decreases exponentially with time. In other oscillators, which will be discussed later, we shall find that energy is not always removed from the oscillation. Here the function Gv will be found to be positive, at least in some parts of the phase plane, and in these regions the total mechanical energy of the oscillator will increase. Under such conditions the relation $Gv = 0$ defines a curve in the phase plane which divides the plane into regions of negative and positive values of Gv, which we shall call *dissipation* and *activation* regions, respectively. For example, if $G(x,v) = Av - Bv^3$, it follows that $v > \sqrt{A/B}$ defines a dissipation region, and $v < \sqrt{A/B}$ an activation region, if A and B are positive.

When both dissipation and activation regions exist, so-called *self-sustained* oscillations can be produced. For such oscillations, the phase-plane trajectory of such a periodic oscillation must pass through both the dissipation and the activation region, so that the energy dissipated in one region is regained in the other. In the particular case referred to above, in which $G = Av - Bv^3$, it follows that the velocity amplitude of any self-sustained oscillation must have a velocity amplitude which is larger than $\sqrt{A/B}$; otherwise the phase-plane trajectory would not be able to enter the dissipation region. We shall return to these oscillations later, with special examples.

Graphical analysis

In the presence of a nonconservative force it is no longer possible to obtain an integral solution similar to that in Eq. (14.1.3); generally, we have to resort to graphical or other numerical methods to construct solutions to the equation of motion. In the graphical procedure it is convenient to write the equation of motion in the form

$$\frac{dv}{dx} = \frac{F(x) + G(x,v)}{Mv} \tag{14.1.9}$$

from which the slope of the phase-plane trajectory can be calculated directly

from F and G. The trajectory can then be constructed when the initial conditions are given.

The graphical construction becomes particularly simple in the special case when the conservative force is linear and the nonconservative force depends only on v. Then, with $F(x) = -Kx$, we obtain $dv/dx = (-\omega_0^2 x + g)/v$, where $\omega_0^2 = K/M$ and $g = G/M$. Furthermore, if the new coordinate $x_1 = \omega_0 x$ is introduced, the equation reduces to

$$\frac{dv}{dx_1} = \frac{-x_1 + (g/\omega_0)}{v} = -\frac{x_1 - g_1(v)}{v} \tag{14.1.10}$$

where $g_1 = g/\omega_0$. The slope dv/dx_1 can now be constructed at a certain

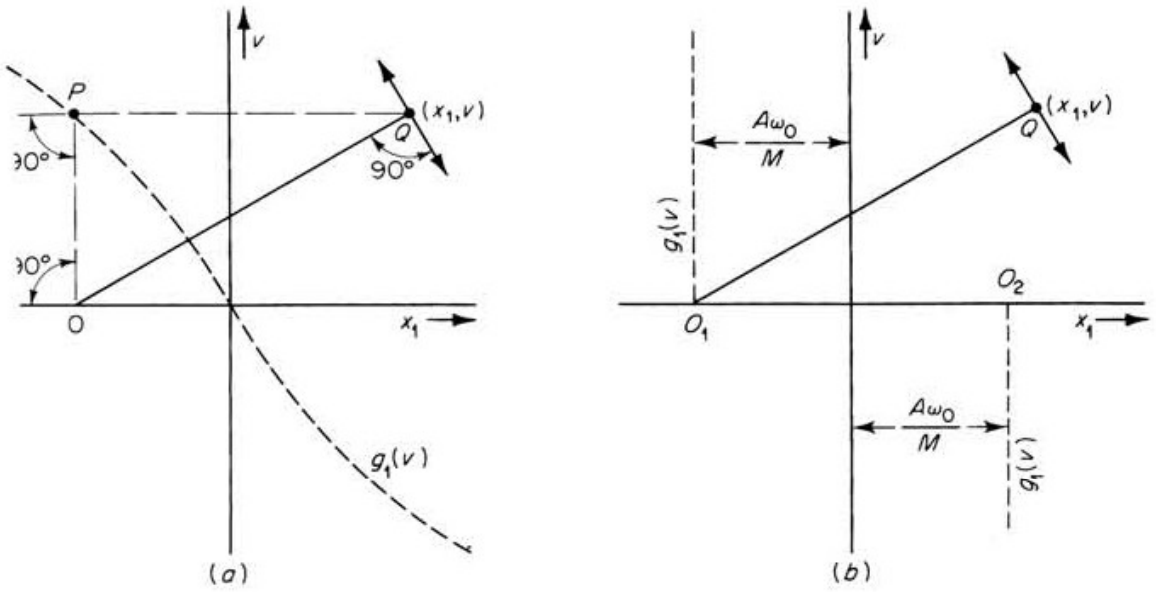


FIGURE 14.3
Graphical construction of slope dv/dx_1 at Q.

point x_1, v in the following manner: First plot the function $g_1(v)$ as indicated schematically in Fig. 14.3*a*. The horizontal line through the point $Q(x_1,v)$ intersects g_1 in the point P. The distance PQ is then equal to $x_1 - g_1(v)$. Draw the vertical through P and mark the intersection O with the x_1 axis. The required slope of the trajectory through x_1, v is then perpendicular to the line OQ. It is noted that the trajectory must have infinite and zero slopes when it crosses the x_1 axis and the "guide" curve $g_1(v)$, respectively.

In the special case when $G(x_1,v)$ is a constant friction force of magnitude A, the solution is particularly simple. The guide curve is represented by the straight lines $g_1(v) = -(A/\omega_0 M)$ for $v > 0$ and $g_1(v) = +(A/\omega_0 M)$ for $v < 0$. We find that the amplitude of oscillation decreases along a spirallike phase-plane trajectory which consists of a succession of semicircles with centers alternating between the points O_1 and O_2 given by $x_1 = -(A/\omega_0 M)$

and $x_1 = +(A/\omega_0 M)$, as indicated in Fig. 14.3*b*. At the points O_1 and O_2, the magnitudes of the spring force and the friction force are equal. Between these points the friction force is larger than the spring force, and once the velocity has become zero in this region, the oscillations cannot be continued. Thus, in the motion illustrated in the figure, the amplitude of the oscillations decays until the mass comes to rest between O_1 and O_2, at which point the oscillator stops.

Slowly varying amplitudes

If the nonconservative and nonlinear forces on an oscillator are small compared with the linear one (which is often the case at sufficiently small amplitudes of oscillation), the motion can be analyzed formally by various approximate methods. It is not our intention here to attempt a systematic survey of these various methods; we shall select one which is simple, yet typical of many situations encountered in practice.

We start by considering again a linear oscillator of mass m. If we use the same coordinates in the phase plane as in Eq. (14.1.10), that is, v and $\omega_0 x$, the trajectory representing the linear oscillator will be the circle

$$v^2 + (\omega_0 x)^2 = \frac{2E}{m}$$

with a radius equal to the velocity amplitude $v_0 = \sqrt{2E/m}$. Actually, if we wish the projections of the radius on the two axes to represent the displacement and the velocity, respectively, we should use as the coordinates $-v$ and $\omega_0 x$, since $\omega_0 x = v_0 \cos(\omega_0 t)$ and $-v = -dx/dt = v_0 \sin(\omega_0 t)$.

A nonlinearity in the oscillator produces a deviation from the circular trajectory and makes both the length of the radius vector A and its angular velocity in the phase plane time-dependent. Instead of the constant velocity amplitude v_0 and the phase $\omega_0 t$, the time-dependent polar coordinates A and ψ of the point $(\omega_0 x, v)$ in the phase plane will be used. In many cases of practical interest the fractional changes of A and $\psi - \omega_0 t$ during the course of a period $T_0 = 2\pi/\omega_0$ are both small, and under these conditions it is possible, as we shall see, to obtain simple approximate expressions for the rates of change of the amplitude and the angular velocity. To see how these relations are obtained, we start by expressing the equations of motion in terms of the polar coordinates A and ψ, defined by $\omega_0 x = A \cos \psi$ and $-v = A \sin \psi$.

This appears, at first, to be substituting two independent variables, A and ψ, for one, x. This is not so, since we have required that the velocity

$$v = \frac{dx}{dt} = \frac{1}{\omega_0}\frac{dA}{dt}\cos\psi - \frac{A}{\omega_0}\frac{d\psi}{dt}\sin\psi$$

be equal to $-A \sin \psi$. The second equation determining A and ψ is, of

course, the equation of motion

$$\frac{dv}{dt} \equiv -\frac{dA}{dt}\sin\psi - A\frac{d\psi}{dt}\cos\psi = -\omega_0^2 x + g(x,v)$$
$$= -\omega_0 A\cos\psi + g(A,\psi)$$

As before $g = G(x,v)/m$, where $G(x,v)$ is the nonlinear or nonconservative part of the force on the oscillator. Solving these two equations simultaneously for dA/dt and $d\psi/dt$, we obtain

$$\frac{dA}{dt} = g(A,\psi)\sin\psi$$
$$A\left(\frac{d\psi}{dt} - \omega_0\right) = g(A,\psi)\cos\psi \qquad (14.1.11)$$

These equations reduce to the linear oscillator equations $A = \text{const}$ and $\psi = \omega_0 t$ when force $g(x,v) = 0$. Since $g(x,v) \equiv g(\omega_0^{-1}A\cos\psi, -A\sin\psi)$, it then follows that the right-hand members of the equations are both periodic functions of ψ, and thus can be expanded in Fourier series.

The first terms in these series, those independent of ψ, are, respectively,

$$a(A) = \frac{1}{2\pi}\int_0^{2\pi} g(A,\psi)\sin\psi\, d\psi$$

and (14.1.12)

$$b(A) = \frac{1}{2\pi}\int_0^{2\pi} g(A,\psi)\cos\psi\, d\psi$$

These terms represent the average, or *secular*, rate of change of A and $\psi - \omega_0 t$ over a period. If A and $\psi - \omega_0 t$ vary slowly, so that their fractional change is much less than unity in a period, these average rates should be good approximations for the determination of the time dependence of A and ψ.

