

If we now introduce this expression for $(\xi_x)_{m,m}$ in the source term

$$\zeta_x[(\eta_x^2 + \zeta_x^2) + 2\xi_x]$$

we find the rather interesting result that the (m,m) component of $(\eta_x^2 + \zeta_x^2)$ is exactly canceled by the (m,m) component of $2\xi_x$. In other words, the fact that longitudinal motion necessarily is excited by transverse motion plays an important role also in regard to the lateral-lateral coupling problems.

14.4 THE SECOND-ORDER EQUATIONS OF SOUND

In previous chapters we have repeatedly pointed out that the nonlinear terms in the equations of motion have been neglected but that the resulting linearized equations adequately describe the acoustic wave motion one usually encounters. Although, in most of this linear analysis, we omitted viscosity and heat conduction, they were included in the study of wave attenuation in Chap. 6. As we pointed out there, the effects of viscosity and heat conduction can be regarded as small perturbations in the equations of motion. We now ask, under what conditions is it justified to omit nonlinear terms compared with the viscous and heat-conduction terms? The role of nonlinearity is not as simple to foresee as is the effect of viscosity; we shall find that its effect makes itself felt even at low amplitudes, in connection with quantities such as mass flow.

Relative magnitudes

To compare the relative importance of the viscous and nonlinear terms, it is sufficient to consider the one-dimensional form of the momentum equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{\mu'}{\rho} \frac{\partial^2 u}{\partial x^2} \quad (14.4.1)$$

where $\mu' = \frac{4}{3}\mu + \eta$ [Eq. (6.4.15)]. Since we are interested in the relative order of magnitude of the terms in this equation in connection with the propagation of sound waves, we introduce for the characteristic time and length the period T and the wavelength λ of a harmonic sound wave, so that $\partial/\partial t \sim 1/T$ and $\partial/\partial x \sim 1/\lambda$. Then, to obtain the relative orders of magnitude of the different terms, we use the properties of plane waves discussed in Chap. 6, where we have seen that if the perturbation of the pressure in the wave is p , the corresponding perturbations in density, fluid velocity, and temperature are $\delta = p/c^2 = (u/c)\rho$, $u = p/\rho c$, and $\theta = (\gamma - 1)(u/c)T = (\gamma - 1)pT/\gamma P$.

The order of magnitude of the first and third terms in Eq. (14.4.1) is then u/T , and $p/\rho\lambda = u/T$; that is, they are of the same order of magnitude. The second term, on the other hand, which is nonlinear, is seen to be of the order of $\frac{u}{c} \frac{u}{T}$. Thus the relative orders of magnitude of the three terms on

the left-hand side of Eq. (14.4.1) are 1, u/c , and 1. Similarly, the order of magnitude of the viscous term, using the first term as reference, is

$$\frac{(\mu/\rho)(\partial^2 u/\partial x_2^2)}{\partial u/\partial t} \simeq \frac{(\mu/\rho)(u/\lambda^2)}{u/T} = \frac{\nu T}{\lambda^2} \simeq \frac{\nu\omega}{c^2}$$

where ν is the kinematic viscosity. It follows, then, that the viscous and nonlinear terms are of the same order of magnitude when at a wave amplitude given by $u = \nu\omega/c$. If we introduce the fluid particle displacement $\xi = u/\omega$ and the characteristic length $\nu/c = l$, which is of the order of the mean-free-path in the gas (page 272), we find that the nonlinear and viscous terms are of the same order of magnitude when the particle displacement in the wave is of the same order of magnitude as the mean-free-path. Thus the omission of the nonlinear terms is justified in comparison with viscous terms only when $\xi \ll l$. In air at atmospheric pressure the mean-free-path is of the order of 10^{-5} cm, and if we consider a sound wave with a frequency of 1,000 cps, the condition $\xi = l$ corresponds to a particle velocity of about 0.1 cm per sec and a sound pressure of about 4 dynes per cm^2 (85 db). In other words, nonlinear effects as compared with viscous ones become significant at a rather modest sound-pressure level. At higher levels, in an analysis of a problem such as the attenuation of a harmonic wave, an omission of nonlinear effects in comparison with viscosity may lead to noticeable errors, particularly at low frequencies.

The second-order equations

After these introductory remarks concerning the relative importance of nonlinear and viscous terms, we shall consider more quantitatively the influence of these effects on sound propagation, and we shall start by deriving the acoustic equations to second order in the field variables. The meaning of this needs some explanation. As we know, the basic equations of fluid motion are nonlinear; they contain terms that are products of the field variables and their derivatives. Complete solutions of these equations are rare. However, if we are interested only in small perturbations of the field variables, linear equations in the perturbations can be developed, as we have seen, and these equations form the basis of the linear theory of sound. Then, if the unperturbed value of a field variable ψ is ψ_0 , the linearized equations contain terms only of first order in the perturbation $\psi_1 = \psi - \psi_0$; terms of the order of ψ_1^2 and higher-order terms are omitted. To develop the theory of sound one step further, second-order terms, of the order of ψ_1^2 , will be included, in addition to the linear terms. To do this in a systematic manner, each field variable ψ is written

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots \quad (14.4.2)$$

i.e., as a sum of the unperturbed value ψ_0 , a first-order term ψ_1 (which is

defined by the fact that it is a solution of the linearized equations), and a second-order term ψ_2 , which is of the order of ψ_1^2 .

If we expand each field variable in this way and insert the expansions into the general equations of fluid motion (Chap. 6) and require that the equations are to be satisfied separately to each order in ψ , we can extract sets of equations containing only zeroth-, first-, and second-order terms. It should be pointed out that a procedure of this kind is meaningful only if the expansion (14.4.2) is a succession of decreasing terms, so that $\psi_2 \ll \psi_1 \ll \psi_0$. Actually, it can be regarded as a Taylor expansion of ψ in which the expansion parameter is a measure of the strength of the wave. Thus, if the exact ψ is assumed to be of the form $\psi(x, t, s)$, where s is an amplitude parameter, we have $\psi_0 = \psi(x, t, 0)$, $\psi_1 = s \left(\frac{\partial \psi}{\partial s} \right)_0$, $\psi_2 = \frac{s^2}{2} \left(\frac{\partial^2 \psi}{\partial s^2} \right)_0$, etc.

We start from the general equations in the form given in Chap. 6.

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\ \frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot (\mathfrak{T} + P\mathcal{J} + \mathfrak{D}) &= 0 \end{aligned} \quad (14.4.3)$$

where $\mathfrak{T}$ = momentum tensor of Eq. (6.2.10)

$P\mathcal{J}$ = pressure tensor

$\mathfrak{D}$ = viscous-stress tensor of Eq. (6.4.8)

$$\mathcal{J}_{ij} = \delta_{ij}$$

The equation of state is assumed to be expressible in the form

$$\rho = \rho(P) \quad (14.4.4)$$

so that the density is a function of pressure only.

In the present discussion we shall assume that the coordinate system is chosen so that the unperturbed fluid is at rest; that is, $\mathbf{V}_0 = 0$. Then, if we insert $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, $\rho = \rho_0 + \rho_1 + \rho_2$, etc., into Eqs. (14.4.3) and (14.4.4) and extract the corresponding first-order equations, we get

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{V}_1 = 0 \quad (14.4.5)$$

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} + \nabla \cdot (P_1 \mathcal{J} + \mathfrak{D}_1) = 0 \quad (14.4.6)$$

$$\rho_1 = \left(\frac{\partial \rho}{\partial P} \right)_0 P_1 = \frac{1}{c^2} P_1 \quad (14.4.7)$$

If we use the expression for the viscous-stress tensor given in Chap. 6 and eliminate $\mathbf{V}_1$ from these equations by taking the time derivative of the first

equation and the divergence of the second, the following wave equation for the first-order pressure results.

$$\nabla^2 \left(P_1 + \frac{\nu'}{c^2} \frac{\partial P}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 P_1}{\partial t^2} = 0 \quad (14.4.8)$$

where $\rho\nu' = (4\mu/3) + \eta$, μ and η being the shear and bulk viscosities [Eq. (6.4.10)]. Similarly, for the second-order acoustic quantities we obtain the following equations:

$$\frac{\partial \rho_2}{\partial t} + \nabla \cdot \rho_0 \mathbf{V}_2 = -\nabla \cdot \rho_1 \mathbf{V}_1 \quad (14.4.9)$$

$$\rho_0 \frac{\partial \mathbf{V}_2}{\partial t} + \nabla P_2 + \nabla \cdot \mathfrak{D}_2 = -\nabla \cdot (\mathfrak{J}_2) - \frac{\partial(\rho_1 \mathbf{V}_1)}{\partial t} \quad (14.4.10)$$

$$\rho_2 = \left(\frac{\partial P}{\partial \rho} \right)_0 P_2 + \frac{1}{2} \left(\frac{\partial^2 \rho}{\partial P^2} \right)_0 P_1^2 = \frac{1}{c_0^2} P_2 + \frac{\Gamma}{\rho_0 c_0^4} P_1^2 \quad (14.4.11)$$

where $\Gamma = \frac{1}{2} \rho_0 c^4 (\partial^2 \rho / \partial P^2)$, which, for an ideal gas, is $\Gamma = -(\gamma - 1)/2\gamma$. We have used here the notation in Eq. (6.4.10), so that in component form,

$$\nabla \cdot \mathfrak{D} = \sum_j \frac{\partial D_{ij}}{\partial x_j} \quad \text{and} \quad \nabla \cdot \mathfrak{D} \cdot \nabla = \sum_i \sum_j \frac{\partial^2 D_{ij}}{\partial x_i \partial x_j}$$

with similar expressions for the tensor $\mathfrak{J}$.

The corresponding wave equation for P_2 is

$$\begin{aligned} \nabla^2 \left(P_2 + \frac{\nu'}{c^2} \frac{\partial P_2}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 P_2}{\partial t^2} \\ = -\nabla \cdot \mathfrak{J}_2 \cdot \nabla - \frac{\Gamma}{\rho_0 c_0^4} \frac{\partial}{\partial t} \left(\frac{\partial P_1^2}{\partial t} + \nu' \nabla^2 P_1^2 \right) \end{aligned} \quad (14.4.12)$$

Since in the present case we assume the unperturbed fluid to be homogeneous and at rest, the zeroth-order equations are identically satisfied. We note that the equations for the first- and second-order contributions to the field variables are of the same form except for the occurrence of *source terms* in the second-order equations. These source terms are products of first-order terms, and if the first-order solutions can be regarded as known, these equations are sufficient for the calculation of the second-order contribution to the sound field. For example, the solution to the wave equation (14.4.12) for P_2 can be obtained in a standard manner, using an appropriate Green's function.

Mass and energy flow

As a first application of Eqs. (14.4.9) to (14.4.12) we shall discuss the mass-flow paradox that was mentioned briefly in Chap. 6. On the basis of linear acoustics, one can be led to the erroneous conclusion that the time average of the mass flow in the sound field is $\langle \rho_1 \mathbf{V}_1 \rangle$, which for a plane

harmonic wave is different from zero. The error lies in the fact that $\langle \rho_1 \mathbf{V}_1 \rangle$ is a quantity of second order and represents only one part of the correct expression for the mass flow, including terms up to second order, $\langle \rho \mathbf{V} \rangle = \langle \rho_0 \mathbf{V}_1 \rangle + \langle \rho_1 \mathbf{V}_1 \rangle + \langle \rho_0 \mathbf{V}_2 \rangle$. For a harmonic time dependence the first term is zero. The second term, on the other hand, is different from zero since, in a traveling plane wave, ρ_1 and $\mathbf{V}_1$ are in phase, as we have already indicated. The third term, $\langle \rho_0 \mathbf{V}_2 \rangle$, follows directly from the time average of Eq. (14.4.9), which becomes $\nabla \cdot \langle \rho_0 \mathbf{V}_2 \rangle = -\nabla \cdot \langle \rho_1 \mathbf{V}_1 \rangle$, since, for a finite value of a field variable ψ , we must have $\langle \partial \psi / \partial t \rangle = 0$. Then, if the flow is irrotational, it follows that $\langle \rho_0 \mathbf{V}_2 \rangle = -\langle \rho_1 \mathbf{V}_1 \rangle$, and therefore the mass flow $\langle \rho_0 \mathbf{V}_1 \rangle + \langle \rho_1 \mathbf{V}_1 \rangle + \langle \rho_0 \mathbf{V}_2 \rangle$, in a sound wave, is zero to this order.

Although the expression for the mass flow obtained on the basis of linear theory led to an erroneous result, we have nevertheless accepted without concern the expression $\langle P_1 \mathbf{V}_1 \rangle$ for the acoustic intensity, and this has been used throughout the book. This quantity, however, like the mass flow $\langle \rho_1 \mathbf{V}_1 \rangle$, is of second order, and there is no guarantee that it is the true value of the intensity, correct to second order. We are now in a position to reexamine this question, and to do this we start from the general expression for the energy-flow vector in a fluid, in the absence of heat conduction and viscosity;

$$\mathbf{I} = \left\langle \left(\rho \frac{V^2}{2} + \rho E \right) \mathbf{V} + P \mathbf{V} \right\rangle \quad (14.4.13)$$

The first term represents the convection of the energy density in the fluid, and the second term represents the work done by pressure. E is the internal energy of the fluid per unit mass. Since we are restricting ourselves to a loss-free fluid, the first law of thermodynamics gives a simple relation between the variation in internal energy and the density $dE = P d\rho / \rho^2$. Since the unperturbed fluid is at rest, the velocity entering into Eq. (14.4.13) is due solely to the sound wave; that is, $\mathbf{V}_0 = 0$. Then, to collect all the terms up to second order that contribute to the acoustic intensity, we introduce the expansions $E = E_0 + E_1 + E_2 + \cdots$, $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots$, $P = P_0 + P_1 + P_2$, $\rho = \rho_0 + \rho_1 + \rho_2$, into (14.4.13), and collect the terms of zero, first, and second order. Since $\mathbf{V}_0 = 0$, we have no zeroth-order contributions. Since we assume harmonic time dependence of the first-order quantities, their time average is zero. Those remaining are of second order, and we have

$$\mathbf{I} = E_0 (\langle \rho_1 \mathbf{V}_1 \rangle + \rho_0 \langle \mathbf{V}_2 \rangle) + \langle E_1 \mathbf{V}_1 \rangle \rho_0 + \langle P_1 \mathbf{V}_1 \rangle + P_0 \langle \mathbf{V}_2 \rangle \quad (14.4.14)$$

Since $\langle \rho_1 \mathbf{V}_1 \rangle = -\rho_0 \langle \mathbf{V}_2 \rangle$, as we have seen from Eq. (14.4.9), the term containing E_0 is zero. Furthermore, from $dE = P d\rho / \rho^2$ follows $E_1 = (P_0 / \rho_0^2) \rho_1$, and therefore $\langle E_1 \mathbf{V}_1 \rangle \rho_0 = (P_0 / \rho_0) \langle \rho_1 \mathbf{V}_1 \rangle = -P_0 \langle \mathbf{V}_2 \rangle$. Consequently, the only surviving term in the expression for the acoustic intensity is

$$\mathbf{I} = \langle P_1 \mathbf{V}_1 \rangle \quad (14.4.15)$$

This, then, is the proof that the expression for the acoustic intensity that we have used in the linear theory, indeed, is correct. It should be emphasized, however, that this result refers to a coordinate system with respect to which the unperturbed fluid velocity is zero. If this is not the case, the expression for the intensity is considerably more complicated.

The *momentum flux tensor* $\mathfrak{T}$ with components $\rho V_i V_j$ correct to second order is simply $\rho_0 \mathbf{V}_1 \mathbf{V}_1$ if the unperturbed fluid is at rest so that $\mathbf{V}_0 = 0$. The corresponding rate of momentum transfer in the direction $\mathbf{n}$ per unit area has the components $\sum_j \rho_0 V_i V_j n_j$, and the time average of this is sometimes called the acoustic radiation pressure on a perfectly absorbing (no-reflection) surface. For one-dimensional motion the radiation pressure in the direction of propagation is $\langle \rho V_1^2 \rangle$, which is equal to the energy density in the wave. However, one should carefully distinguish between this quantity and the *total* time-average pressure, or rate of momentum transfer to a surface, which is $\langle \sum_j [(P - P_0) \delta_{ij} + \rho V_i V_j] n_j \rangle$, which, for one-dimensional motion, is $\langle P - P_0 \rangle + \langle \rho_0 V_1^2 \rangle$. To evaluate it, we must determine also $\langle P - P_0 \rangle$, and we shall make some comments on it under the following heading.

Other time averages

In addition to the time averages of mass, momentum, and energy flow discussed above, some properties of the time averages of the field variables themselves can be inferred from the second-order equations (14.4.9) to (14.4.11). For example, if we consider irrotational flow and set the viscous stress equal to zero, we find from Eq. (14.4.9) that the time average of the (Eulerian) velocity is given by $\langle \mathbf{V}_2 \rangle = -\langle \rho_1 \mathbf{V}_1 \rangle / \rho_0$. For a traveling harmonic sound wave with $V_1 = u \sin(\omega t - kx)$ and $\rho_1 = \delta \sin(\omega t - kx)$, where $\delta = p/\rho c = (u/c)\rho_0$, this time average becomes $\langle V_2 \rangle = -u^2/2c$. This is in agreement with the result obtained in Chap. 6 in connection with the discussion of the relationship between the Lagrangian and the Eulerian description of fluid motion.

Similarly, on the assumption of irrotational motion, we see from Eq. (14.4.10) that the time average of the pressure in one-dimensional motion is $\langle P_2 \rangle = -\rho_0 \langle V_1^2 \rangle$. This result suggests that in a traveling beam of sound the average pressure in the beam should be lower than the pressure in the unperturbed surrounding fluid. Such a pressure difference will induce secondary flow, however, and the assumption of irrotational flow and the omission of viscosity in the discussion are hardly justified. Under such conditions the question of whether a pressure decrease in the sound field will result or not will depend on boundary conditions. Actually, this question is of importance in the interpretation of experiments designed to measure acoustic radiation pressure.

In the special case, however, when we have a sound wave in a closed tube, so that there is no communication with the outside unperturbed fluid,

our expression for the time-average pressure is expected to be a good approximation to the actual behavior in the tube. Thus, if the tube is terminated at $x = 0$ and $x = L$, a standing wave with the velocity field $u = u_0 \sin(\pi x/L) \sin(\omega t)$ will produce a time-average pressure field $\langle P \rangle = P_0 - \rho(u_0^2/2) \times \sin^2(\pi x/L) = P_0 - \rho_0(u_0/2)^2[1 - \cos(2\pi x/L)]$. In other words, the time-average pressure has minima at the velocity maxima, and the distance between these minima is half a wavelength. Such a pressure distribution, indeed, is found to exist in the tube, and an amusing way to demonstrate it is to let the standing wave act on a water surface (by partially filling the tube with water, the tube being horizontal). The water is pulled up in the region of the pressure minima and is maintained in this way by the sound field.

Vorticity

In our discussion of the linear theory of a viscous fluid in Chap. 6 we have already pointed out that the velocity field can be considered to be a superposition of two contributions. The first, representing the actual sound field, is related to the density fluctuations and is curl-free; that is, $\nabla \times \mathbf{V}_1 = 0$ and $\text{div } \mathbf{V}_1 = -(1/\rho_0) \partial \rho_1 / \partial t$. The second part is a shear motion, and there are no density fluctuations associated with it, so that $\text{div } \mathbf{V}'_1 = 0$. The vorticity, on the other hand, is different from zero; in fact, with $\boldsymbol{\Omega}_1 = \nabla \times \mathbf{V}'_1$, we have

$$\mu \nabla^2 \boldsymbol{\Omega}_1 - \frac{\partial \boldsymbol{\Omega}_1}{\partial t} = 0$$

This velocity field, as we have seen, is present close to boundaries, where it serves to bring the total tangential velocity in to zero at the boundary.

So far in our discussion of second-order effects in a sound field, we have been concerned, basically, with the curl-free velocity field related to the actual sound field. It remains to make some comments about second-order effects related to the second type of motion involving the vorticity. The equation for the second-order vorticity vector $\boldsymbol{\Omega}_2 = \nabla \times \mathbf{V}_2$ is obtained from the second-order momentum equation (14.4.10). The divergence of the viscous-stress tensor that occurs in this equation is

$$\nabla \cdot \mathbf{D}_2 = \mu \nabla^2 \mathbf{V}_2 + (\frac{1}{3}\mu + \eta) \text{grad}(\text{div } \mathbf{V}_2)$$

Thus, by taking the curl of Eq. (14.4.10), we obtain, for the second-order vorticity $\boldsymbol{\Omega}_2 = \text{curl } \mathbf{V}_2$,

$$\mu \nabla^2 \boldsymbol{\Omega}_2 - \frac{\partial \boldsymbol{\Omega}_2}{\partial t} = \nabla \times \nabla \cdot \mathfrak{J} + \frac{\partial}{\partial t} [\nabla \times (\rho_1 \mathbf{V}_1)]$$

This diffusion equation, in contrast with the equation for $\boldsymbol{\Omega}_1$, contains a volume distribution of sources involving the first-order acoustic field. Thus, assuming that the first-order field for the viscous fluid has been determined

from Eqs. (14.4.5) to (14.4.7), we can in principle solve the equation for Ω_2 . It is interesting to note that, in general, Ω_2 has a nonzero time average, given by

$$\mu \nabla^2 \langle \Omega_2 \rangle = \nabla \times \nabla \cdot \langle \mathfrak{J} \rangle$$

Steady vortex patterns can readily be observed in the standing wave in a tube and about cylinders and other objects with which the sound wave interacts. A quantitative analysis of these vortex patterns requires, first, that a solution be found for the first-order sound field in a viscous fluid satisfying the boundary conditions of zero tangential velocity on the boundaries. After this field has been determined, the inhomogeneous vorticity equation for $\langle \Omega_2 \rangle$ has to be solved. As with other second-order effects, the general nature of the phenomenon is of interest, but detailed solutions, for specific situations, do not usually merit the considerable additional computational effort.

Wave interaction

As another application of the second-order equations we shall discuss some solutions to the second-order wave equation (14.4.12) when viscosity is neglected. The source term in this equation contains the square of first-order acoustic field variables, and therefore a harmonic sound wave with a frequency ω will result in a source term which contains a term with frequency 2ω . Thus the harmonic sound wave will continuously generate a second-order wave field of frequency 2ω as it propagates through the fluid. As we shall see shortly, this second harmonic wave will grow with distance. This wave of frequency 2ω will interact with the primary wave of frequency ω to produce a source term containing the frequency 3ω , etc. In other words, the initially single harmonic wave will be distorted as it travels. Our quantitative analysis of this problem of wave (self) interaction, however, is limited only to comparatively small distortions, since we have developed the acoustic equations only to second order. Under the next heading we shall discuss the behavior of the other extreme case of a largely distorted wave.

In addition to its application in the problem of a sound wave interacting with itself, Eq. (14.4.12) also allows the calculation of the second-order field resulting from the interaction between two or more sound fields, P_a , P_b , etc., emitted from different sources. In that case, $P_1 = P_a + P_b + \dots$, and it follows that if these waves are harmonic, with frequencies ω_a , ω_b , ..., the resulting wavefield will contain combination tones with frequencies $\omega_a \pm \omega_b$, etc. In the present analysis we shall, for simplicity, consider only two such waves, P_a and P_b , and assume the sources of these waves to be at the same positions.

In carrying out such calculations it is found convenient to introduce a change of variables which will in effect eliminate $\mathfrak{J}$ from the source term

in Eq. (14.4.12). Thus, if we set

$$\begin{aligned} P_2 &= P'_2 - \frac{\rho_0 V_1^2}{2} - \frac{P_1^2}{2\rho_0 c_0^2} - \frac{1}{\rho_0 c_0^2} \frac{\partial P_1}{\partial t} \int P_1 dt \\ V_2 &= V'_2 - \frac{1}{\rho_0 c_0^2} \frac{\partial}{\partial t} \left(V_1 \int P_1 dt \right) \\ \rho_2 &= \rho'_2 - \frac{\rho_0 V_1^2}{2c_0^2} - (1 - \Gamma) \frac{P_1^2}{2\rho_0 c_0^4} - \frac{1}{\rho_0 c_0^2} \frac{\partial P_1}{\partial t} \int P_1 dt \end{aligned} \quad (14.4.16)$$

the wave equation for P'_2 is then, simply (for Γ see page 866),

$$\nabla^2 P'_2 - \frac{1}{c_0^2} \frac{\partial^2 P'_2}{\partial t^2} = -(1 - \Gamma) \frac{1}{\rho_0 c_0^4} \frac{\partial^2 P_1^2}{\partial t^2} \quad (14.4.17)$$

The problem of determining the second-order sound field thus has been reduced to a problem, familiar from linear acoustics, of determining a field produced by a source which can be interpreted as a monopole distribution. As indicated earlier, we shall consider the second-order field produced when the primary first-order pressure wave P_1 consists of two harmonic components, P_a and P_b , with frequencies ω_a and ω_b . As has been the practice throughout most of this book, we assume that these first-order field components are expressed in complex form. For example, in the case of a plane wave, we have $P_a = A \exp(ik_a x - i\omega_a t)$.