Under these conditions we can approximate Eqs. (14.1.11) by

$$\frac{dA}{dt} = a(A) \qquad A\left(\frac{d\psi}{dt} - \omega_0\right) = b(A) \qquad (14.1.13)$$

The first of these equations can be integrated directly, yielding t as a function of A; then ψ can be determined from the second equation.

$$\psi = \omega_0 t + \int \frac{b(A)}{A}\, dt \qquad (14.1.14)$$

Before we apply these results to some specific cases, we observe that if the system is conservative, g depends only on $x = \omega_0^{-1}\cos\psi$, and not on v, and is consequently an even function of ψ. As a result, $\int_0^{2\pi} g(A,\psi)\sin\psi = 0$,

and according to the first of Eqs. (14.1.12), the average rate of change of the amplitude A vanishes. Thus, to the approximation considered here, the amplitude of oscillation remains constant, $A = v_0$, and only the phase is affected by a conservative perturbation of the linear oscillator. The phase is then given by

$$\psi = \omega_0 t + \left[\frac{b(v_0)}{v_0}\right] t = \left[\omega_0 + \frac{1}{2\pi m v_0}\int_0^{2\pi} G(A,\psi)\cos\psi\, d\psi\right] t \qquad (14.1.15)$$

The period of the oscillation, i.e., the time required to change the phase by an angle 2π, is then

$$T = \frac{2\pi}{\omega_0 + (b/v_0)} = \frac{T_0}{1 + (b/\omega_0 v_0)}$$

The quantity $(b/\omega_0 v_0)$ can be interpreted as the ratio between the mean value of the nonlinear force, averaged over the motion, and the maximum linear restoring force $m\omega_0 v_0$. If this average nonlinear force is negative (soft spring), the period is increased; if it is positive (hard spring), the period is decreased by the nonlinearity.

Frictional forces

Having considered the effect of a conservative perturbation, we next treat the case when the force g depends only on the velocity. We get $b = (2\pi)^{-1}\int_0^{2\pi} g(-A\sin\psi)\cos\psi\, d\psi$, and if g is an odd function of v, this is zero. Therefore, according to Eqs. (14.1.13), only the amplitude, and not the phase, is influenced by the perturbation force g.

It is instructive to use the present approximate method on the familiar example of a linear oscillator in which the damping force is $G = -Rv$. With $v = -A\sin\psi$ and $g = G/m$, we have $g = (2/\tau)A\sin\psi$, where $2/\tau = R/m$. The rates of change of the amplitude and the phase of the oscillator then reduce to $a(A) = (2\pi)^{-1}\int_0^{2\pi} g\sin\psi\, d\psi = (2\pi)^{-1}(2/\tau)\int_{v_0}^{2\pi}\sin^2\psi\, d\psi = A/\tau$ and $b(A) = 0$; that is,

$$\frac{dA}{dt} = -\frac{A}{\tau} \qquad A\left(\frac{d\psi}{dt} - \omega_0\right) = 0$$

The solutions are $A = v_0 e^{-t/\tau}$ and $\psi = \omega_0 t + \phi_0$. Thus the expression for the velocity amplitude agrees with the exact solution in Chap. 2, whereas the angular velocity $\psi = \omega_0$ falls short of the exact solution by a factor $1/\sqrt{1 - (\omega_0\tau)^{-2}}$. This deviation from the exact solution is negligible for practical purposes when the "lifetime" τ of the oscillation is long compared with the period $T_0 = 2\pi/\omega_0$ of the undamped oscillator, as we have assumed.

As another illustration of the method, we consider again the problem of the transverse mass-spring oscillator already shown in Fig. 14.1. In that case, from Eq. (14.1.7), the restoring force for sufficiently small displacements

is of the form $kx[1 + (x/d)^2]$. The nonlinear part is then $G(x) = kx^3/d^2$, or $g(x) = G/m = \omega_0^2 x^3/d^2$, which, with $x = (A/\omega_0)\cos\psi$, becomes $g(A,\psi) = (A^3/\omega_0 d^2)\cos^3\psi$. Since in this case $a(A) = 0$, the amplitude of the oscillation remains constant, $A = v_0$. The expression for the rate of change of the phase, according to Eqs. (14.1.12) and (14.1.13), is then

$$b(A) = (2\pi)^{-1}\frac{v_0^3}{\omega_0 d^2}\int_0^{2\pi}\cos^4\psi\, d\psi = \frac{3}{8}\frac{v_0^3}{\omega_0 d^2}$$

and Eq. (14.1.15) gives the corresponding expression for the period of oscillation,

$$T = \frac{T_0}{1 + \frac{3}{8}(x_0/d)^2} = T_0\left[1 - \frac{3}{8}\left(\frac{x_0}{d}\right)^2 + \cdots\right]$$

where we have introduced $v_0 = \omega_0 x_0$, x_0 being the displacement amplitude. It is interesting to see that this result for the period is the same as that in Eq. (14.1.8), obtained from the exact solution of the equation of motion.

As a final example, let us consider an oscillator with a linear restoring force and with a friction force of the form $G = Rv + Cv^3 = Rv[1 + (v/\mu)^2]$. Here the quantity μ is the velocity at which the linear friction force is equal to the nonlinear one. Introducing the polar coordinates, we have $v = -A\sin\psi$ and $g = G/m = -(2/\tau)A\sin\psi + \mu^{-2}A^3\sin^3\psi$, where $\tau = 2m/R$. After having performed the required integrations for the calculation of $a(A)$, we obtain, according to Eqs. (14.1.11) and (14.1.12),

$$\frac{dA}{dt} = -\frac{1}{\tau}A\left[1 + \frac{3}{4}\left(\frac{A}{\mu}\right)^2\right] \tag{14.1.16}$$

It so happens that this equation can be solved exactly, in closed form; the solution is

$$A = v_0 e^{-t/\tau}\left[1 + \frac{3}{4}\left(\frac{v_0}{\mu}\right)^2(1 - e^{-2t/\tau})\right]^{-\frac{1}{2}} \tag{14.1.17}$$

where v_0 is the initial velocity amplitude. Thus

$$x \simeq \left(\frac{1}{\omega_0}\right)A(t)\cos(\omega_0 t + \Phi)$$

neglecting higher terms in the Fourier series for x and v, which, as we saw, is allowable as long as $\tau \gg 1/\omega_0$, that is, as long as the Q of the oscillator, $\sqrt{Km/R^2} = \omega_0 m/R$ of Eq. (2.2.4), is a large number. The amplitude of motion diminishes with time, a little faster than exponentially at first, because of the nonlinear term.

Self-sustained oscillations

When the dissipation function $vG(x,v)$ is positive as well as negative here is the opportunity for the oscillator to gain energy, as well as to lose it.,

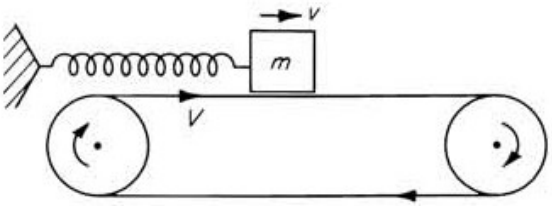


FIGURE 14.4
Example of a self-sustained oscillator.

And when the path in phase space traverses activation regions (where vG is positive) as well as dissipation regions (where vG is negative), steady-state motion is again possible if the amount of energy gained just equals the energy lost per cycle. A large number of mechanical and electronic systems display this characteristic. In some of these, self-sustained oscillations are produced by means of some "feedback" mechanism, the energy required to sustain the motion being derived from a time-independent energy source. Electronic tube oscillators belong to this category. In other systems the oscillations are driven more or less directly by some steady force; wind instruments and aircraft wings are driven into oscillation by an air blast, violin strings by the friction of the moving bow, and clocks by the steady force of the mainspring. The activation of the oscillator by these forces in most cases can be regarded conveniently as a negative resistance, and the conditions for self-sustained oscillations then can be expressed by the requirement that the total resistance in the oscillator should be zero (or negative).

As an illustration, let us consider the oscillator shown in Fig. 14.4. A mass m is attached to the end of a spring which provides a linear restoring force. The mass slides on a horizontal surface which is moving with a velocity V. We assume that air friction or some other damping mechanism produces a damping force $-Rv$ proportional to the velocity v of the mass. In addition, there is the friction force from the belt, a force $F(V - v)$, which depends on the relative velocity $V - v$ between the mass and the belt. The equation of motion of m is

$$m\frac{d^2x}{dt^2} = -Kx - Rv + F(V - v) \qquad v = \frac{dx}{dt}$$

We shall consider $v < V$ and expand, $F(V - v) = F(V) - F'(V)v + F''(V)(v^2/2) + \cdots$. The contact friction force decreases with increasing relative velocity (i.e., with *decrease* of v) so that $F'(V) < 0$. If we set $F'(V) = -R'$ and neglect higher-order terms in v, we have $F(V - v) \simeq F(V) + R'(dx/dt)$, and the resistive force in the equation of motion becomes $-(R - R')(dx/dt)$. In other words, the effect of the moving belt, for small velocities v, can be interpreted as a negative resistance R', and if $R' > R$, the total resistance in the oscillator becomes negative and the oscillations grow with time. These oscillations do not grow indefinitely because the higher-order terms eventually will introduce further damping, and the amplitude of oscillation will be limited, as we shall see in the following

discussion. The mechanism illustrated by this example describes in principle the oscillations of a bowed violin string or the squeaking of a door or the brake on a car.

To demonstrate the amplitude-limiting effect of the higher-order terms in the expression for the contact friction, we shall consider only the third-order term. We are interested now in the case when the total linear resistance $R - R'$ in the oscillator is negative, and we set $R - R' = -R_1$. The total resistive force is then of the form

$$-R_1 v + Cv^3 = -R_1 v\left[1 - \left(\frac{v}{\mu}\right)^2\right]$$

where μ is the velocity at which the magnitudes of the linear and nonlinear terms are the same.

Except for the difference in sign, this nonconservative force is of the same form as in the example discussed earlier in this section [Eq. (14.1.16)]. If this force is small compared with the linear restoring force, the same approximate method of solution can be used here. Under these conditions the expression for the velocity amplitude of the oscillation can be written down directly from the result already given in Eq. (14.1.17). The only change required is to replace τ by $-\tau$ and μ^2 by $-\mu^2$, and the time dependence of the (velocity) amplitude of the oscillations becomes

$$A(t) = v_0 e^{t/\tau}\left[1 + \frac{3}{4}\left(\frac{v_0}{\mu}\right)^2 (e^{2t/\tau} - 1)\right]^{-\frac{1}{2}} \tag{14.1.18}$$

where $\tau = 2m/R_1$, and v_0 is the velocity amplitude at $t = 0$. The "relaxation time" τ must be much larger than $1/\omega_0$ for the approximation

$$x = (1/\omega_0)A(t)\cos(\omega_0 t + \Phi)$$

to be valid. We see that, no matter what its initial value v_0, the velocity amplitude A eventually becomes $\sqrt{\frac{4}{3}}\,\mu$; if v_0 is larger than this, the amplitude decreases; if v_0 is smaller, the oscillator picks up energy to reach the stable amplitude. This differs from the purely dissipative case of Eq. (14.1.17), for which the amplitude goes to zero, no matter what its initial value was.

This result is consistent with our previous findings. The nonconservative force in the present case is $G(v) = R_1 v[1 - (v/\mu)^2]$, and the power transferred to the oscillation is $Gv^2[1 - (v/\mu)^2]$. Thus, when $v < \mu$, the energy transfer is positive, and when $v > \mu$, it is negative; the region $|v| < \mu$ in phase space is the activation region, and the region $|v| > \mu$ is the dissipation region. Therefore, when the oscillation is stationary, the phase-space trajectory must go through both these regions, and consequently the velocity amplitude must be larger than μ.

When the oscillation amplitude has reached the stationary value $\sqrt{\frac{4}{3}}\,\mu$, the rate of change of the amplitude is zero, as can be seen directly from

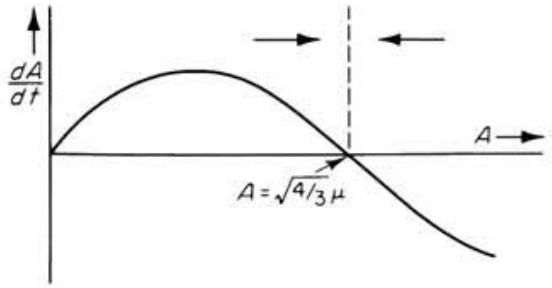


FIGURE 14.5
Plot of time rate of change of amplitude.

Eqs. (14.1.12) and (14.1.13), which in this case result in

$$\frac{dA}{dt} = a(A) = \frac{1}{\tau} A\left[1 - \frac{3}{4}\left(\frac{A}{\mu}\right)^2\right] \tag{14.1.19}$$

Clearly, the rate of change dA/dt is zero also at $A = 0$. At this point, however, the motion is unstable, which is characterized by the fact that the curve dA/dt versus A has a positive slope at that point, as shown in Fig. 14.5. A small disturbance from the point $A = 0$ makes the rate of change of A increase further in the same direction. At $A = \sqrt{\frac{4}{3}}\,\mu$, on the other hand, the slope is negative, and an increase of A beyond this point will be reduced until the amplitude is brought back to the value $\sqrt{\frac{4}{3}}\,\mu$; likewise, a reduction of A will cause a countertrend to increase A again. In other words, regardless of the starting point in the phase plane, all phase-space trajectories end up in the same stable path $A = \sqrt{\frac{4}{3}}\,\mu$, circular to this order of approximation.

There are other types of force fields, producing self-sustained oscillations, for which this is not the case, forces resulting in a rate of change of the amplitude which depends on A, in the manner shown in Fig. 14.6. In this case, if the initial amplitude of oscillation is less than A_1, the oscillation will decay to the stable equilibrium point $A = 0$. If, on the other hand, the initial amplitude is larger than A_1, the oscillation will be pulled toward a stable motion in which the amplitude is A_2. A familiar example of an oscillation of this kind is an ordinary pendulum clock. To get the clock going, the pendulum must be started with an amplitude that exceeds a certain value A_1; otherwise the motion of the pendulum will decay.

Graphical solutions

When the nonconservative forces in the oscillator are not small compared with the restoring force, the approximate formulas of Eqs. (14.1.13) are no

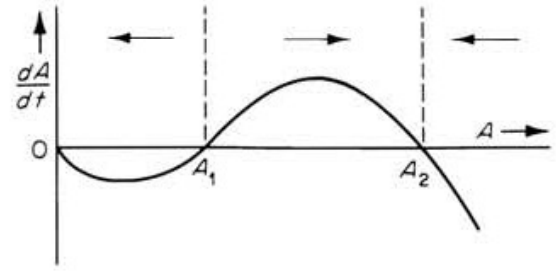


FIGURE 14.6
Oscillator with two distinct points of stable equilibrium, 0 and A_2.

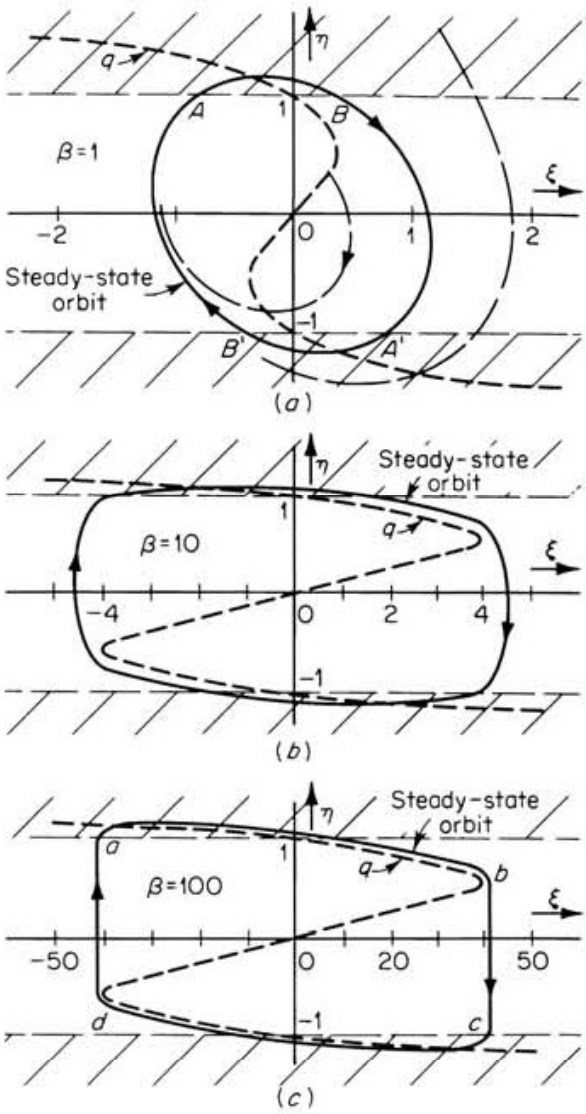


FIGURE 14.7
Phase-plane trajectories for systems with negative and strong positive resistive forces.

longer satisfactory. Rather than make use of improved versions of this method or other analytical techniques, we shall construct solutions graphically, using the method described in connection with Fig. 14.3. For this purpose it is convenient to rewrite the equation of motion $mv(dv/dx) = -Kx + Rv[1 - (v/\mu)^2]$ in terms of the normalized coordinates $\xi = \omega_0 x/\mu$ and $\eta = v/\mu$. In this way we reduce the number of parameters in the equation to one, and obtain

$$\frac{d\eta}{d\xi} = \frac{-\xi + \beta\eta(1 - \eta^2)}{\eta} \tag{14.1.20}$$

where $\beta = 2/\omega_0\tau = R/\omega_0 m = 1/Q$ is the reciprocal of the Q of the oscillator. We are now assuming that Q is not very large; therefore β is not necessarily small. This equation is the same as Eq. (14.1.10), with η instead of v, ξ instead of x_1, and $q = \beta\eta(1 - \eta^2)$ instead of $g_1(v)$.

In the graphical solution of this equation the curve $q = \beta\eta(1 - \eta^2)$ is shown as the dotted line of Fig. 14.7*a*. Using it as a guide curve, we

construct the slope of the trajectory following the procedure described earlier in Fig. 14.3. Trajectories thus obtained for the case $\beta = 1$ are shown in Fig. 14.7*a*. Two trajectories with different initial oscillation amplitudes are shown to be "pulled" in toward the same stable orbit, as expected. We recall from the analytical treatment that the phase-space trajectory for small values of β is approximately circular, with an amplitude $A = \sqrt{\frac{4}{3}}\,\mu$, that is, $\eta = \sqrt{\frac{4}{3}}$. The graphically constructed limiting orbit for $\beta = 1$ is not too far different from this circular path.

In Fig. 14.7*a* are also shown the lines $\eta = 1$ and $\eta = -1$ which divide the phase plane into the dissipation and activation regions, $|\eta| > 1$ and $|\eta| < 1$, respectively. In the dissipative region the amplitude of oscillation decreases, and in the activation region it increases. When the limit cycle is reached, the net energy exchange is zero, so that the motion becomes periodic; the dissipation along the paths $A - B$ and $A' - B'$ is then compensated by the energy gain in the regions $B' - A$ and $B - A'$.

As β is increased, the steady-state orbit will deviate more and more from the circular path and cling to the guide curve, as illustrated in Fig. 14.7*b*, where the steady-state orbit for $\beta = 10$ is shown. To study Eq. (14.1.20) for larger values of β, it is convenient to introduce $\zeta = \xi/\beta$ and obtain $\eta(d\eta/d\zeta) = \beta^2[\eta(1 - \eta^2) - \zeta]$. Then, if $\beta = \infty$, the slope $d\eta/d\zeta$ is infinite at all points in the phase plane, except on the guide curve $\zeta = \eta(1 - \eta^2)$. In view of this fact, and with reference to Fig. 14.7*a* and *b*, it is reasonable to assume (and it can indeed be shown rigorously) that as β approaches infinity, the limit cycle will approach the q curve, as indicated in Fig. 14.7*c*.