The pressure P_1 in Eq. (14.4.17) then is

$$P_1 = \text{Re}(P_a + P_b) = \frac{1}{2}[(P_a + P_a^*) + (P_b + P_b^*)]$$

where P_a^* is the complex conjugate of P_a . The second-order component P'_2 then will contain harmonic terms with frequencies $2\omega_a$, $2\omega_b$, and $\omega_a \pm \omega_b$. The first two of these correspond to the interaction of the wave components with themselves, whereas the mutual interaction results in the frequencies $\omega_+ = \omega_a + \omega_b$ and $\omega_- = \omega_a - \omega_b$. It is sufficient to consider here only the portion of the source term that contains these combination frequencies, since the result thus obtained reduces to the case of self-interaction if we put $\omega_a = \omega_b$ and $P_a = P_b$. The corresponding portion of P_1^2 that contains ω_+ is $\frac{1}{2}(P_a P_b + P_a^* P_b^*) = \text{Re}(P_a P_b)$, and similarly, the ω_- term is $\frac{1}{2}(P_a P_b^* + P_a^* P_b) = \text{Re}(P_a P_b^*)$.

Then, if we express the solution $P'_{2\pm}$ of Eq. (14.4.17) that corresponds to the frequency $\omega_{\pm}$ as $\text{Re } P_{\pm}$, the equation for $P_{\pm}$ is

$$(\nabla^2 + k_{\pm}^2)P_{\pm} = \frac{1 - \Gamma}{\rho_0 c^2} k_{\pm}^2 P_a P_{b\pm} \quad (14.4.18)$$

where $k_+ = \omega_+/c = (\omega_a + \omega_b)/c$, and $P_{b+} = P_b$, $P_{b-} = P_b^*$ (complex conjugate). To obtain the corresponding equation for P_- , we replace k_+ by k_- and P_b by P_b^* .

The interaction fields produced by various primary waves P_a and P_b

can now be determined by solving this equation. As illustrations we shall consider here the important cases when the primary waves are plane, cylindrical, and spherical. For two plane waves traveling in the same direction along the x axis, we have $P_a = A \exp(ik_a x - i\omega_a t)$ and $P_b = B \exp(ik_b x - i\omega_b t)$, where A and B are real; the equation for $P_{\pm}$ is

$$(\nabla^2 + k_{\pm}^2)P_{\pm} = (\rho_0 c_0^2)^{-1}(1 - \Gamma)k_{\pm}^2 AB \exp(ik_{\pm} x - i\omega_{\pm} t) \\ \equiv C_{\pm} \exp(ik_{\pm} x - i\omega_{\pm} t) \quad (14.4.19)$$

This equation is of the same type that we encountered in the discussion of stimulated Brillouin scattering; the particular solution, for the plus sign, is

$$P'_{2+} = \text{Re } P_+ = \text{Re } \frac{C_{+x}}{2ik_+} \exp(ik_+ x - i\omega_+ t) \\ = \frac{1 - \Gamma}{2\rho_0 c_0^2} AB (k_+ x) \sin(k_+ x - \omega_+ t) \quad (14.4.20) \\ = (1 - \Gamma) \frac{\rho_0}{2} v_A v_B (k_+ x) \sin(k_+ x - \omega_+ t)$$

where v_A and v_B are the velocity amplitudes of the two primary waves. In other words, the interaction field amplitude increases with distance; there is a continuous energy transfer from the primary harmonic components to the combination tone.

The experimental arrangement that corresponds to this case of two interacting plane waves (which could very well be two harmonic components of one and the same wave) might be a plane piston moving with a velocity $V_x = V_a \exp(-i\omega_a t) + V_b \exp(-i\omega_b t)$. Similarly, we may consider the combination tones in the cylindrical and spherical wavefields as being produced by a pulsating cylinder and a sphere, respectively. Thus, consider a cylinder of radius a pulsating so that the radial surface velocity of the cylinder is

$$V_r = V_a \exp(-i\omega_a t) + V_b \exp(-i\omega_b t)$$

Referring to the discussion of the radiation from a cylinder in Chap. 7, we find that the corresponding sound pressure field is

$$P_{a,b} = -\rho_0 c_0 V_{a,b} \frac{iH_0^{(1)}(k_{a,b} r)}{H_1^{(1)}(k_{a,b} a)} \exp(-i\omega_{a,b} t) \quad (14.4.21)$$

We now proceed in a manner completely analogous to the plane-wave case and determine the second-order pressures $P_{\pm}$ from Eq. (14.4.18), using the values for P_a and P_b given in Eq. (14.4.21). The following expressions for the combination tones $\omega_+ = \omega_a + \omega_b$ and $\omega_- = \omega_a - \omega_b$ are found to satisfy the wave equation (14.4.18):

$$P_{\pm} = \mp \frac{\rho_0 V_a V_b (1 - \Gamma) k_{\pm} r}{2} \frac{H_1^{(1)}(k_a r) H_{0\pm}(k_b r) \pm H_0^{(1)}(k_a r) H_{1\pm}(k_b r)}{H_1^{(1)}(k_a a) H_{1\pm}(k_b a)} \\ \times \exp(-i\omega_{\pm} t) \quad (14.4.22)$$

and the actual pressure is $P'_{2\pm} = \text{Re } P_{\pm}$. Here $H_0^{(1)}$ and $H_1^{(1)}$ are Hankel functions of the first kind, of zeroth and first order. The plus sign and the minus sign on the Hankel functions indicate functions of first and second kind, $H_- = H^{(2)}$ being the complex conjugate of $H_+ = H^{(1)}$. This difference between the expressions for P_+ and P_- occurs because of the difference in the source terms as shown in Eq. (14.4.18).

It is interesting to investigate the asymptotic form of these waves when $k_{\pm}r \rightarrow \infty$. If the cylinder radius is small, $k_{\pm}a \ll 1$, this asymptotic form is

$$P'_{2\pm} \Rightarrow \mp \rho_0 V_a V_b \frac{\pi}{2} (1 - \Gamma)(k_a k_b)^{1/2} k_{\pm} a^2 \begin{cases} \cos(k_+ r - \omega_+ t) \\ \sin(k_- r - \omega_- t) \end{cases} \quad (14.4.23)$$

This pressure amplitude is independent of r , and the total energy flux in this wave grows with the distance from the source, just as in the case of plane waves. However, the solution ceases to be valid beyond a distance at which the amplitude is of the same order of magnitude as that of the primary wave pressures.

Finally, we consider a sphere of radius a pulsating with a radial velocity $V_r = V_a \exp(-i\omega_a t) + V_b \exp(-i\omega_b t)$. The corresponding pressure fields P_a and P_b , with frequencies ω_a and ω_b , are

$$P_{a,b} = -\frac{i\rho_0 c_0 k_{a,b} a^2}{1 - ik_{a,b}a} V_{a,b} \exp(ik_{a,b}r - i\omega_{a,b}t)$$

Again, using these expressions for P_a and P_b in Eq. (14.4.18), we find (by direct insertion into the equation) the following function:

$$P_{\pm} = \pm(1 - \Gamma)\rho_0 V_a V_b \frac{ik_{\pm}k_a k_b a^4}{(1 - ik_a a)(1 - ik_b a)} \times \frac{\exp i(k_{\pm}r - \omega_{\pm}t)}{r} \left[\ln \frac{r}{a} - e^{-i\phi(r)} \right] \quad (14.4.24)$$

where $\phi(r) = 2k_{\pm}r \int_a^r r^{-1} \exp(2ik_{\pm}r) dr$. For a small radiating sphere, $k_{\pm}a \ll 1$, the asymptotic form of the corresponding pressure field $P'_{2\pm} = \text{Re } P_{\pm}$ for large values of r is

$$P'_{2\pm} \rightarrow \pm(1 - \Gamma)\rho_0 V_a V_b (k_{\pm}k_a k_b a^4) \frac{\ln(k_{\pm}r)}{2r} \sin(k_{\pm}r - \omega_{\pm}t) \quad (14.4.25)$$

As in the cylindrical case, the amplitude of the combination tone does not decrease as rapidly with distance as do the primary-wave components, the radial dependence being of the form $r^{-1} \ln(k_{\pm}r)$ as compared with r^{-1} for the primary waves. The range of validity of the solution, of course, is subject to the same restriction as in the plane and cylindrical cases, that the second-order pressure field be small compared with the primary pressures.

14.5 SHOCKED SOUND WAVES

As has already been pointed out, the analysis in the preceding section is restricted to comparatively weak nonlinear effects and does not lend itself very well to the study of large distortions encountered, for example, in shock waves. Actually, a detailed analysis of the continuous large-amplitude distortion of a sound wave, and the formation of a shock front, is a problem not as yet fully clarified in all its details. However, a good deal of understanding of the process of wave distortion and the properties of shocks can be obtained from comparatively simple considerations; we shall devote this section to that end.

Wave steepening and shock formation

If we neglect the effects of viscosity and heat conduction, the distortion of a sound wave is a cumulative effect resulting from the fact that the sound wave produces a small perturbation of the medium so that the wave speed varies somewhat from one point to another in the wave. As a result, different portions of the wave will travel with different speeds, and wave distortion results.

To investigate this phenomenon somewhat more closely, we consider a traveling plane wave with a fluid velocity field $u = u[t - (x/c)]$. Associated with this wave, as we have seen in Chap. 6, is a pressure, a density, and a temperature wave. Of particular interest here is the temperature wave, which is

$$\theta = \frac{u}{c_0} (\gamma - 1) T_0 \quad (14.5.1)$$

where T_0 is the unperturbed temperature of the fluid, and c_0 the corresponding speed of sound. In the plane sound wave these waves are all in phase.

Since the speed of sound is proportional to the square root of the temperature, it follows that at a point in the wave where the excess temperature produced by the sound field is θ , the speed of sound at that point is $c = c_0(1 + \theta/2T_0)$. At the same point (the velocity and temperature waves being in phase), the fluid velocity in the wave is $u = \frac{\theta}{T_0} \frac{c_0}{\gamma - 1}$. The total speed of propagation of the disturbance at that point is the sum of the local sound speed and the fluid velocity

$$c + u = c_0 \left(1 + \frac{\theta}{2T_0} \right) + u = c_0 + \frac{\gamma + 1}{2} u \quad (14.5.2)$$

Thus, if the plane wave is harmonic with a velocity amplitude u_0 , the wave speed in the crest of the wave is $c_0 + \frac{1}{2}u_0(\gamma + 1)$ and in the trough it is $c_0 - \frac{1}{2}u_0(\gamma + 1)$. As a result, the wave will be distorted, the crest

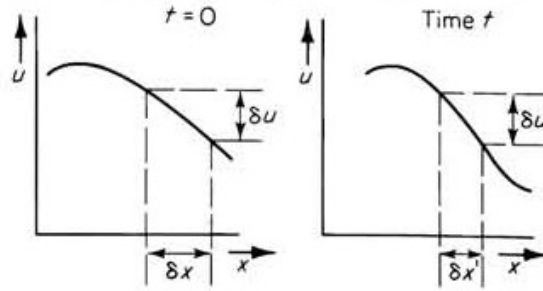


FIGURE 14.15
Increasing steepness of acoustic wavefront
of finite size.

eventually will overtake the trough, and the velocity gradient in the wave will be infinite. This corresponds to shock formation.