In the sections *bc* and *da* of this curve, the position coordinate remains constant (at $\zeta = \pm 2/3\sqrt{3}$). (The corresponding values of η are $\eta_b = 1/\sqrt{3}$, $\eta_d = -1/\sqrt{3}$, $\eta_a = 2/\sqrt{3}$, $\eta_c = -2/\sqrt{3}$.) Therefore, in the calculation of the period of oscillation, these sections do not contribute. Since the times between the points a and b and between c and d are the same, the period of oscillation for large values of β, say, $\beta > 10$, can be approximated by $T = 2\int_a^b |dx/v|$. Introducing $\xi = \omega_0 x/\mu$ and $\eta = v/\mu$, we get

$$T = (2/\omega_0)\int d\xi/\eta$$

which, with $\xi = \beta\eta(1 - \eta^2)$, can be expressed as

$$\begin{aligned} T &= \frac{2\beta}{\omega_0}\int_{\eta_a}^{\eta_b} |1 - 3\eta^2|\,\frac{d\eta}{\eta} \\ &= (3 - 2\ln 2)\,\frac{\beta}{\omega_0} \simeq 1.61\,\frac{\beta}{\omega_0} = \frac{T_0}{10.2\tau}\,T_0 \end{aligned} \qquad (14.1.21)$$

where $T_0 = 2\pi\sqrt{m/K}$, $\tau = 2m/R_1$, and $q = \sqrt{mK/R_1^2} = 1/\beta$. As pointed out above, this expression is a good approximation only for sufficiently

large values of β, say, $\beta > 10$, which means that the value of τ must be less than about $T_0/10\pi$ for the validity of Eq. (14.1.21). We can also express this limitation directly in terms of T. With $\beta > 10$ inserted in Eq. (14.1.21), we see that the period T should be larger than three times T_0.

14.2 FORCED MOTION

So far we have studied the effect of nonlinearities only upon the free and self-sustained motion of an oscillator. Of at least equal interest is the response of such an oscillator to an external harmonic driving force. The forced motion of an undamped linear oscillator, as we know, has infinite amplitude at resonance; therefore it is to be expected that, in practice, nonlinear effects should be of importance, at least in the neighborhood of the resonance frequency. The study of the nonlinear distortion of the amplitude-frequency response curve is of particular interest in this connection. Nonlinearities are responsible also for other phenomena, such as the generation of motion containing harmonics, and under certain conditions even subharmonics, of the driving frequency. Furthermore, when the driving force contains two or more frequencies, the resulting motion will contain frequencies which are linear combinations of the driving frequencies.

A simple example

In the following discussion of these questions, we shall consider only a special example, an oscillator with a "hard" restoring force of the form $Kx[1 + (x/d)^2]$ and with a linear damping force $R(dx/dt)$. The equation of motion is then

$$m\frac{d^2x}{dt^2} + R\left(\frac{dx}{dt}\right) + Kx\left[1 + \left(\frac{x}{d}\right)^2\right] = F\cos(\omega t) \qquad (14.2.1)$$

where we must use the cosine, instead of the exponential $e^{-i\omega t}$, since the equation is nonlinear. It is convenient to rewrite this equation in dimensionless form. Two characteristic displacements are involved, the static displacement $x_{st} = F/K$ corresponding to the force amplitude F, and the displacement d at which the nonlinear restoring force becomes equal to the linear one. We pick the former for the normalization of the displacement, and introduce $\xi = x/x_{st}$. Furthermore, we set $\omega_0 = \sqrt{K/m}$, $\theta = \omega_0 t$, $\gamma = \omega/\omega_0$, and $\epsilon = (x_{st}/d)^2$. First let us consider the resistance to be negligible, in which case the equation of motion reduces to

$$\frac{d^2\xi}{d\theta^2} + \xi(1 + \epsilon\xi^2) = \cos(\gamma\theta) \qquad (14.2.2)$$

We seek a periodic solution with the fundamental frequency equal to the driving frequency. Since the right-hand side is symmetrical with respect

to $\theta = 0$ and antisymmetrical with respect to $\gamma\theta = \pi/2$, the Fourier expansion of the left-hand side must have the same properties; that is, ξ must be of the form

$$\xi = \sum_n \xi_n \cos(n\gamma\theta)$$

where $n = 1, 3, 5, \ldots$.

The left-hand side of the equation will be of the form $A_n \cos(n\gamma\theta)$, where the A_n's are functions of ξ_n; to satisfy the equation we must have $A_1 = 1$ and $A_n = 0$ for all other values of n. This condition corresponds to an infinite set of equations from which, in principle, we can determine the amplitudes ξ_n. As a first approximation, we consider now only two terms in the expansion and set $\xi = \xi_1 \cos(\gamma\theta) + \xi_3 \cos(3\gamma\theta)$. By using the trigonometric relation $4\cos^3(\gamma\theta) = 3\cos(\gamma\theta) + \cos(3\gamma\theta)$, we find that the coefficient for the $\cos(\gamma\theta)$ term on the left-hand side of Eq. (14.2.2) is

$$A_1 = (1 - \gamma^2)\xi_1 + \tfrac{3}{4}\epsilon\xi_1(\xi_1^2 + \xi_1\xi_3 + 2\xi_3^2)$$

and similarly, the coefficient for the $\cos(3\gamma\theta)$ term is

$$A_3 = (1 - 9\gamma^2)\xi_3 + \tfrac{1}{4}\epsilon(\xi_1^3 + 6\xi_1^2\xi_3 + 3\xi_3^3)$$

In addition to these terms, we also get a series of terms containing the frequencies 5γ, 7γ, and 9γ, namely,

$$\frac{\epsilon\xi_3}{4}[3\xi_1(\xi_1 + \xi_3)\cos(5\gamma\theta) + 3\xi_1\xi_3\cos(7\gamma\theta) + \xi_3^2\cos(9\gamma\theta)]$$

These remaining terms are small compared with A_1 and A_3 if $\epsilon \ll 1$ and $|\xi_3| \ll |\xi_1|$, and only under these conditions will our approximation be good. The terms containing the product $\epsilon\xi_3$ in A_1 and A_3 also can be neglected, and

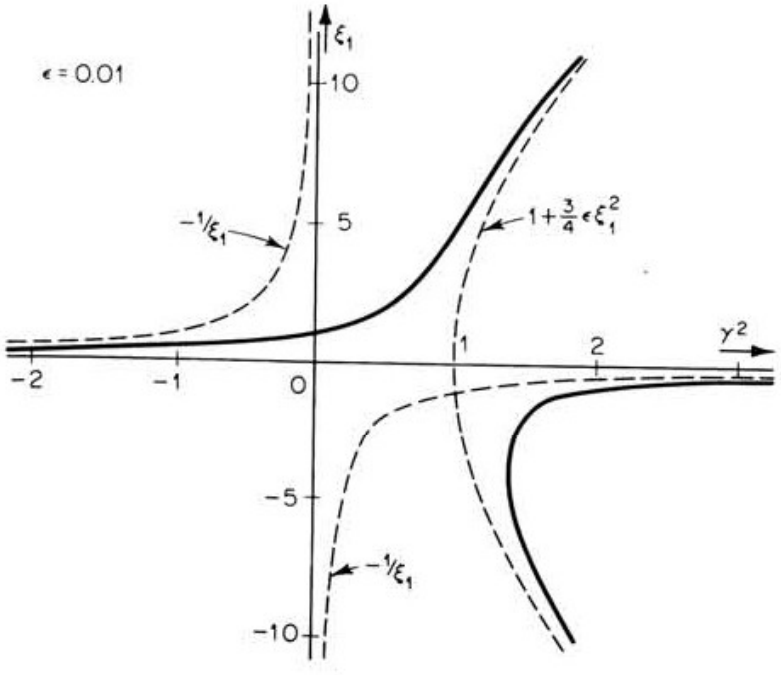


FIGURE 14.8
Construction of amplitude-frequency response curves. Solid curves are plots of $\gamma^2 = (1 + \frac{3}{4}\epsilon\xi_1^2) - (1/\xi_1)$ versus ξ_1.

we obtain the following equations for the determination of the coefficients ξ_1 and ξ_3:

$$1 - \gamma^2 + \tfrac{3}{4}\epsilon\xi_1^2 = \frac{1}{\xi_1} \qquad (1 - 9\gamma^2)\xi_3 + \tfrac{1}{4}\epsilon\xi_1^3 = 0 \tag{14.2.3}$$

The first of these equations represents the amplitude-frequency response for the fundamental frequency in terms of the nonlinearity parameter ϵ.

The amplitude of the third harmonic follows from Eqs. (14.2.3).

$$\xi_3 = -\frac{\epsilon\xi_1^3}{4(1 - 9\gamma^2)}$$

a valid approximation as long as $|\xi_3| < |\xi_1|$. If we wish to obtain improved

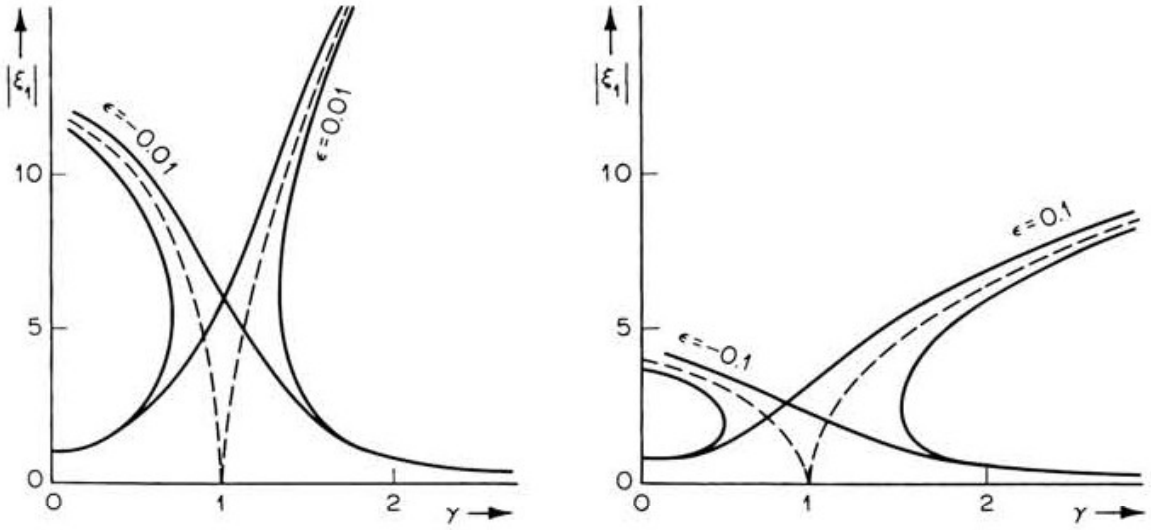


FIGURE 14.9
Amplitude-frequency response curves for nonlinear oscillator. Amplitude ξ_1 of fundamental plotted against driving frequency $\omega/2\pi = \gamma\omega_0/2\pi$.

expressions for ξ_1 and ξ_3, we start with three terms in the initial expansion and satisfy Eq. (14.2.2) for terms including the fifth harmonic. From the three equations then obtained, the amplitudes ξ_1, ξ_3, and ξ_5 can be determined. Higher approximations are obtained in a similar manner.

The dependence of the amplitude ξ_1 of the fundamental frequency in Eqs. (14.2.3) on the frequency ratio $\gamma = \omega/\omega_0$ is displayed in Fig. 14.8 as the sum of the parabola $1 + \frac{3}{4}\epsilon\xi_1^2$ and the hyperbola $-1/\xi_1$. As in the case of the linear oscillator, the response curve contains two branches; in one the displacement is in phase, in the other it is out of phase with the driving force. Both branches approach the curve $1 + \frac{3}{4}\epsilon\xi_1^2$ asymptotically. Response curves obtained in this manner for $\epsilon = 0.01$ and $\epsilon = 0.1$ are shown in Fig. 14.9.

The important difference between the linear and nonlinear oscillators is

exhibited in the response curve; for the nonlinear oscillator it is multivalued over part of the frequency range. For a hard nonlinearity, $\epsilon > 0$, this effect occurs above, and for a soft nonlinearity, $\epsilon < 0$, below the linear resonance frequency, as can be seen in Fig. 14.9. This result suggests that in these frequency regions the motion is unstable; the oscillator amplitude can jump suddenly from one value to another.

The distortion of the frequency-response curve is consistent with the observation made earlier in the discussion of the free oscillations, that the characteristic frequency of the oscillator changes with the amplitude of oscillation. As the driving frequency increases from zero, the amplitude of oscillation increases, and if $\epsilon > 0$, the "resonance" frequency increases also. The maximum of the displacement, therefore, is not reached at the customary value $\gamma = 1$, but is pushed toward higher frequencies. In fact, when damping is absent, the maximum value of the curve is moved out to $\gamma = \infty$.

The effects of friction

Just as in the linear oscillator, damping will limit the amplitude of oscillation. It is again useful to determine the amplitude and the corresponding frequency in terms of the oscillator parameters. For this purpose we rewrite the equation of motion (14.2.1), containing a linear damping term, in terms of the dimensionless variables ξ and θ, as in the foregoing analysis, and obtain

$$\frac{d^2\xi}{d\theta^2} + \frac{1}{Q}\frac{d\xi}{d\theta} + \xi(1 + \epsilon\xi^2) = \cos(\gamma\theta) \tag{14.2.4}$$

where $\xi = Kx/F$

$$\theta = \omega_0 t = t\sqrt{K/m}$$

$$\epsilon = (F/Kd)^2$$

$$Q = (1/R)\sqrt{Km} = m\omega_0/R$$

The frequency dependence of the amplitude of the fundamental follows from this equation by inserting $\xi = \xi_1 \cos(\gamma\theta)$. The left-hand side of this equation then will be of the form $A\cos(\gamma\theta) + B\sin(\gamma\theta)$, where $A = \xi_1(1 - \gamma^2) + \frac{3}{4}\epsilon\xi_1^3$ and $B = -(\gamma/Q)\xi_1$. The expression for A, of course, is the same as in the previous analysis. The amplitude of the right-hand side is unity, and to satisfy the equation we must have $A^2 + B^2 = 1$. To satisfy the phase relations in the equation, a phase angle should be added in the terms in the Fourier series for ξ, but this does not alter the relation between the amplitudes, which is

$$\left(\frac{\gamma}{Q}\right)^2 \xi_1^2 + (1 - \gamma^2 + \tfrac{3}{4}\epsilon\xi_1^2)^2\xi_1^2 = 1 \tag{14.2.5}$$

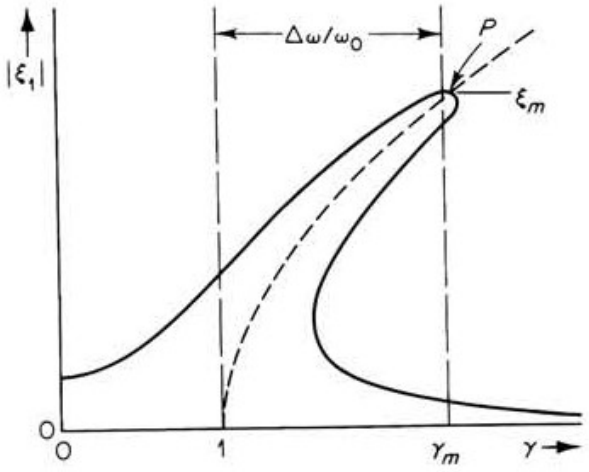


FIGURE 14.10
Response curve for oscillator with nonlinear force proportional to x^3 plus friction. Resonance is at point P.

This response curve intersects the former asymptotic curve $1 - \gamma^2 + \frac{3}{4}\epsilon\xi_1^2 = 0$ at the point P (Fig. 14.10), with the coordinates $\gamma^2 = \frac{1}{2}(1 + \sqrt{1 + 3\epsilon Q^2})$ and $\xi_1^2 = Q^2\gamma^2$. In the case of a linear oscillator ($\epsilon = 0$), these coordinates are simply $\gamma = 1$ and $\xi_1 = Q$, and the Q value [see discussion of Eq. (2.3.2)] can be interpreted as the ratio between the amplitude of oscillation x_1 at resonance and the static displacement x_{st}, $Q = x_1/x_{st}$. We recall that the parameter ϵ is defined by $\epsilon = (x_{st}/d)^2$, where d is the displacement at which the linear and nonlinear restoring forces are equal. The quantity ϵQ^2 thus can be given the simple interpretation $(x_1/d)^2$; it is a measure of the degree of nonlinearity present in the region of maximum amplitude. If $x_1/d \ll 1$, the coordinates of the point P reduce to $\gamma^2 \simeq 1 + (3\epsilon Q^2/4)$ and $\xi_1 \simeq Q(1 + \frac{3}{8}\epsilon Q^2)$.

The maximum of ξ_1, which appears a little to the left of the point P in Fig. 14.10, is found to have the coordinates

$$\gamma_m^2 = \frac{1}{2}\left(1 - \frac{1}{Q^2}\right)\left(1 + \sqrt{1 + 3\epsilon Q^2\,\frac{Q^2 + 1}{Q^2 - 1}}\right)$$
$$\xi_m^2 = (Q^2 + 1)\left(1 + \frac{3\epsilon Q^2}{4}\,\frac{Q^2 + 1}{Q^2 - 1}\right) \qquad (14.2.6)$$

For small values of ϵQ^2 these expressions reduce to

$$\gamma_m^2 - 1 \simeq \tfrac{3}{4}\epsilon(Q^2 + 1) - \frac{1}{Q^2}$$
$$\xi_m \simeq (\sqrt{1 + Q^2})(1 + \tfrac{3}{8}\epsilon Q^2)$$

If we set $\gamma_m^2 - 1 \simeq 2(\gamma - 1) = 2\Delta\omega/\omega_0$, and if $Q \gg 1$, the frequency corresponding to the maximum of the response curve can be written $\omega_0 + \Delta\omega$, where

$$\frac{\Delta\omega}{\omega_0} = \frac{3}{8}\,\epsilon Q^2 = \frac{3}{8}\left(\frac{x_1}{d}\right)^2 \qquad (14.2.7)$$

To investigate the properties of the response curves further, we turn our attention to the points D_1 and D_2 in Fig. 14.11, where the slope of the curves is infinite. The section of the curve between D_1 and D_2, as a detailed analysis will show, represents an unstable motion of the oscillator, and as the frequency is increased continuously from zero, the oscillator amplitude will jump suddenly from the point D_1 to D_1' when the frequency γ_m is reached, as illustrated schematically in the figure. Similarly, if the frequency is decreased continuously, the amplitude of oscillation follows the lower branch of the response curve until the point D_2 is reached, at which the amplitude

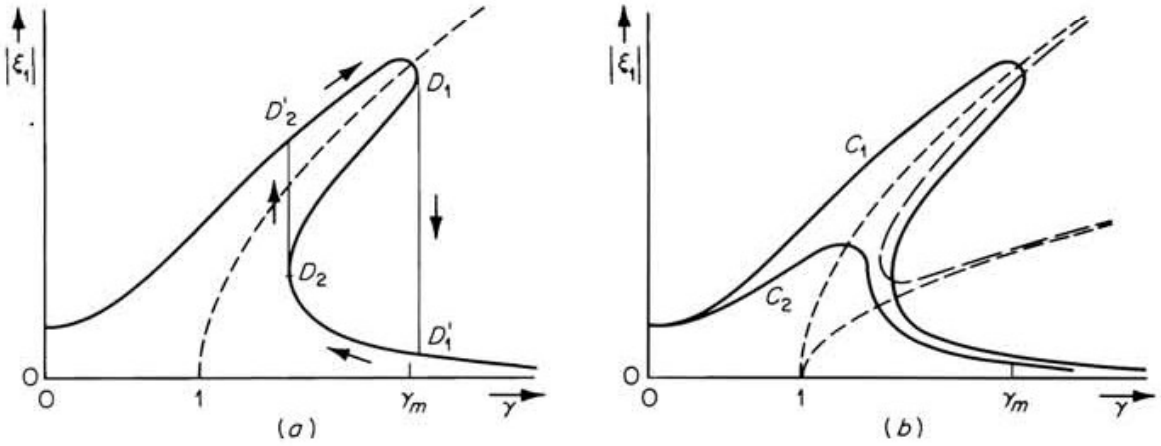


FIGURE 14.11
Illustrating the instabilities of the nonlinear oscillator near resonance. Dashed curve of *b* is locus of the points D_1 and D_2, where jumps occur.

jumps suddenly to the point D_2'. As the nonlinearity in the oscillator is decreased, the region of instability is decreased until, at a certain value of ϵ, the two points D_1 and D_2 coincide.