To determine the distance of wave travel required for shock formation, we consider two points in the fluid with an initial difference in fluid velocity δu and separation δx , as shown in Fig. 14.15. The difference in wave speed between these points, according to Eq. (14.5.2), is $\frac{1}{2}(\gamma + 1)\delta u$. Thus, after a time t , the separation δx of the points characterized by the velocity difference δu has changed to

$$\delta x' = \delta x + (\gamma + 1) \frac{\delta u}{2} t$$

and the initial velocity gradient $\delta u/\delta x$ has changed to $\delta u/\delta x'$. The separation $\delta x'$ becomes zero, and the velocity gradient infinite, at a time $t = -2[(\gamma + 1)(\partial u/\partial x)]^{-1}$. In other words, the larger the initial velocity gradient, the shorter the time required for shock formation. If, initially, the largest negative gradient in the wave is $(-\partial u/\partial x)_m$, the time of travel to shock formation is

$$t_s = \frac{2}{(\gamma + 1)(-\partial u/\partial x)_m} \quad (14.5.3)$$

As an example we consider an initially harmonic wave $u = u_0 \sin(kx - \omega t)$. We have $|\partial u/\partial x|_m = ku_0 = 2\pi u_0/\lambda$, and the corresponding distance of travel to the shocked state is

$$c_0 t_s = \frac{1}{\pi(\gamma + 1)} \frac{c}{u_0} \lambda = \frac{\gamma}{\gamma + 1} \frac{1}{\pi} \frac{P_0}{p} \lambda$$

where λ is the wavelength, $P_0 = \rho c^2/\gamma$ is the static pressure, and $p = \rho c u_0$ is the sound pressure. Thus, with a sound-pressure amplitude about one-tenth of the static pressure, a shock front will form after only a few wavelengths of wave travel.

Factors opposing shock formation

This phenomenon of nonlinear wave distortion is often referred to as a result of “convection,” which implies that fluid velocity contributes to the signal speed. This is not the only effect; the change in the local temperature, and hence in the local sound speed, also contributes to the distortion.

When an infinite velocity gradient has been reached at a certain point, the properties of the fluid have a discontinuity at this point, not only in the flow velocity, but also in the state variables. Continued influence of the convective wave distortion would lead to a multivalued function, since the wave crest would overtop the wave trough. This, naturally, is not physically possible. Thus the wave appears to be unstable, and on the basis of this discussion, an *ideal compressible* fluid should be unstable for any (one-dimensional) arbitrarily small perturbation, since eventually a discontinuity and a multivalued function would result from it.

The wave-steepening convective effect is not, however, the only one involved. Other competing processes tend to relax the buildup of the gradients in the fluid. The most apparent is the diffusion of velocity and temperature resulting from viscosity and heat conduction. In addition, relaxation of internal degrees of freedom in the fluid resists rapid changes in the temperature, which again tends to limit the gradients.

To estimate the significance of this effect, consider the influence of viscosity and heat conduction on the waveform of a plane wave-pulse. We recall that a harmonic wave is attenuated by the factor $\exp(-\alpha x)$, where $\alpha \simeq \omega^2 l / c^2$, with l of the order of the mean-free-path. An arbitrary wave, which at $x = 0$ has a time dependence $u_0(t)$,

$$u_0(t) = \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega t} d\omega$$

then will have the form

$$u(t, x) = \int_{-\infty}^{+\infty} F(\omega) \exp\left(-i\omega \frac{t-x}{c}\right) \exp\left(\frac{-\omega^2 l x}{c^2}\right) d\omega$$

at the position x . For simplicity, let us make $u_0(t)$ a delta-function pulse $u_0(t) = \delta(t)$. The spectral density $F(\omega)$ is then simply $1/2\pi$, and we obtain for $u(t, x)$

$$\begin{aligned} u(t, x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-iA\omega) \exp(-B^2\omega^2) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[-B^2\left(\omega + \frac{iA}{2B^2}\right)^2 - \frac{A^2}{4B^2}\right] d\omega \end{aligned}$$

Since $\int_{-\infty}^{+\infty} \exp(-ay^2) dy = \sqrt{\pi/a}$, we obtain, with $A = t - x/c$, $B = \sqrt{x l / c^2}$,

$$u(t, x) = \frac{c}{\sqrt{2\pi(2lx)}} \exp\left[-\frac{(t - x/c)^2 c^2}{4lx}\right]$$

In other words, the wave is a Gaussian distribution with a standard deviation, $\sqrt{lx}$, which increases as the square root of the travel distance. The fact that the gradient-relaxing effect by diffusion is proportional to $\sqrt{x}$

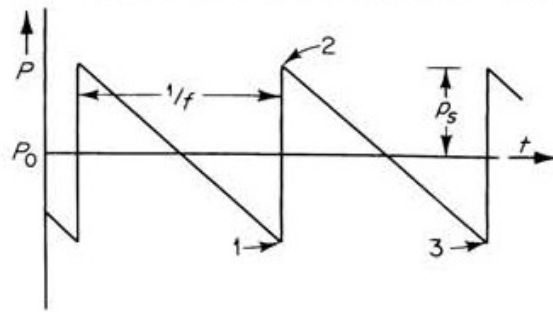


FIGURE 14.16
Pressure fluctuations in a shocked wave.

is significant. The wave distortion produced by the convective effect is proportional to x , as we have seen. Thus, for very small x , the diffusion effects predominate. If we start out with a discontinuity at $t = 0$ at $x = 0$, the convective effects, which tend to make the wave multivalued, are more than checked by the diffusive effects, and the slope of the wavefront initially is decreased. After this initial predominant influence of diffusion, its effect decreases, and the competing convective effects may be able to balance the diffusive effects and thus establish a stable waveform. From these considerations we expect that a sufficiently strong sound wave eventually will reach a *stable* sawtoothlike shape, as portrayed in Fig. 14.16. We shall call such a wave a shocked sound wave, or a repeated shock wave.

Shock relations

We shall now assume that a balance has been struck between the competing effects of nonlinearity and diffusion so that a stable shocked sound wave has been established. In each of the shock fronts in the wave the state of the fluid changes quite rapidly; a fluid element which is carried through the shock experiences a sudden change in velocity, density, pressure, and temperature and, as we shall see later, also in entropy. For the present we shall not be interested in the spatial dependence of this change of state. Instead, we ask what can be said about the relationship between the state of the fluid element before and after the passage through the shock.

The situation is much the same as in the study of collisions in particle dynamics, where, without the knowledge of the interaction, a good deal can be said about the relationship between the state of motion before and after the collision merely by applying conservation laws. Similar results can be obtained for the fluid. To show this, it is convenient to describe the fluid motion with respect to a coordinate system that moves with the shock wave. In this coordinate system the fluid moves into the shock on the front side and leaves it on the other side. If we integrate the equations of mass, momentum, and energy balance over a volume element that includes the shock, we find (replacing the volume integral of a divergence with a surface integral, by means of Green's theorem) that the components normal to the shock front of the mass, momentum, and energy fluxes are the same on both sides of the shock. This is an obvious result, since there are no sources of mass, momentum, or energy in the shock.

In component form the expressions for the mass, momentum, and energy fluxes are $j_i = \rho V_i$, $P\delta_{ij} + \rho V_i V_j$, and $(\rho E + \frac{1}{2}\rho V^2 + P)V_i$, where ρE is the internal energy of the fluid per unit volume. In order to save some writing, we shall indicate by square brackets the difference of a quantity behind the shock and in front of the shock. The conservation of mass, momentum, and energy flux then takes the form, if $\mathbf{n}$ is the direction normal to the shock front,

$$\begin{aligned} \sum_i [\rho V_i n_i] &= 0 & \text{or} & & \sum_i [j_i n_i] &= 0 \\ \sum_j [P\delta_{ij} + \rho V_i V_j] n_j &= 0 & \text{or} & & \sum_j [(P\delta_{ij} + j_i V_j) n_j] &= 0 \\ \sum_i [(\rho E + \frac{1}{2}\rho V^2 + P)V_i n_i] &= 0 & \text{or} & & \sum_i [(H + \frac{1}{2}V^2)j_i n_i] &= 0 \end{aligned} \quad (14.5.4)$$

where $H = E + (P/\rho)$ is the enthalpy. These five equations, usually called the Rankine-Hugoniot relations, together with an equation of state, are sufficient to determine the six quantities V_1 , V_2 , V_3 , ρ , and P on one side of the shock in terms of the corresponding quantities on the other side. Although we shall be concerned with normal shock waves, in which the flow velocity is perpendicular to the shock front, Eqs. (14.5.4) apply to more general situations. To write the equations in a more explicit form, we choose the x_1 direction as the direction of propagation so that the components of n_i are (1,0,0). Equations (14.5.4) then take the simpler form

$$[\rho V_1] = 0 \quad (14.5.5)$$

$$[P + \rho V_1^2] = 0 \quad [\rho V_1 V_2] = 0 \quad [\rho V_1 V_3] = 0 \quad (14.5.6)$$

$$[(H + \frac{1}{2}V^2)\rho V_1] = 0 \quad (14.5.7)$$

If we have a velocity component V_1 different from zero, perpendicular to the shock, it follows from Eq. (14.5.5) that ρV_1 is continuous across the shock, and from Eqs. (14.5.6) that also the tangential components V_2 and V_3 are continuous. If V_2 and V_3 are zero, we have a *normal* shock, which is of interest here. Otherwise the directions of propagation and fluid motion are different, and we have an *oblique* shock.

Since we shall restrict ourselves here to the normal shock traveling in the x direction, we can drop the coordinate subscripts, and we shall use the subscript 0 to designate the state of the fluid in front of the shock. Then, with $j = \rho V = \rho_0 V_0$, we obtain from Eqs. (14.5.6), with $\rho V^2 = j^2/\rho$,

$$j^2 = \frac{P - P_0}{\rho_0^{-1} - \rho^{-1}}$$

The velocity of the fluid in front of the shock, $V_0 = j/\rho_0$, follows directly from this equation, and since our frame of reference moves with the speed of the shock wave, it follows that V_0 is the speed of the shock with respect

to the laboratory coordinate system, with respect to which the fluid in front of the shock is at rest. In other words, the shock speed is

$$U = |V_0| = \sqrt{\frac{P - P_0}{\rho - \rho_0} \frac{\rho}{P_0}} = \frac{1}{\rho_0} \sqrt{\frac{P - P_0}{\rho_0^{-1} - \rho^{-1}}} \quad (14.5.8)$$

For a very weak shock wave this expression reduces to $\sqrt{\partial P / \partial \rho}$, which, if the flow is isentropic, is the same as the speed of sound.

In order to express the shock speed in terms of the pressure ratio (P/P_0), we must obtain the relation between ρ and P . To obtain this relation, we may first use Eq. (14.5.7) to obtain $H - H_0 = \frac{1}{2}(V_0^2 - V^2)$, which, with $V = j/\rho$, becomes

$$H - H_0 = \frac{j^2}{2} \left(\frac{1}{\rho_0^2} - \frac{1}{\rho^2} \right) = \frac{1}{2}(P - P_0)(\rho_0^{-1} + \rho^{-1}) \quad (14.5.9)$$

This expression for $H - H_0$ does not contain the flow velocities, and relates the thermodynamic state variables on the two sides of the shock. If the enthalpy function H is known in terms of the state variables P and ρ , we have the desired relation between ρ and P . If we specialize to an ideal gas, we have

$$H = E + \frac{P}{\rho} = C_v T + (C_p - C_v)T = C_p T = \frac{\gamma P}{\rho(\gamma - 1)}$$

and Eq. (14.5.9) can be expressed as

$$\frac{\rho}{\rho_0} = \frac{P + P_0[(\gamma - 1)/(\gamma + 1)]}{P_0 + P[(\gamma - 1)/(\gamma + 1)]} \quad (14.5.10)$$

We now return to Eq. (14.5.8) for the shock speed. Using Eq. (14.5.10) for the density ratio, we can express the shock speed in terms of a single shock-strength parameter, which we choose to be $\eta = (P - P_0)/P_0$, so that $P/P_0 = 1 + \eta$. The expression for the shock speed can then be written

$$U_0 = c_0 \sqrt{1 + \frac{(\gamma + 1)\eta}{2\gamma}} \quad (14.5.11)$$

where c_0 is the speed of sound in the unperturbed fluid in front of the shock. It is interesting to express also other properties of the shock in terms of this parameter. For example, the flow velocity behind the shock, measured with respect to the laboratory coordinate system, is found to be

$$V_L = c_0 \frac{\eta}{\sqrt{1 + [(\gamma + 1)\eta/2\gamma]}} \quad (14.5.12)$$

and the temperature and the corresponding local speed of sound in the shock are obtained from

$$\frac{T}{T_0} = \frac{c^2}{c_0^2} = (1 + \eta) \frac{2\gamma + (\gamma + 1)\eta}{2\gamma + (\gamma - 1)\eta} \quad (14.5.13)$$

The derivation and further study of these relations are left as one of the problems.