To determine this critical value of ϵ, we proceed to find the coordinates of the points D_1 and D_2. Differentiating Eq. (14.2.5) with respect to ξ, we obtain, for $d\gamma/d\xi = 0$,

$$(1 - \gamma^2 + \tfrac{3}{4}\epsilon\xi_1^2)(1 - \gamma^2 + \tfrac{9}{4}\epsilon\xi_1^2) + \left(\frac{\gamma}{Q}\right)^2 = 0 \qquad (14.2.8)$$

This relation is represented by the dashed curve in Fig. 14.11*b*. The points D_1 and D_2 are given by the intersections between the curves (14.2.5) and (14.2.8). These intersections merge at the point where the function of Eq. (14.2.8) has a vertical tangent. The response curve which passes through that point will not be multivalued, and the corresponding value of ϵ then gives the limit below which no jump phenomenon occurs. After differentiation of Eq. (14.2.8) with respect to ξ_1 and with $d\gamma/d\xi_1 = 0$, we find that $1 - \gamma^2 = -9\epsilon\xi_1^2/8$, and the merging point is found to be given by $1 - \gamma^2 = -(3/2Q^2) \times [1 \pm \sqrt{1 + (4Q^2/3)}]$ and $\epsilon\xi_1^2 = (4/3Q^2)[1 \pm \sqrt{1 + (4Q^2/3)}]$. (The minus

sign corresponds to negative ϵ.) Finally, if these values are inserted into Eq. (14.2.5), the particular value of ϵ is found below which no jump phenomenon will occur. The complete expression is somewhat lengthy, but if $Q \gg 1$, we find that the amplitude-frequency-response curve is single-valued and jump-free if

$$|\epsilon| < \frac{16}{9}(\sqrt{4/3})\frac{1}{Q^3} \simeq \frac{2}{Q^3}$$

Since $\epsilon = (x_{st}/d)^2$, we can also express this condition as $x_{st} < d(1.4/Q^{\frac{3}{2}})$, which means that the amplitude of the driving force must be smaller than $Kd(1.4Q^{-\frac{3}{2}})$.

Subharmonics

Which steady-state motions are actually produced depends on the initial conditions; relatively slight changes may result in quite different behavior. All we can do here is to mention some of the alternative steady-state solutions which may eventuate. It is not difficult to see that in the particular oscillator with a restoring force $Kx[1 + (x/d)^2]$, the equation of motion can be satisfied with a periodic motion with a fundamental frequency which is one-third of the driving frequency. Assuming the existence of such a periodic motion, we shall now investigate the conditions that must be satisfied. The calculations are analogous to those in the foregoing section, and as we set the amplitude of the subharmonic component equal to zero, we should, of course, obtain the result found earlier.

Starting with the equation of motion without damping,

$$\frac{d^2\xi}{d\theta^2} + \xi(1 + \epsilon\xi^2) = \cos(\gamma\theta)$$

as in Eq. (14.2.2), where the normalized coordinates are defined, we set $\xi = \xi_{\frac{1}{3}} \cos(\gamma\theta/3) + \xi_1 \cos(\gamma\theta) + \cdots$ and use in the first approximation only the first two terms in this expansion. The left-hand side of the equation then will be of the form $A_{\frac{1}{3}} \cos(\gamma\theta/3) + A_1 \cos\gamma\theta + \cdots$, where the coefficients are functions of $\xi_{\frac{1}{3}}$ and ξ_1. As a first approximation, we then get the equations $A_{\frac{1}{3}} = 0$ and $A_1 = 1$, to satisfy the equation of motion. These equations are

$$\begin{aligned} \xi_{\frac{1}{3}}\left[1 - \frac{\gamma^2}{9} + \tfrac{3}{4}\epsilon(\xi_{\frac{1}{3}}^2 + 2\xi_1^2 + \xi_{\frac{1}{3}}\xi_1)\right] &= 0 \\ (1 - \gamma^2)\xi_1 + \frac{\epsilon}{4}(\xi_{\frac{1}{3}}^3 + 6\xi_1^2\xi_1 + 3\xi_1^3) &= 1 \end{aligned} \tag{14.2.9}$$

From these equations we can determine the amplitudes $\xi_{\frac{1}{3}}$ and ξ_1 as functions of the frequency parameter γ. From the first equation we obtain

$$\xi_{\frac{1}{3}} = -\tfrac{1}{2}\xi_1\left[1 \pm \sqrt{\frac{16}{27\epsilon\xi_1^2}(\gamma^2 - 9) - 7}\right]$$

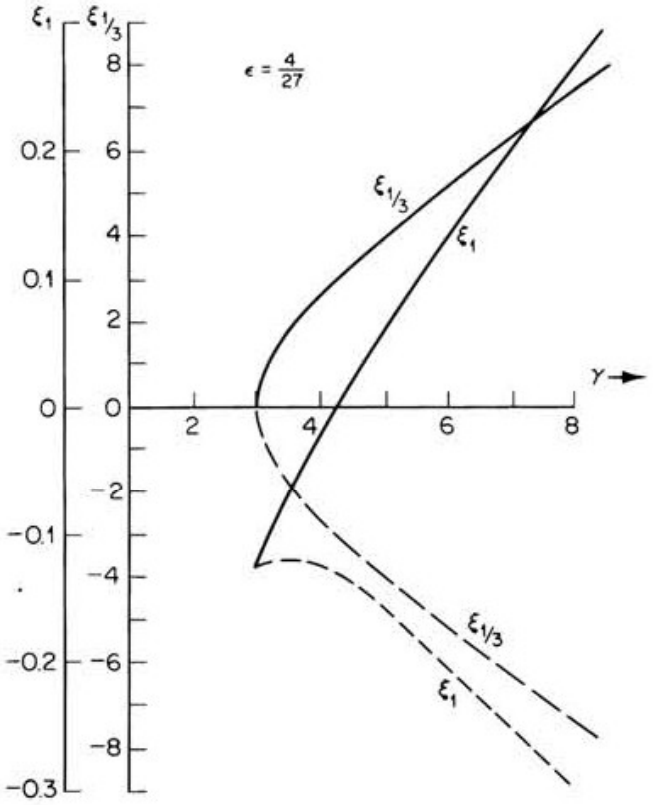


FIGURE 14.12
Amplitudes of the fundamental and subharmonic displacements of forced motion of a nonlinear oscillator.

In order that $\xi_{\frac{1}{3}}$ be real, the condition $\gamma^2 > 9 + (189\epsilon\xi_1^2/16)$ must be satisfied. For small values of ϵ this reduces to $\gamma > 3$; for example, the driving frequency must be larger than three times the (linear) resonance frequency $\omega_0/2\pi$.

If ϵ is small, the amplitude of the subharmonic is large, approximately equal to $\pm\sqrt{(4/27\epsilon)(\gamma^2 - 9)}$. To obtain a corresponding approximation for the amplitude ξ_1 of the fundamental, we must solve the second of Eqs. (14.2.9). The largest term on the left-hand side is $(\epsilon/4)\xi_{\frac{1}{3}}^3$, which is of order $1/\sqrt{\epsilon}$, but to compute ξ_1 we must also include the terms $(1 - \gamma^2)\xi_1$ and $(2\epsilon/3)\xi_{\frac{1}{3}}^2\xi_1$, which are independent of ϵ; terms in positive powers of ϵ can be neglected. To this approximation, then,

$$\xi_{\frac{1}{3}} \simeq \pm\sqrt{\frac{4}{27\epsilon}(\gamma^2 - 9)} \qquad \epsilon \ll 1 \qquad (14.2.10)$$

$$\xi_1 \simeq \frac{\frac{3}{8}\epsilon\xi_{\frac{1}{3}}^3 - \frac{3}{2}}{1 + \frac{11}{9}\gamma^2}$$

Figure 14.12 shows a typical set of curves for the two amplitudes vs. the driving-frequency parameter $\gamma = \omega/\omega_0$. The solid curves are one solution, the dashed curves the other; which pair is used depends on the initial conditions. (Note, however, that a combination of both solutions is *not* a solution; the steady-state motion corresponds either to the solid curves of Fig. 14.12

or to the dashed ones.) For the case of the solid curves, the fundamental amplitude goes to zero when

$$\gamma^2 \simeq 9 + \frac{1}{27}\left(\frac{4}{\epsilon}\right)^{\frac{1}{2}}$$

At this driving frequency, and for this mode of vibration, the driven motion is almost entirely at the subharmonic frequency.

Bias forces

In the study of the motion of a linear oscillator we are accustomed to disregard the influence of a constant force since it merely produces a change of the equilibrium position of the motion. For example, a mass-spring oscillator will have the same natural frequency when the motion is along a vertical line, under the influence of gravity, as when the motion is along a horizontal line. For a nonlinear oscillator, however, this is no longer the case; a constant force, which may be called a "bias" force, does influence the response of the oscillator to a periodic driving force.

As an illustration, let us again study the oscillator with a restoring force of the form $Kx[1 + (x/d)^2]$, where K is the spring constant for small displacements, and d is the characteristic displacement at which the linear and nonlinear contributions to the restoring force are the same. Let this oscillator be acted upon by a constant force F plus a periodic force f. If $f \ll F$, the equilibrium position X of the oscillator is determined solely by the time-independent force, according to the equation

$$\frac{K}{d^2}X^3 + KX = F$$

which has the solution

$$X = d[(a + b)^{\frac{1}{3}} + (a - b)^{\frac{1}{3}}]$$

where $a = F/Kd$, and $b = \sqrt{\frac{1}{27} + (F/2Kd)^2}$. A small periodic force produces an oscillation about this equilibrium, and as long as the corresponding displacement is sufficiently small, the response of the oscillator is the same as that of a linear oscillator, with a spring constant given by the slope of the force vs. displacement curve at the point X. The slope of the curve at X is

$$\frac{dF}{dX} = K\left[1 + 3\left(\frac{X}{d}\right)^2\right] = K_1$$

and the effective spring constant of the oscillator at that point is then $K_1 = K[1 + 3(X/d)^2]$. The frequency of small oscillations about the equilibrium point is then $\omega_1 = \omega_0\sqrt{1 + 3(X/d)^2}$, where ω_0 is the natural frequency of small oscillations without a bias force.

A bias force of some acoustical importance is produced by a moving airstream. This force depends on the velocity of the stream relative to the

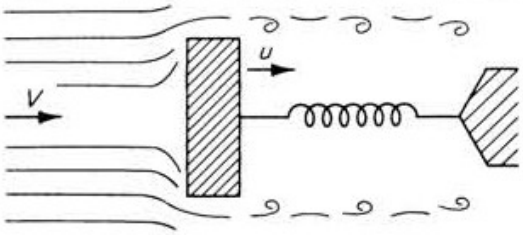


FIGURE 14.13
Airstream impinging on an oscillator produces additional damping of the oscillator.

oscillator. It produces not merely a change in the effective stiffness of the oscillator (assuming, of course, that the restoring force is nonlinear), but also an additional damping in the system. To demonstrate this effect, consider as a simple example a mass-spring oscillator in an airstream moving along the line of oscillation of the oscillator, as shown in Fig. 14.13. If the velocity of the airstream is V, the force produced on the oscillator is of the form

$$F = AC_d(\rho V^2) \tag{14.2.11}$$

where A is the area of the body (or rather, the projection of the area on a plane perpendicular to the airstream), and C_d is a constant, the so-called drag coefficient. This coefficient, which has to be determined experimentally, depends on the geometry of the body and, in general, also upon V. In some cases, however, C_d is essentially independent of velocity over a wide range of velocities; we shall assume this to be the case in these calculations. (Clearly, at very low velocities, C_d varies inversely as the velocity since, in the limit $V \to 0$, the force must become the viscous force, which is proportional to the velocity. We consider velocities large enough so that viscous effects are not important.)

If the oscillator has a velocity u, the relative velocity between the airstream and the oscillator is $V - u$, and the force in Eq. (14.2.11) becomes

$$F + \Delta F = AC_d\rho(V^2 + u^2) - 2\rho AC_dVu$$

If $u \ll V$, we see that, in addition to the time-independent force F, we now have also the time-dependent force $\Delta F = -(2AC_d\rho V)u$, which is proportional to the velocity u and is equivalent to a frictional force. In the more general case, when C_d is velocity-dependent, the equivalent frictional force can be expressed as $(dF/dV)_X u$. This aerodynamically produced frictional force in the oscillator has a considerable influence on various acoustical resonators operating in relative motion with the surrounding medium. The free oscillations of cavity resonators, strings, and membranes, etc., have been found to decay more rapidly when in motion than when at rest. The reason for this behavior is much the same as in the present simple example.

14.3 LARGE-AMPLITUDE STRING WAVES

In keeping with the treatment in the rest of the book, in this chapter we have first dealt with the nonlinear behavior of a point mass; next we should

take up nonlinear waves on a string, and then come to the nonlinear acoustics of a fluid. This section is concerned with the wave motion of a string when its amplitude of transverse displacement is large enough so that the simple linear wave equation does not adequately describe its motion.

Of course we shall not relax all the idealizations imposed in Chap. 4. We still assume that the string is uniform in mass density and is perfectly flexible, though corrections to the formulas to allow for stiffness can be applied analogous to those of Sec. 5.1. We also shall here continue to assume that the end supports, holding the string and applying the tension, are perfectly rigid, though the extension of the theory to allow for support motion is not difficult—indeed, is an obvious extension of the discussions on pages 112 and 172.

The nonlinearity discussed in this section comes about when the transverse displacement of the string is large enough to modify the tension by a material amount; the string is stretched more in some parts than in others. We thus must consider the longitudinal motion of the string as well as its transverse motion—in fact, the two are coupled—and we must introduce the relationship between longitudinal strain and stress resulting from the differences in stretch, the string's equation of state, so to speak. In contrast to the last two sections of this chapter, where the effects of nonlinearities of the equation of state of the fluid are discussed, in this section we shall assume that the linear relationship of Hooke's law holds for the equation of state of the stretched string.

General wave motion

In equilibrium we suppose the string to be stretched along the x axis and its longitudinal strain $\Delta l/l_0$ to be produced by being stretched between rigid supports, so that the equilibrium stress is tension $T_0 = QA(\Delta l/l_0)$, Q being the Young's modulus [Eq. (5.1.1)] of the string and A its cross-sectional area. In this stretched condition, at rest along the x axis, the mass of the string per unit length is ϵ_0. When in motion, the element of string which was, in equilibrium, at point $(x,0,0)$ will be at point $(x + \xi, \eta, \zeta)$, its displacement from equilibrium being given by the vector $\mathbf{r}$, so that the vector from the origin to its instantaneous position is

$$\mathbf{R}(x,t) = (x + \xi)\mathbf{e}_x + \eta\mathbf{e}_y + \zeta\mathbf{e}_z = x\mathbf{e}_x + \mathbf{r}(x,t) \qquad (14.3.1)$$

where $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors along the coordinate axes. Thus we are using the Lagrange description, shown in Fig. 6.2, to specify the motion.

The element of length dx in equilibrium will now have the length $|\partial\mathbf{R}/\partial x|\,dx$, where

$$\left|\frac{\partial\mathbf{R}}{\partial x}\right| = \sqrt{\left(1 + \frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial x}\right)^2} \qquad (14.3.2)$$

$$\epsilon_0 = \epsilon\left|\frac{\partial\mathbf{R}}{\partial x}\right|$$

Therefore the instantaneous mass density of the string, originally at $(x,0,0)$, is $\epsilon_0/|\partial\mathbf{R}/\partial x|$, and its tension $T(x,t)$ will be modified by the change in length. If Hooke's law holds, the change will be linear, so that

$$T(x,t) = T_0 + QA\left(\left|\frac{\partial \mathbf{R}}{\partial x}\right| - 1\right) \tag{14.3.3}$$

The force on the displaced element (Fig. 4.2) will arise, not only because the pull across the two ends may be in slightly different directions, but also because the magnitude of T differs slightly between the two ends. If $\mathbf{s}(x,t)$ is the unit vector pointing along the tangent to the string at the point labeled x,

$$\mathbf{s} = \frac{\partial\mathbf{R}/\partial x}{|\partial\mathbf{R}/\partial x|} = \frac{[1 + (\partial\xi/\partial x)]\mathbf{e}_x + (\partial\eta/\partial x)\mathbf{e}_y + (\partial\zeta/\partial x)\mathbf{e}_z}{\sqrt{[1 + (\partial\xi/\partial x)]^2 + (\partial\eta/\partial x)^2 + (\partial\zeta/\partial x)^2}}$$

then the equation of motion of the element of string which was of length dx and at point $(x,0,0)$ in equilibrium (and thus always has mass $\epsilon_0\,dx$) is

$$\epsilon_0 \frac{\partial^2\mathbf{R}}{\partial t^2} = \frac{\partial(T\mathbf{s})}{\partial x} = \mathbf{s}\frac{\partial T}{\partial x} + T\frac{\partial \mathbf{s}}{\partial x} \tag{14.3.4}$$

In Chap. 4 we neglected $\partial T/\partial x$ and assumed that $\mathbf{s} = \mathbf{e}_x + (\partial\eta/\partial x)\mathbf{e}_y$. The exact equation includes longitudinal, as well as transverse, forces.

Inserting Eq. (14.3.3) into this equation of motion and using the formula

$$\frac{\partial \mathbf{s}}{\partial x}\left|\frac{\partial\mathbf{R}}{\partial x}\right| = \frac{\partial^2\mathbf{R}}{dx^2} - \mathbf{s}\frac{\partial\,|\partial\mathbf{R}/\partial x|}{\partial x}$$

we obtain

$$\frac{\partial^2\mathbf{R}}{\partial t^2} = c_l^2\frac{\partial^2\mathbf{R}}{\partial x^2} - (c_l^2 - c_t^2)\frac{\partial\mathbf{s}}{\partial x}$$
$$c_l^2 = \frac{AQ}{\epsilon_0} \qquad c_t^2 = \frac{T_0}{\epsilon_0} \tag{14.3.5}$$

When $\mathbf{s} \simeq \mathbf{e}_x + (\partial\eta/\partial x)\mathbf{e}_y \simeq \partial\mathbf{R}/\partial x$, this reduces to the linear form of Eq. (4.1.2). Constants c_l and c_t are the familiar longitudinal and transverse wave speeds for small-amplitude waves. It is interesting to note that the nonlinearity is all contained in the second term on the right-hand side of the equation of motion, and that this term is proportional to the difference in the squares of the longitudinal and transverse wave speeds. In other words, a string for which these speeds are equal would be linear; it would transmit large-amplitude waves in just the same way as small-amplitude waves. However, to obtain such a string, with $c_l = c_t$, the tension T_0 must be large enough to extend the string to twice its relaxed length. Except possibly for rubber strings or coil springs, this condition can hardly be satisfied for most strings used in practice; generally, c_l is considerably larger than c_t. The general wave equation (14.3.5) is written in vector form, but if we use the explicit expression for $\mathbf{R}$ and $\mathbf{s}$ as they were defined, we obtain three coupled

differential equations in terms of the components ξ, η, ζ. Before we discuss these equations, it is interesting to consider, first, some general properties of motion.

Constant tension

If the boundary conditions of the string are properly chosen, it seems possible that a wave can propagate on the string without altering the initial tension. In order to investigate whether this is consistent with the equation of motion, we first observe from Eq. (14.3.4) that on the assumption of Hooke's law, a constant tension implies a constant value of $|\partial\mathbf{R}/\partial x|$ and also a constant value of ϵ. In that case we get $\partial\mathbf{s}/\partial x = \dfrac{\partial^2\mathbf{R}}{\partial x^2}\Big/\left|\dfrac{\partial\mathbf{R}}{\partial x}\right|$, and the wave equation (14.3.3) then can be expressed in the form

$$\epsilon\frac{\partial^2\mathbf{R}}{\partial t^2} = T\frac{\partial^2\mathbf{R}}{\partial x^2} \qquad \epsilon, T \text{ const} \tag{14.3.6}$$

where we have introduced ϵ instead of ϵ_0 according to Eqs. (14.3.2). This equation is just like the ordinary equation for small-amplitude motion; the only difference is that we use the actual values of ϵ and T of the deformed string rather than the unperturbed values ϵ_0 and T_0. Equation (14.3.6) is valid for arbitrary amplitudes, but limited to constant values of T (and ϵ), and it follows that under these conditions all portions of a large-amplitude wave will travel with the same wave speed $\sqrt{T/\epsilon}$; no distortion of the wave-form will take place. It is easy to see, however, that this conclusion is consistent with the requirement of a constant value of the quantity $|\partial\mathbf{R}/\partial x|^2$ (which is equivalent to constant tension) only for a single traveling wave along the string. For such a wave we can maintain a constant value of $|\partial\mathbf{R}/\partial x|$, once established, because, as we have seen, the wave will travel without change in its shape and therefore without change of $\partial\mathbf{r}/\partial x$; the mere translation of the wave pattern that occurs in the traveling wave cannot alter the constant value of $|\partial\mathbf{R}/\partial x|$. For waves traveling in the opposite direction, on the other hand, a constant value cannot be maintained; the wave pattern, and consequently the derivatives $\partial\xi/\partial x$, etc., will vary as different parts of the two waves interact.