For given values of ρ_0 and P_0 , Eq. (14.5.10) defines the relation between ρ and P in the shock. It is interesting to note that the density ratio has a maximum value $(\gamma + 1)/(\gamma - 1)$ which corresponds to $P/P_0 = \infty$. For a monatomic gas this maximum value is 4, and for a diatomic gas it is 6. Clearly, this relation is not isentropic; therefore, as a fluid particle passes through the shock, its entropy will increase. To calculate the entropy increase in the case of an ideal gas, we use the equation of state $S = S(\rho, P)$,

$$S - S_0 = C_v \log \left[\frac{P}{P_0} \left(\frac{\rho_0}{\rho} \right)^\gamma \right]$$

and use Eq. (14.5.10) to express ρ/ρ_0 in terms of P/P_0 . Then, if we introduce $\eta = (P - P_0)/P_0$ as a measure of the shock strength, we can express $S - S_0$ in terms of η . In a power series expansion of $S - S_0$ in terms of η , the contributions from η and η^2 are found to be absent, and the first nonzero term is of third order in η . Thus, for $\eta < 1$, we obtain

$$S - S_0 \simeq \mathcal{R} \frac{\gamma + 1}{12\gamma^2} \eta^3 \quad \eta = \frac{P - P_0}{P_0} \quad (14.5.14)$$

where $\mathcal{R}$ is the gas constant, and $\gamma = C_p/C_v$. Physically, this increase in entropy must correspond to the diffusion processes caused by viscosity and heat conduction in the shock front, where large gradients of velocity and temperature are maintained. If these gradients were known, the heat generation and the corresponding entropy production could have been calculated explicitly. It is interesting to see, however, that even without this detailed knowledge of the structure of the shock, the conditions imposed by the conservation laws are strong enough to determine the entropy production in the shock front. This reminds one of the similar situation in an inelastic one-dimensional collision between two particles in classical mechanics. There the conservation of momentum enables one to determine the amount of kinetic energy transformed into heat without the detailed knowledge of the interaction forces.

Attenuation of a shocked sound wave

We shall now apply the result in Eqs. (14.5.14) to the calculation of the attenuation of a shocked sound wave. The basic idea in this calculation is to determine the rate of entropy production in the fluid per unit volume as a result of the passage of the shocked sound wave. The time dependence of the state of the fluid element is indicated schematically in Fig. 14.16. The particle goes through the shock at point 1, and the state changes to the value corresponding to point 2.

A fluid element passes through a shock front f times a second, where f is the frequency of the sound wave, and each time the entropy is increased by

an amount that can be determined from Eqs. (14.5.14). The amplitude parameter η was defined in this equation, and in the present case we have $\eta = (P_2 - P_1)/P_1$. Thus, if the pressure amplitude of the shocked sound wave is p_s , so that $P_2 = P_0 + p_s$ and $P_1 = P_0 - p_s$, the amplitude parameter is $\eta = 2p_s/(P_0 - p_s) \simeq 2p_s/P_0$. The rate of heat production per unit volume in the fluid, $dQ/dt = \rho T f \Delta S$, resulting from the heating of the fluid by the shock fronts, then can be expressed as

$$L_1 = \rho \frac{dQ}{dt} = \rho T f \Delta S = \rho \mathcal{R} T \frac{\gamma + 1}{12\gamma^2} \eta^3 f = P_0 \frac{\gamma + 1}{12\gamma^2} \frac{8p_s^3}{P_0^3} f \quad (14.5.15)$$

In the region between successive shocks (between points 2 and 3 in Fig. 14.16) the gradient in the fluid is comparatively small, and to a first approximation the entropy here may be neglected. If we do wish, however, to account for the entropy production also in this region, we may use the general expression developed in Eqs. (6.4.18) and (6.4.21), which, for one-dimensional motion, becomes

$$\rho T \frac{dS}{dt} = \mu' \left(\frac{\partial u}{\partial x} \right)^2 + \frac{K}{T} \left(\frac{\partial T}{\partial x} \right)^2 \quad (14.5.16)$$

where $\mu' = (4\mu/3) + \eta[(6.4.14)]$, and K is the heat conductivity. In the present case the gradient is constant in the regions between successive shocks, as shown in Fig. 14.16, and therefore, if the particle velocity amplitude in the wave is u_0 , the velocity gradient is $\partial u/\partial x = 2u_s/\lambda$, where λ is the wavelength. Similarly, the gradient in temperature is $2\theta_s/\lambda$, where θ_s is the temperature amplitude in the wave.

For modestly large amplitudes we may use the plane-wave acoustic relations between the field variables of Eqs. (6.4.23) and express both u_s and θ_s in terms of the pressure amplitude p_s of the shocked sound wave. The energy loss in the region between shocks, from Eq. (14.5.16), then can be written in the form

$$L_2 = \frac{\delta}{\rho c \lambda^2} \frac{p_s^2}{\rho_0 c} \quad (14.5.17)$$

where $\delta = 4[\mu' + (\gamma - 1)K/C_p]$. The total energy loss is $L_1 + L_2$, and in order to determine the attenuation caused by it, we have to calculate the average energy density in the wave. If p is the excess pressure in the linearly varying portion of the wave between the shock fronts, the potential wave energy is $p^2/2\rho_0 c^2$, and with an equal amount of kinetic energy, the total energy density is $p^2/\rho_0 c^2$.

Since the pressure varies linearly with position between the shocks from 0 to a maximum value p_s in a distance of $\lambda/2$, we have, at a position x from the zero point, $p = 2xp_s/\lambda$. The average energy density in the wave is then

$$W = \frac{1}{\lambda/2} \int_0^{\lambda/2} \frac{1}{\rho c^2} \frac{p_s^2}{(\lambda/2)^2} x^2 dx = \frac{1}{3} \frac{p_s^2}{\rho c^2} \quad (14.5.18)$$

The spatial rate of change of W is related to the loss rate $L_1 + L_2$ through $c dW/dx = -(L_1 + L_2)$, and from Eqs. (14.5.16) to (14.5.18), the corresponding equation for p_s becomes

$$\frac{2}{3} \frac{1}{\rho c^2} \frac{dp_s}{dx} = -\frac{2}{3} \frac{\gamma + 1}{\gamma^2 \lambda} \frac{p_s^2}{P_0^2} - \frac{\delta}{\rho c \lambda^2} \frac{p_s}{\rho_0 c^2}$$

This equation is of the form

$$\frac{dp_s}{(Ap_s + B)p_s} = -dx$$

with the solution

$$\frac{p_s p_{s0} + (B/A)}{p_{s0} p_s + (B/A)} = \exp[-B(x - x_0)] \quad B = \frac{3}{2} \frac{l}{\lambda^2} \quad (14.5.19)$$

where $l = \mu'/\rho c$ is of the order of the mean-free-path, and

$$A = \frac{\gamma + 1}{\gamma \lambda P_0} \quad \frac{B}{A} = \frac{3}{2} \frac{l}{\lambda^2} \frac{P_0 \lambda}{\gamma + 1} = \frac{3}{2} \frac{\gamma}{\gamma + 1} \frac{l}{\lambda} P_0$$

In this equation p_{s0} is the value of p_s at $x = x_0$.

For very small values of B and B/A , we set $\exp(-Bx) = 1 - Bx$, and Eqs. (14.5.19) reduces to

$$\frac{1}{p_{s0}} - \frac{1}{p_s} = \frac{\gamma + 1}{\gamma} \frac{1}{P_0} \frac{(x - x_0)}{\lambda} \quad (14.5.20)$$

The dependence of p_s on x is not exponential in this limiting case, since here the loss rate is not proportional to the energy density. The pressure amplitude p_s decreases from its initial value p_{s0} , at $x = x_0$, to half this value a distance $(P_0/2p_{s0})[\gamma/(\gamma + 1)]$ wavelengths further on. Eventually, p_s is inversely proportional to $x - x_0$.

In the other limit, on the other hand, when $p_{s0} \ll (l/\lambda)P_0$, or correspondingly, $\xi_0 < l$, where ξ_0 is the particle displacement in the wave, the decay is exponential, $p_s = p_{s0} \exp[-B(x - x_0)]$. However, in this limit, $\xi < l$, as we have seen in the introductory comparison between the relative importance of nonlinearity and diffusive effects, the nonlinear wave steepening is more than checked by diffusion, and the sound wave no longer can maintain its shocked state, but decays into an ordinary sound wave.

Problems

- I A simple oscillator of mass M has a conservative restoring force acting on it, $F(x) = -(2lM/\pi)\omega_0^2 \sin(\pi x/2l)$, where $\omega_0/2\pi$ is the frequency of small-amplitude oscillations. Plot the ratio (ω/ω_0) of natural frequency to small-amplitude frequency, as function of the amplitude of oscillation A , from $A = 0$ to $A = 2l$. What happens when $A = 2l$? Use tables of the elliptic integral F , if you wish.

- 2 A simple oscillator of mass M is under the influence of a conservative potential energy $V = Ae^{-2x/l} - 2Ae^{-x/l}$. Sketch V as a function of x/l . If the energy of the system is $E = -\gamma A$ ($0 < \gamma < 1$), show that the displacement of the oscillator is given by the equation

$$\frac{x}{l} = \ln \left[\frac{1}{\gamma} + \frac{\sqrt{1-\gamma}}{\gamma} \sin \left(\sqrt{\frac{2\gamma A}{m}} \frac{t - t_0}{l} \right) \right]$$

(Hint: Set $x/l = -\ln y$ in the integral for t .) What is the period T of oscillation? Plot x/l against t/T for one-half cycle, for $\gamma = 0.8$. What is the equation relating x and t when E is a positive quantity? What is then the period of oscillation?

- 3 If we account for the nonlinear dissipative effects in the neck of a Helmholtz cavity resonator, an approximate equation for free oscillations of such a resonator is of the form

$$m \frac{d^2x}{dt^2} + R \frac{dx}{dt} \left[1 + \beta \left(\frac{dx}{dt} \right)^2 \right] + Kx = 0$$

With $Q = \sqrt{Km}/R = 30$, and $\beta = 2 \times 10^{-5} (\text{cm per sec})^{-2}$, determine graphically the phase-space trajectory of free oscillations, using the method described in the text. The initial conditions are $dx/dt = 700$ cm per sec, and $x = 0$. From this construction show that the maximum excursion of the oscillator in this motion is approximately 0.15 cm.

- 4 A body A of mass m is attracted by a fixed body B by a force which is inversely proportional to the distance between the bodies. Body A is restrained by a spring (force constant K). Thus, if the distance between A and B is x_0 when the spring is relaxed, the force between A and B is of the form $A/(x_0 - x)$, where x is the displacement of A from the relaxed position of the spring. Plot phase-space trajectories for the motion of A for different values of the total energy of motion. For what energies is motion periodic? Show that no periodic motion is possible if $A > Kx_0^2/4$.
- 5 Solve Eq. (14.1.10) graphically, for the case

$$g_1(v) = \begin{cases} -v & v > 1 \\ -1 & 1 > v > 0 \\ +1 & 0 > v > -1 \\ -v & -1 > v \end{cases}$$

for the phase-plane trajectory starting upward from the initial point $v = 0$, $x_1 = -3$.

- 6 A string of length L , clamped at both ends, oscillates in its n th harmonic mode, with a displacement amplitude η_0 , in the xy plane. Use the procedure indicated in the text to determine the longitudinal motion induced in the string by this transverse motion. What is the spatial perturbation in the mass density and tension? What is the change in resonant frequency produced by the interaction?
- 7 A transverse traveling wave $\eta = \eta_0 \sin(k_\perp x - \omega t)$ is generated by a transverse displacement $-\eta_0 \sin \omega t$ at the end $x = 0$. By taking the coupling between transverse and longitudinal motion into account to the third order, show that the wave speed is not the square root of T_0/ϵ_0 , but equals the square root of $(T_0/\epsilon_0) \{1 + \frac{1}{4} [(AQ/T_0) - 1](k_\perp \eta_0)^2\}$.