A wave of this sort can be generated by the rather impractical method illustrated in Fig. 14.14. The string is displaced from equilibrium along the portion from a to b by being fitted inside a frictionless tube, as shown, originally at rest. This transverse displacement will of course stretch the string so that the tension throughout its length will equalize to a new value $T > T_0$, and its density will also take the uniform value $\epsilon < \epsilon_0$. At point P of the tube, where the radius of curvature is ρ, the element ds of string will have acting on it a net force $T\,ds/\rho$ toward the center of curvature.

Now set the string into motion, maintaining tension, with a velocity v, when the element ds will also have acting a centrifugal force $\epsilon v^2\,ds/\rho$ away

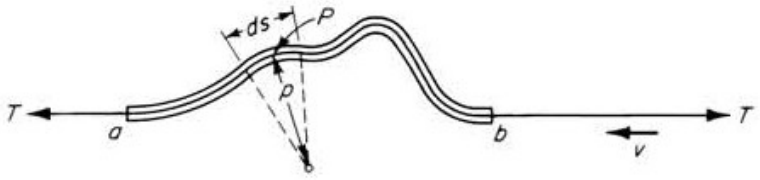


FIGURE 14.14
Large-amplitude constant-tension traveling wave in string.

from the center of curvature. If the speed happens to equal $c = \sqrt{T/\epsilon}$, these two forces will exactly balance, *no matter what value* ρ *has.* In other words, at this speed the tube could be removed, and the stationary wave would remain; at every point the forces balance, and there would be no change in waveform, and thus no change in string length or tension or density. Of course, the situation of the string moving with velocity c and the waveform standing still is no different dynamically from the case of the string standing still (the part outside of *ab*, that is) and the wave moving with velocity c. Thus the wave, started in this artificial manner, will travel with constant shape and velocity and will maintain constant tension throughout the string. The wave must go in just one direction, however; one could not start two interpenetrating waves, going in opposite directions, by this method. Also, if two such waves, originally noninterpenetrating, going in opposite directions, should meet, the conditions would no longer hold, the wave would change form, and the tension would no longer be uniform.

Third-order equations

Apart from the situation just outlined, of constant tension, or when $c_l = c_t$, exact solutions of Eq. (14.3.5) are not known. With waves of modest amplitudes, however, an approximate form of the equation can be obtained from a power expansion of the nonlinear term $\partial \mathbf{s}/\partial x$ in Eq. (14.3.5), in which only the first few terms are retained. The equation thus obtained is a convenient starting point for perturbation calculations of nonlinear behavior of these waves.

In obtaining the power series expansion, we start from the explicit form of $\mathbf{s} = \dfrac{\partial \mathbf{R}/\partial x}{|\partial \mathbf{R}/\partial x|}$ in terms of the slopes $\partial\xi/\partial x$, etc., as given in Eq. (14.3.5). Thus, after having expanded the square root in this expression and performed the differentiation $\partial \mathbf{s}/\partial x$, we obtain, after some algebra, the following three equations for the components ξ, η, and ζ:

$$\frac{\partial^2\xi}{\partial t^2} - c_l^2 \frac{\partial^2\xi}{\partial x^2} = \tfrac{1}{2}(c_l^2 - c_t^2) \frac{\partial}{\partial x}[(\eta_x^2 + \zeta_x^2)(1 - 2\xi_x)] \tag{14.3.7}$$

$$\frac{\partial^2\eta}{\partial t^2} - c_l^2 \frac{\partial^2\eta}{\partial x^2} = \tfrac{1}{2}(c_l^2 - c_t^2) \frac{\partial}{\partial x}[\eta_x(\eta_x^2 + \zeta_x^2) + 2\eta_x\xi_x(1 - \xi_x)] \tag{14.3.8}$$

$$\frac{\partial^2\zeta}{\partial t^2} - c_t^2 \frac{\partial^2\zeta}{\partial x^2} = \tfrac{1}{2}(c_l^2 - c_t^2) \frac{\partial}{\partial x}[\zeta_x(\eta_x^2 + \zeta_x^2) + 2\zeta_x\xi_x(1 - \xi_x)] \tag{14.3.9}$$

where we have here written $\partial\eta/\partial x$ as η_x, etc. In these equations the nonlinear terms have been written as "source terms" in the wave equations for the three modes of motion, the longitudinal mode and the two transverse modes. These modes are all coupled with each other as a result of the nonlinearity.

Even these equations cannot be solved exactly, and we have to resort to approximate methods of analysis much in the same way as in the case of the nonlinear one-dimensional oscillator in Sec. 14.1. In many cases we can start from the first-order motions being solutions of the linear string equations. Using these first-order solutions for the evaluation of the nonlinear source terms in Eqs. (14.3.7) to (14.3.9), nonlinear corrections to the linear solutions can be obtained, including the longitudinal-transverse and transverse-transverse mode-coupling effects.

The generation of a longitudinal wave mode by a transverse wave is to be expected on account of the stretching of the string which is associated with the transverse motion. This effect is expressed by Eq. (14.3.7), which shows that longitudinal motion is induced by transverse motion through a "source strength" of second order in the transverse motion, proportional to $\partial(\eta_x^2 + \zeta_x^2)/\partial x$. Less obvious is the coupling between the two transverse modes η and ζ. This coupling consists of two contributions; the first is a "direct" coupling, which for the η motion corresponds to the source term $\partial(\zeta_x^3 + \zeta_x\eta_x^2)/\partial x$, and the second is a coupling carried via longitudinal motion and expressed by the source term proportional to $\partial(\zeta_x\xi_x)/\partial x$. This term is also of third order in the transverse field variables since, according to Eq. (14.3.7), ξ_x is of second order in η and ζ. Consequently, in any study of interactions in lateral motions of a string, the *longitudinal motion generally cannot be neglected.*

Transverse-longitudinal mode coupling

As an example, we shall determine the longitudinal motion induced by the (first-order) fundamental mode of a transverse motion of a string clamped at both ends. Thus, let the first-order transverse motion be $\eta_1 = \eta_0 \sin(\pi x/L)\sin(\omega_t t)$, where $\omega_t/2\pi = c_t/2L$ is the fundamental frequency. This transverse motion generates a longitudinal motion, which we find from Eq. (14.3.7). By using η_1 in the source term of this equation, we get for the second-order longitudinal motion ξ_2,

$$\frac{\partial^2\xi_2}{\partial t^2} - c_l^2\frac{\partial^2\xi_2}{\partial x^2} = \tfrac{1}{2}(c_l^2 - c_t^2)\frac{\partial}{\partial x}\eta_x^2$$

$$= (c_l^2 - c_t^2)(k_t\eta_0^2)\frac{\partial}{\partial x}[\cos^2(k_t x)\sin^2(\omega_t t)]$$

where $k_t = \omega_t/c_t$. The steady-state solution to this equation is

$$\xi_2 = \frac{\pi}{16}\frac{\eta_0}{L}\eta_0\left[\sin(2k_t x)\cos(2\omega_t t) - \left(1 - \frac{c_t^2}{c_l^2}\right)\sin(2k_t x)\right] \tag{14.3.10}$$

The time-dependent part of ξ_2 varies with twice the transverse frequency; the string stretches at every transverse excursion of the string, regardless of its direction. The second term represents the time-average displacement $\langle \xi_2 \rangle$. Its space average must be zero since, for fixed supports, the projection of the string on the x axis is not altered by the motion. The axial displacement $\langle \xi_2 \rangle$ is negative on the left half of the string, where each point of the string moves slightly to the left as the string makes its transverse displacement. The center point of the string moves vertically, so that ξ_2 is zero there, and the points on the right half of the string move slightly to the right, making $\langle \xi_2 \rangle$ positive. We leave it for one of the problems to calculate the corresponding changes in the mass per unit length and the tension.

It is interesting to find that the amplitude of this solution is independent of the relation between the "driving frequency" and the resonance frequency ω_l of the longitudinal motion; one may expect resonance excitation to occur if $\omega_t = \omega_l$. However, the corresponding resonant denominator is canceled by the fact that the nonlinearity of the string, as we have seen, is proportional to $c_l^2 - c_t^2$.

Lateral-lateral mode coupling

In much the same manner we can proceed to calculate the coupling between transverse modes, for example, the nonlinear distortion of a plane transverse motion, starting, say, from the fundamental η mode. Or we can determine the coupling between an η mode and a ζ mode. Although we shall not carry out these calculations here, we merely point out that even when we deal with lateral-lateral coupling problems, the associated excitation of longitudinal motion cannot be neglected as a coupling mechanism. Suppose, for example, we are interested in the excitation of higher order ζ motion by (first-order) η and ζ motion. The source term to be considered is then proportional to the x derivative of $\zeta_x[(\eta_x^2 + \zeta_x^2) + 2\xi_x]$, neglecting the term involving ξ_x^2, which turns out to be of higher order than the other terms. The longitudinal motion ξ_x excited by the first-order η and ζ motions can be determined from Eq. (14.3.7), which, after differentiation with respect to x, can be written

$$\frac{\partial^2 \xi_x}{\partial t^2} - c_l^2 \frac{\partial^2 \xi_x}{\partial x^2} = \frac{1}{2}(c_l^2 - c_t^2) \frac{\partial^2}{\partial x^2}(\eta_x^2 + \zeta_x^2) \qquad (14.3.11)$$

where we have neglected the terms containing ξ_x on the right-hand side, being of higher order. Suppose now that $\eta_x^2 + \zeta_x^2$ contains terms of the form $\sin(mk_t x) \sin(m\omega_t t)$ or $\sin(mk_t x) \cos(m\omega_t t)$ or some other combination of these trigonometric functions. We denote this part of $\eta_x^2 + \zeta_x^2$ by $(\eta_x^2 + \zeta_x^2)_{m,m}$. The longitudinal motion induced by these terms, as expressed by the solution ξ_x in Eq. (14.3.11), will have the same x, t dependence. In fact, the solution is

$$(\xi_x)_{m,m} = -\tfrac{1}{2}(\eta_x^2 + \zeta_x^2)_{m,m}$$