- 8 The $g_1(v)$ of Eq. (14.1.10) for an idealized force having both activation and dissipation regions is

$$g_1(v) = \begin{cases} -\sqrt{3} & v > 1 \\ +\sqrt{3} & 1 > v > 0 \\ -\sqrt{3} & 0 > v > -1 \\ +\sqrt{3} & -1 > v \end{cases}$$

By piecing the phase-plane trajectory out of parts of circles, show that the steady-state self-sustaining oscillations have displacement amplitude $(x_1)_{\max} = 2\sqrt{2}$. What is the velocity amplitude?

- 9 Suppose the frictional force $F(V - v)$ of the discussion of Fig. 14.4 is

$$F(u) = \begin{cases} \Delta - R'u & u > 0 \\ -\Delta - R'u & u < 0 \end{cases} \quad u = V - v$$

Use Eqs. (14.1.19) to show that the velocity amplitude of steady-state oscillations of the mass is

$$A \simeq V \left[1 + \frac{1}{2} \left(\frac{\pi R' V}{4\Delta} \right)^2 \right] \quad \text{if } \frac{\pi R' V}{4\Delta} \ll 1$$

What is the frequency of oscillation, to first order, in $\pi R' V / 4\Delta$?

- 10 Work out the solution of Eq. (14.2.2), to first approximation in ϵ , including both the third harmonic and the $\frac{1}{3}$ subharmonic; i.e., set

$$\xi = \xi_{\frac{1}{3}} \cos \frac{\gamma\theta}{3} + \xi_1 \cos(\gamma\theta) + \xi_3 \cos(3\gamma\theta)$$

and solve for all three coefficients to first approximation.

- 11 In the Lagrangian description of fluid motion as discussed in Chap. 6, $\phi_L(t, x)$ designates the value of the field variable ϕ at time t for a fluid element which, at $t = 0$, was at the position x . If in time t the displacement of this fluid element from this position is ξ , the position of the element at time t is $x + \xi$. The value of ϕ at this point can be expressed in the Eulerian description as $\phi(x + \xi, t)$, and it follows that the relation between the Lagrangian and the Eulerian description can be expressed as

$$\phi_L(t, x) = \phi(x + \xi, t)$$

Suppose φ is a perturbation of the field variable ϕ so that $\phi = \phi_0 + \varphi$. Expand ϕ in a Taylor series, and show that, to second order in the field variable φ , the relation between the Lagrangian and Eulerian values ϕ_L and ϕ is

$$\phi_L = \phi_0 + \sum_i (\xi_{1i} + \xi_{2i}) \frac{\partial \phi_0}{\partial x_i} + \sum_i \xi_{1i} \frac{\partial \varphi_1}{\partial x_i} + \sum_{ij} \frac{\xi_{1i} \xi_{1j}}{2} \frac{\partial^2 \phi_0}{\partial x_i \partial x_j} + \dots$$

As in the text, ξ_1 and φ_1 represent the solutions to the linearized equations of motion, whereas ξ_2 , φ_2 are of second order in ξ_1 and φ_1 .

As a specific example, let φ_1 be the velocity component in a plane sound wave $\varphi_1 = u_0 \sin(\omega t - kx)$. From the result above show that

$$\langle u \rangle - \langle u_L \rangle = -\frac{u_0^2}{2c}$$

consistent with the result obtained in Chap. 6.

- 12 Consider the equations for mass and momentum conservation for isentropic motion of an ideal gas. Introduce the local speed of sound c as a dependent variable, and show that these equations can be written

$$(a) \quad \frac{1}{\gamma - 1} \frac{\partial c}{\partial t} + \frac{V}{\gamma - 1} \frac{\partial c}{\partial x} + \frac{c}{2} \frac{\partial V}{\partial x} = 0$$

$$(b) \quad \frac{1}{2} \frac{\partial V}{\partial t} + \frac{V}{2} \frac{\partial V}{\partial x} + \frac{c}{\gamma - 1} \frac{\partial c}{\partial x} = 0$$

Next introduce the “characteristic” variables $R_+ = (V/2) + [c/(\gamma - 1)]$ and $R_- = -(V/2) + [c/(\gamma - 1)]$, and show that the equations for R_+ and R_- are

$$\frac{\partial R_+}{\partial t} + (V + c) \frac{\partial R_+}{\partial x} = 0$$

$$\frac{\partial R_-}{\partial t} + (V - c) \frac{\partial R_-}{\partial x} = 0$$

(R_+ and R_- are the Riemann invariants.) From $dR_{\pm} = (\partial R_{\pm}/\partial x)dx + (\partial R_{\pm}/\partial t)dt$, show that $R_{\pm}$ is constant along the “characteristics” in the xt plane, defined by the differential equation

$$\frac{dx}{dt} = V \pm c$$

- 13 Show that the speed of a shock wave has a value that lies between the local sound speeds c_0 and c_1 in the unperturbed fluid in front of the shock and in the fluid behind the shock. In particular, for weak shock, show that the shock speed is $(c_0 + c_1)/2$.
- 14 With reference to Eq. (14.5.10), sometimes called the “shock adiabat,” make a rough graph of P as a function of $1/\rho$, starting the graph from the unperturbed state ρ_0, P_0 to the point ρP , representing the state of the fluid in the shock. Draw a straight line between these points, and show that the slope of this line is a direct measure of the speed of the shock wave. Also, show that the ordinary adiabatic curve through $\rho_0 P_0$ is tangent to the shock adiabat and that the slope of this curve represents the speed of sound in the unperturbed fluid.
- 15 With reference to the calculation of the distance to shock formation in a plane wave in the text, show that the corresponding distance in a spherical wave, which at $x = x_0$ has a maximum “slope” $\delta u_0/\partial x_0$, is given by

$$\log \frac{x_s}{x_0} = \frac{2}{\gamma + 1} \frac{c}{x_0} \frac{1}{\partial u_0/\partial x_0}$$

Also in analogy with the calculation carried out for the plane wave, determine the attenuation of the spherical shocked sound wave.

- 16 In the text, both in the linear and nonlinear analyses of sound waves, we have for the most part used the Eulerian description of motion. In the study of one-dimensional nonlinear wave motion, however, it is sometimes convenient to use the Lagrangian description. Show that under isentropic conditions the equations for mass and momentum balance, the equation of state, and the resulting “wave equation” are

$$\rho \left(1 + \frac{\partial \xi}{\partial x} \right) = \rho_0 \quad \rho_0 \frac{\partial^2 \xi}{\partial t^2} = - \frac{\partial p}{\partial x} \quad p \left(1 + \frac{\partial \xi}{\partial x} \right)^\gamma = p_0$$

$$c_0^2 \frac{\partial^2 \xi}{\partial x^2} = \left(1 + \frac{\partial \xi}{\partial x} \right)^{\gamma+1} \frac{\partial^2 \xi}{\partial t^2}$$

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GLOSSARY OF SYMBOLS

In general, italic, roman, and Greek symbols are used for scalar quantities (real or complex), boldface roman and Greek for vectors, and German capitals for vector operators (dyadics, tensors). The symbol $||$, in general, means "the magnitude of"; $|\mathbf{A}| = A$ is equal to the square root of the sum of the squares of the components of the vector $\mathbf{A}$; $|f|$ is the square root of the sum of the squares of the real and imaginary parts of the complex number $f = u + iv$. The asterisk indicates the complex conjugate, $f^* = u - iv$. The symbol $\simeq$ means "approximately equal to"; the symbol $\rightarrow$ means "approaches, in the limit"; the symbol $\equiv$ means "equals, by definition." For a property f of a fluid, the expression df/dt represents the time rate of change of f , moving with the fluid; the expression $\partial f/\partial t$ represents the time rate of f at a fixed point in space [Eq. (6.1.10)].

The list of symbols given here includes only those used in more than a single section. The commonly used meaning or meanings are given, followed by the equation or page number where the symbol is defined (if a definition is not obvious).

- a Radius of cylinder or disk or sphere
- A Amplitude of vibration
- b_e Electron mobility (12.1.1)
- $B_m = |dh_m(\xi)/d\xi|$ Radiation amplitude for a sphere (7.2.15), Table VIII
- $\mathbf{B}$ Magnetic induction (12.3.1)
- c Wave velocity (4.1.3), (6.1.8)
- c_ϕ Phase velocity (9.1.17)
- c_g Group velocity (9.1.18)
- $c(\infty)$ Signal velocity (see page 479)

- $\cos z$ Cosine function (1.2.3), Tables I and II
 $\cosh z$ Hyperbolic cosine (1.2.10), Tables I and II
 C Amplitude (1.2.8)
 $C = 0.8905$ (7.3.2), (10.1.6)
 C_a Analogous capacitance (9.1.11), (9.1.21)
 C_p, C_v Heat capacities at constant pressure or volume (6.1.3), (6.4.3)
 C_r Reflection coefficient (6.3.5), (6.3.14)
 d_v, d_h Thermal and viscous boundary thickness (6.4.31)
 D Viscous loss rate (6.4.10)
 D_i Components of $\mathbf{D}$ (7.1.7)
 D_{ij} Components of $\mathfrak{D}$ (6.4.8)
 $\mathbf{D}$ Dipole strength (7.1.7), (7.1.13)
 $\mathfrak{D}$ Shear-stress tensor (6.4.8)
 e Electronic charge (12.1.2)
 $e = 2.71828 \cdots$ Base of natural logarithms
 E Energy of system (1.3.2), internal energy (see page 296)
 $E_m = |dH_m(u)/du|$ Radiation amplitude for a cylinder (7.3.7), Table V
 $\mathbf{E}$ Electric intensity (12.1.1)
 f General function, sometimes a force
 $f(x, t)$ Force density (4.4.3)
 F or $\mathbf{F}$ Mechanical force (1.1.1), (6.2.11)
 $F(\omega)$ Fourier transform of $f(t)$ (1.3.16)
 $F_L(s)$ Laplace transform of $f(t)$ (1.3.23), Table 2.1
 F_0 Magnitude of F
 $g(x | x_0)$ Green's function for infinite string or plate (4.4.11), (5.3.17)
 $g_\omega(\mathbf{r} | \mathbf{r}_0) = e^{ikR}/4\pi R$ Green's function for infinite space (7.2.31), (7.3.14)
 G General Green's function (4.4.7), (7.1.15)
 G_ω Green's function for frequency $\omega/2\pi$
 h Half-thickness of a plate (5.3.1), (10.1.21)
 $h_n(x)$ Spherical Hankel function (7.2.9), Table VII
 H Hamiltonian function (1.3.2), (4.1.10)
 $H_m^{(1)}(z)$ Cylindrical Hankel function (10.1.37), Table III
 i An integer
 $i = \sqrt{-1} = -j$ (see page 12)
 $I_m(x)$ Hyperbolic Bessel function (5.3.3), Table IV
 $\text{Im} f$ Imaginary part of f
 $\mathbf{I}$ Intensity (6.2.15)
 $\mathcal{I}$ Identity tensor, $(\mathcal{I})_{ij} = \delta_{ij}$
 j An integer
 $j = -i$ (see page 12)
 $\mathbf{j}$ Current density (12.3.1)
 $J_m(x)$ Bessel function (1.2.5), (5.2.20), Table III
 $j_n(z)$ Spherical Bessel function (7.2.12), Table VII
 $\mathbf{J}$ Flux vector (6.1.15)

- $\mathfrak{J}$ Momentum flux tensor (6.2.10)
 k Damping constant (2.2.1), (3.2.8)
 $k = \omega/c = 2\pi/\lambda$ Wavenumber (4.3.4), (7.1.5)
 $\mathbf{k}$ Wavenumber vector (8.1.14)
 K Stiffness constant of spring (2.1.1), (3.1.1)
 K Thermal conductivity (6.4.1)
 $\mathbf{K}$ Wavenumber vector in spatial Fourier transform (7.1.31), (8.1.15)
 l Length of string (4.3.9)
 $l_h = k/\rho c C_p$ Characteristic thermal length (6.4.20)
 $l_v = \mu/\rho c$ Characteristic viscous length (6.4.20)
 $\ln z$ Natural logarithm of z
 L Lagrange density (4.1.8), (6.2.13)
 L_a Analogous inductance (9.1.10), (9.1.22)
 L_{bh}, L_{bv} Power loss at surface (6.4.37)
 m An integer
 m Mass (1.1.1)
 M mass
 M Molecular weight (6.1.4)
 $M = V/c$ Mach number (11.1.1)
 $\mathbf{M}$ Wave momentum (6.2.16)
 n An integer
 $n_m(z)$ Spherical Neumann function (7.2.12), Table VII
 N Number of particles per unit volume (12.1.2)
 $N_m(x)$ Neumann function (5.3.15), (7.3.2), Table III
 p Acoustic pressure (6.1.2)
 P Acoustic-pressure amplitude
 P Equilibrium pressure (6.1.2)
 $P_n(\eta)$ Legendre function (7.2.4), Table VI
 $\mathfrak{P}$ Stress tensor (6.2.10), (6.4.9)
 q Rate of production of fluid (7.1.19)
 $q = \mu - iv$ Standing-wave constant for rectangular duct (9.2.14), Plate IV
 $q = \sqrt{k^2 - K^2}$ (10.1.11), (10.1.23)
 Q Young's modulus (5.1.1)
 $Q = \omega_0 m/R$ The Q of the system (2.2.4)
 Q_{ij} Components of $\mathfrak{Q}$ (7.1.9)
 $\mathfrak{Q}$ Quadrupole strength tensor (7.1.10)
 r Radial distance from origin (7.1.1); occasionally, radial distance in cylindrical coordinates, when there is no danger of confusion with spherical coordinates
 $\mathbf{r}$ Radius vector from origin
 R Distance between source and observer (7.1.6)
 R Mechanical resistance (2.2.1), (2.3.2)
 R Reynolds number (11.3.28), (12.3.11)
 R_a Analogous resistance (9.1.12), (9.1.23)

$\operatorname{Re} f$ Real part of f

$\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ Radius vector from source to observer (7.1.6)

s Monopole source-strength density (7.1.22)

s Poisson's ratio (5.3.1)

$\sin z$ Sine function (1.2.3), Tables I and II

$\sinh z$ Hyperbolic sine (1.2.10), Tables I and II

S Cross-sectional area (5.1.1), (9.1.10)

S Equilibrium value of entropy (6.4.12)

S Strength of monopole source (7.1.4)

t Time

T Equilibrium temperature (6.1.3)

T Tension (4.1.3), (5.2.1)

T_p Period of oscillation (1.3.4), (14.1.8)

$\mathfrak{T} = \mathfrak{J} + \mathfrak{D}$ (6.2.11), (7.1.20)

u Acoustic velocity (6.1.6)

$u(x)$ Unit step function (1.3.22)

U Kinetic energy (4.3.3)

U Velocity amplitude

v Velocity

dv Volume element

V Potential energy (1.1.4)

V Volume

$V(u)$ Variational expression (4.5.22)

$\mathbf{V}$ Flow velocity

w Acoustic energy density (6.2.15), (9.5.1)

w Radial distance from cylindrical axis, when there is likelihood of confusion with spherical radius r (7.1.1)

$w = v + i\mu$ Standing-wave constant for cylindrical duct (9.2.25), Plate V

x, y, z Rectangular coordinates

X_m Mechanical reactance (2.3.2)

y Displacement of string (4.1.1)

$Y_{mn}^{\sigma}(\theta, \varphi)$ Spherical harmonic (7.2.7)

$z = p/u = \rho c \zeta = \rho c/\beta$ Acoustic impedance (6.3.4)

$Z_m = F/u$ Mechanical impedance (2.3.2), (4.4.17)

Z_a Analogous impedance (9.1.10), (9.1.29)

Z_s Analogous impedance of simple source (7.1.11), (9.2.12)

$\alpha = c/c_s$ (10.1.5); $= c/c_p$ (10.2.1)

α_m, β_m Characteristic numbers for Bessel functions (5.2.23), (9.2.23), Table X

β Coefficient of thermal expansion (6.1.1)

$\beta = \rho c/z = \xi - i\sigma$ Specific acoustic admittance (6.3.4), (7.4.9)

γ Wavenumber parameter for plate (5.3.9), (10.1.21)

γ Ratio of specific heats (6.1.3)

γ_m Phase shift for cylindrical scattering (7.3.7), Table V

- δ Acoustic density change (6.1.2)
- $\delta(x)$ Dirac delta function (1.3.24)
- δ_m Phase shift for spherical scattering (7.2.15), Table VIII
- $\delta_{mn} = 1$ when $m = n$; $= 0$ when $m \neq n$; Kronecker delta symbol (1.3.9)
- ∂ Partial-derivative symbol
- ϵ Dielectric constant (13.1.1)
- ϵ Linear density (3.3.2), (4.1.3)
- $\epsilon_m = 1$ when $m = 0$; $= 2$ when $m \neq 0$; Neumann symbol (7.2.7)
- $\zeta = z/\rho c = \theta - i\chi$ Specific acoustic impedance (6.3.4)
- η Coefficient of bulk viscosity (6.4.8), (6.4.47)
- η Displacement of membrane or plate (5.2.1), (10.1.21)
- ϑ Angle of incidence (6.3.5)
- ϑ Azimuthal angle in spherical coordinates (7.1.1)
- θ Specific acoustic resistance (6.3.4)
- κ Compressibility, usually adiabatic, κ_s (6.1.3)
- κ_T Isothermal compressibility (6.1.1)
- χ Eigenvalue for transverse mode in duct (9.2.6)
- $\lambda = 2\pi/k = 2\pi c/\omega$ Wavelength (4.3.4)
- Λ Wavelength of light (13.1.11)
- Λ_n Mean-square amplitude of cross-duct mode (9.2.8)
- μ Coefficient of viscosity (6.4.4)
- μ Coupling constant (3.1.2)
- μ_n Wave-distribution parameter (9.2.19), Plates IV and V
- ν Kinematic viscosity (page 750), (12.1.16)
- $\nu = \omega/2\pi$ Frequency (page 6), (2.1.3)
- ν_n Wave-distribution parameter (9.2.9), Plates IV and V
- ξ Specific acoustic conductance (6.3.4)
- $\pi = 3.14159 \cdots$ Ratio of circumference to diameter of circle
- Π Total radiated power (7.1.5)
- ρ Mean density of medium (6.1.2)
- ρ Volume density (5.1.5)
- σ Acoustic entropy change (6.4.18)
- σ Areal density (5.2.1)
- σ Specific acoustic susceptance (6.3.4)
- $\sum$ Summation symbol (page 4)
- Σ_s, Σ_a Absorption and scattering cross sections (8.2.16), (8.2.17)
- τ Acoustic temperature change (6.1.3)
- τ Time shift in autocorrelation function (1.3.13)
- Υ Autocorrelation function (1.3.13), (7.1.32)
- Υ Power transmitted (4.5.9)
- φ Axial angle in spherical coordinates (7.1.1)
- ϕ Axial angle for cylindrical coordinates (7.1.1)
- ϕ_n Characteristic function for cross-duct mode (9.3.24)
- Φ Flow resistance (6.2.23)

Φ Phase angle (1.2.8), (9.1.3)

$\Phi_s(\vartheta)$ Angle-distribution factor (8.1.14), (8.2.15)

χ Specific acoustic reactance (6.3.4)

χ Variational trial function (4.5.25), (9.3.13)

ψ_n Characteristic function (5.2.16)

Ψ Velocity potential (6.2.9)

Ψ_n^* Characteristic function for cross-duct mode (9.2.8)

$\omega = 2\pi\nu$ Angular velocity (1.2.1), (page 6)

Ω Porosity (6.2.22)

$\Omega = 2\pi$ times frequency of light (13.1.4)

$\nabla^2 = \text{div grad}$ Laplacian operator (5.2.3), (6.2.8), (7.1.3)

$\nabla \equiv \text{grad}$ Gradient operator (1.1.5), (7.1.1)

$\nabla \cdot \equiv \text{div}$ Divergence operator (6.1.15), (7.1.2)

$\nabla \times \equiv \text{curl}$ Curl operator (6.4.6)

$\mathbf{u} \cdot \nabla \equiv \mathbf{u} \cdot \text{grad}$ Directional derivative (6.1.14)

$\nabla \cdot \mathfrak{D} = \sum_i \partial D_{ij} / \partial x_i; \mathfrak{D} \cdot \nabla = \sum_j \partial D_{ij} / \partial x_j$

TABLE I
Trigonometric and hyperbolic functions [see Eqs. (1.2.3) and (1.2.10)]

x	$\sin(x)$	$\cos(x)$	$\tan(x)$	$\sinh(x)$	$\cosh(x)$	$\tanh(x)$	e^x	e^{-x}
0.0	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	1.0000	1.0000
0.2	0.1987	0.9801	0.2127	0.2013	1.0201	0.1974	1.2214	0.8187
0.4	0.3894	0.9211	0.4228	0.4018	1.0811	0.3799	1.4918	0.6703
0.6	0.5646	0.8253	0.6841	0.6367	1.1855	0.5370	1.8221	0.5488
0.8	0.7174	0.6967	1.0296	0.8881	1.3374	0.6640	2.2255	0.4493
1.0	0.8415	0.5403	1.5574	1.1752	1.5431	0.7616	2.7183	0.3679
1.2	0.9320	0.3624	2.5722	1.5095	1.8106	0.8337	3.3201	0.3012
1.4	0.9854	+0.1700	+5.7979	1.9043	2.1509	0.8854	4.0552	0.2466
1.6	0.9996	-0.0292	-34.233	2.3756	2.5775	0.9217	4.9530	0.2019
1.8	0.9738	-0.2272	-4.2863	2.9422	3.1075	0.9468	6.0496	0.1553
2.0	0.9093	-0.4161	-2.1850	3.6269	3.7622	0.9640	7.3891	0.1353
2.2	0.8085	-0.5885	-1.3738	4.4571	4.5679	0.9757	9.0250	0.1108
2.4	0.6755	-0.7374	-0.9160	5.4662	5.5569	0.9837	11.023	0.0907
2.6	0.5155	-0.8569	-0.6016	6.6947	6.7690	0.9890	13.464	0.0742
2.8	0.3350	-0.9422	-0.3555	8.1919	8.2527	0.9926	16.445	0.0608
3.0	+0.1411	-0.9900	-0.1425	10.018	10.068	0.9951	20.086	0.0498
3.2	-0.0584	-0.9983	+0.0585	12.246	12.287	0.9967	24.533	0.0407
3.4	-0.2555	-0.9668	0.2643	14.965	14.999	0.9978	29.964	0.0333
3.6	-0.4425	-0.8968	0.4935	18.285	18.313	0.9985	36.598	0.0273
3.8	-0.6119	-0.7910	0.7736	22.339	22.362	0.9990	44.701	0.0223
4.0	-0.7568	-0.6536	1.1578	27.290	27.308	0.9993	54.598	0.0183
4.2	-0.8716	-0.4903	1.7778	33.335	33.351	0.9996	66.686	0.0150
4.4	-0.9516	-0.3073	3.0963	40.719	40.732	0.9997	81.451	0.0123
4.6	-0.9937	-0.1122	+8.8602	49.737	49.747	0.9998	99.484	0.0100
4.8	-0.9962	+0.0875	-11.385	60.751	60.759	0.9999	121.51	0.0082
5.0	-0.9589	0.2837	-3.3805	74.203	74.210	0.9999	148.41	0.0067
5.2	-0.8835	0.4685	-1.8856	90.633	90.639	0.9999	181.27	0.0055
5.4	-0.7728	0.6347	-1.2175	110.70	110.71	1.0000	221.41	0.0045
5.6	-0.6313	0.7756	-0.8139	135.21	135.22	1.0000	270.43	0.0037
5.8	-0.4646	0.8855	-0.5247	165.15	165.15	1.0000	330.30	0.0030
6.0	-0.2794	0.9602	-0.2910	201.71	201.71	1.0000	403.43	0.0025
6.2	-0.0831	0.9965	-0.0834	246.37	246.37	1.0000	492.75	0.0020
6.4	+0.1165	0.9932	+0.1173	300.92	300.92	1.0000	601.85	0.0016
6.6	0.3115	0.9502	0.3279	367.55	367.55	1.0000	735.10	0.0013
6.8	0.4941	0.8694	0.5683	448.92	448.92	1.0000	897.85	0.0011
7.0	0.6570	0.7539	0.8714	548.32	548.32	1.0000	1096.6	0.0009
7.2	0.7937	0.6084	1.3046	669.72	669.72	1.0000	1339.4	0.0007
7.4	0.8987	0.4385	2.0493	817.99	817.99	1.0000	1636.0	0.0006
7.6	0.9679	0.2513	3.8523	999.10	999.10	1.0000	1998.2	0.0005
7.8	0.9985	+0.0540	+18.507	1220.3	1220.3	1.0000	2440.6	0.0004
8.0	0.9894	-0.1455	-6.7997	1490.5	1490.5	1.0000	2981.0	0.0003

TABLE II

Trigonometric and hyperbolic functions [see Eqs. (1.2.3) and (1.2.10)]

x	$\sin(\pi x)$	$\cos(\pi x)$	$\tan(\pi x)$	$\sinh(\pi x)$	$\cosh(\pi x)$	$\tanh(\pi x)$	$e^{\pi x}$	$e^{-\pi x}$
0.00	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	1.0000	1.0000
0.05	0.1564	0.9877	0.1584	0.1577	1.0124	0.1558	1.1701	0.8546
0.10	0.3090	0.9511	0.3249	0.3194	1.0498	0.3042	1.3691	0.7304
0.15	0.4540	0.8910	0.5095	0.4889	1.1131	0.4392	1.6019	0.6242
0.20	0.5878	0.8090	0.7265	0.6705	1.2040	0.5569	1.8745	0.5335
0.25	0.7071	0.7071	1.0000	0.8687	1.3246	0.6558	2.1933	0.4559
0.30	0.8090	0.5878	1.3764	1.0883	1.4780	0.7363	2.5663	0.3897
0.35	0.8910	0.4540	1.9626	1.3349	1.6679	0.8003	3.0028	0.3330
0.40	0.9511	0.3090	3.0777	1.6145	1.8991	0.8502	3.5136	0.2846
0.45	0.9877	+0.1564	+6.3137	1.9340	2.1772	0.8883	4.1111	0.2432
0.50	1.0000	0.0000	∞	2.3013	2.5092	0.9171	4.8105	0.2079
0.55	0.9877	-0.1564	-6.3137	2.7255	2.9032	0.9388	5.6287	0.1777
0.60	0.9511	-0.3090	-3.0777	3.2171	3.3689	0.9549	6.5861	0.1518
0.65	0.8910	-0.4540	-1.9626	3.7883	3.9180	0.9669	7.7062	0.1298
0.70	0.8090	-0.5878	-1.3764	4.4531	4.5640	0.9757	9.0170	0.1109
0.75	0.7071	-0.7071	-1.0000	5.2280	5.3228	0.9822	10.551	0.09478
0.80	0.5878	-0.8090	-0.7265	6.1321	6.2131	0.9870	12.345	0.08100
0.85	0.4540	-0.8910	-0.5095	7.1879	7.2572	0.9905	14.437	0.06922
0.90	0.3090	-0.9511	-0.3249	8.4214	8.4806	0.9930	16.902	0.05916
0.95	+0.1564	-0.9877	-0.1584	9.8632	9.9137	0.9949	19.777	0.05056
1.00	0.0000	-1.0000	0.0000	11.549	11.592	0.9962	23.141	0.04321
1.05	-0.1564	-0.9877	+0.1584	13.520	13.557	0.9973	27.077	0.03693
1.10	-0.3090	-0.9511	0.3249	15.825	15.857	0.9980	31.682	0.03156
1.15	-0.4540	-0.8910	0.5095	18.522	18.549	0.9985	37.070	0.02697
1.20	-0.5878	-0.8090	0.7265	21.677	21.700	0.9989	43.376	0.02305
1.25	-0.7071	-0.7071	1.0000	25.367	25.387	0.9992	50.753	0.01970
1.30	-0.8090	-0.5878	1.3764	29.685	29.702	0.9994	59.387	0.01683
1.35	-0.8910	-0.4540	1.9626	34.737	34.751	0.9996	69.484	0.01438
1.40	-0.9511	-0.3090	3.0777	40.647	40.660	0.9997	81.307	0.01230
1.45	-0.9877	-0.1564	+6.3137	47.563	47.573	0.9998	95.137	0.01051
1.50	-1.0000	0.0000	∞	55.654	55.663	0.9998	111.32	0.00898
1.55	-0.9877	+0.1564	-6.3137	65.122	65.130	0.9999	130.25	0.00767
1.60	-0.9511	0.3090	-3.0777	76.200	76.206	0.9999	152.41	0.00656
1.65	-0.8910	0.4540	-1.9626	89.161	89.167	0.9999	178.33	0.00561
1.70	-0.8090	0.5878	-1.3764	104.32	104.33	1.0000	208.66	0.00479
1.75	-0.7071	0.7071	-1.0000	122.07	122.08	1.0000	244.15	0.00409
1.80	-0.5878	0.8090	-0.7265	142.84	142.84	1.0000	285.68	0.00350
1.85	-0.4540	0.8910	-0.5095	167.13	167.13	1.0000	334.27	0.00299
1.90	-0.3090	0.9511	-0.3249	195.56	195.56	1.0000	391.12	0.00256
1.95	-0.1564	0.9877	-0.1584	228.82	228.82	1.0000	457.65	0.00219
2.00	0.0000	1.0000	0.0000	267.75	267.75	1.0000	535.49	0.00187

TABLE III
Bessel functions for cylindrical coordinates
 $J_n(x)$ and $N_n(x)$
[see Eqs. (1.2.5), (5.2.20), (5.3.15), and (7.3.2)]

x	$J_0(x)$	$N_0(x)$	$J_1(x)$	$N_1(x)$	$J_2(x)$	$N_2(x)$
0.0	1.0000	$-\infty$	0.0000	$-\infty$	0.0000	$-\infty$
0.1	0.9975	-1.5342	0.0499	-6.4590	0.0012	-127.64
0.2	0.9900	-1.0811	0.0995	-3.3238	0.0050	-32.157
0.4	0.9604	-0.6060	0.1960	-1.7809	0.0197	-8.2983
0.6	0.9120	-0.3085	0.2867	-1.2604	0.0437	-3.8928
0.8	0.8463	-0.0868	0.3688	-0.9781	0.0758	-2.3586
1.0	0.7652	+0.0883	0.4401	-0.7812	0.1149	-1.6507
1.2	0.6711	0.2281	0.4983	-0.6211	0.1593	-1.2633
1.4	0.5669	0.3379	0.5419	-0.4791	0.2074	-1.0224
1.6	0.4554	0.4204	0.5699	-0.3476	0.2570	-0.8549
1.8	0.3400	0.4774	0.5815	-0.2237	0.3061	-0.7259
2.0	0.2239	0.5104	0.5767	-0.1070	0.3528	-0.6174
2.2	0.1104	0.5208	0.5560	+0.0015	0.3951	-0.5194
2.4	+0.0025	0.5104	0.5202	0.1005	0.4310	-0.4267
2.6	-0.0968	0.4813	0.4708	0.1884	0.4590	-0.3364
2.8	-0.1850	0.4359	0.4097	0.2635	0.4777	-0.2477
3.0	-0.2601	0.3768	0.3391	0.3247	0.4861	-0.1604
3.2	-0.3202	0.3071	0.2613	0.3707	0.4835	-0.0754
3.4	-0.3643	0.2296	0.1792	0.4010	0.4697	+0.0063
3.6	-0.3918	0.1477	0.0955	0.4154	0.4448	0.0831
3.8	-0.4026	+0.0645	+0.0128	0.4141	0.4093	0.1535
4.0	-0.3971	-0.0169	-0.0660	0.3979	0.3641	0.2159
4.2	-0.3766	-0.0938	-0.1386	0.3680	0.3105	0.2690
4.4	-0.3423	-0.1633	-0.2028	0.3260	0.2501	0.3115
4.6	-0.2961	-0.2235	-0.2566	0.2737	0.1846	0.3425
4.8	-0.2404	-0.2723	-0.2985	0.2136	0.1161	0.3613
5.0	-0.1776	-0.3085	-0.3276	0.1479	+0.0466	0.3677
5.2	-0.1103	-0.3312	-0.3432	0.0792	-0.0217	0.3617
5.4	-0.0412	-0.3402	-0.3453	+0.0101	-0.0867	0.3429
5.6	+0.0270	-0.3354	-0.3343	-0.0568	-0.1464	0.3152
5.8	0.0917	-0.3177	-0.3110	-0.1192	-0.1989	0.2766
6.0	0.1507	-0.2882	-0.2767	-0.1750	-0.2429	0.2299
6.2	0.2017	-0.2483	-0.2329	-0.2223	-0.2769	0.1766
6.4	0.2433	-0.2000	-0.1816	-0.2596	-0.3001	0.1188
6.6	0.2740	-0.1452	-0.1250	-0.2858	-0.3119	+0.0586
6.8	0.2931	-0.0864	-0.0652	-0.3002	-0.3123	-0.0019
7.0	0.3001	-0.0259	-0.0047	-0.3027	-0.3014	-0.0605
7.2	0.2951	+0.0339	+0.0543	-0.2934	-0.2800	-0.1154
7.4	0.2786	0.0907	0.1096	-0.2731	-0.2487	-0.1652
7.6	0.2516	0.1424	0.1592	-0.2428	-0.2097	-0.2063
7.8	0.2154	0.1872	0.2014	-0.2039	-0.1638	-0.2395
8.0	0.1716	0.2235	0.2346	-0.1581	-0.1130	-0.2630

TABLE IV
Hyperbolic Bessel functions
 $I_m(z) = i^{-m} J_m(iz)$
[see Eq. (5.3.3)]

z	$I_0(z)$	$I_1(z)$	$I_2(z)$
0.0	1.0000	0.0000	0.0000
0.1	1.0025	0.0501	0.0012
0.2	1.0100	0.1005	0.0050
0.4	1.0404	0.2040	0.0203
0.6	1.0921	0.3137	0.0464
0.8	1.1665	0.4329	0.0843
1.0	1.2661	0.5652	0.1358
1.2	1.3937	0.7147	0.2026
1.4	1.5534	0.8861	0.2876
1.6	1.7500	1.0848	0.3940
1.8	1.9895	1.3172	0.5260
2.0	2.2796	1.5906	0.6890
2.2	2.6292	1.9141	0.8891
2.4	3.0492	2.2981	1.1111
2.6	3.5532	2.7554	1.4338
2.8	4.1574	3.3011	1.7994
3.0	4.8808	3.9534	2.2452
3.2	5.7472	4.7343	2.7884
3.4	6.7848	5.6701	3.4495
3.6	8.0278	6.7926	4.2538
3.8	9.5169	8.1405	5.2323
4.0	11.302	9.7594	6.4224
4.2	13.443	11.705	7.8683
4.4	16.010	14.046	9.6259
4.6	19.097	16.863	11.761
4.8	22.794	20.253	14.355
5.0	27.240	24.335	17.505
5.2	32.584	29.254	21.332
5.4	39.010	35.181	25.980
5.6	46.738	42.327	31.621
5.8	56.039	50.945	38.472
6.0	67.235	61.341	46.788
6.2	80.717	73.888	56.882
6.4	96.963	89.025	69.143
6.6	116.54	107.31	84.021
6.8	140.14	129.38	102.08
7.0	168.59	156.04	124.01
7.2	202.92	188.25	150.63
7.4	244.34	227.17	182.94
7.6	294.33	274.22	222.17
7.8	354.68	331.10	269.79
8.0	427.57	399.87	327.60

TABLE V
Phase angles and amplitudes for radiation and scattering from a cylinder
 $ka = 2\pi a/\lambda = \omega a/c$
[see Eqs. (7.3.7)]

ka	E_0	γ_0	E_1	γ_1	E_2	γ_2	E_3	γ_3	E_4	γ_4
0.0	∞	0.00°	∞	0.00°	∞	0.00°	∞	0.00°	∞	0.00°
0.1	12.92	0.44	63.06	— 0.45	2546	0.00	—	0.00	—	0.00
0.2	6.651	1.71	15.55	— 1.82	318.2	— 0.01	9565	0.00	—	0.00
0.4	3.583	6.28	3.875	— 6.97	39.71	— 0.14	600.7	0.00	—	0.00
0.6	2.585	12.82	1.844	— 13.62	11.72	— 0.69	119.6	— 0.01	1595	0.00
0.8	2.091	20.66	1.199	— 18.73	4.922	— 2.09	38.20	— 0.06	382.9	0.00
1.0	1.793	29.39	0.9283	— 20.50	2.529	— 4.77	15.81	— 0.20	127.3	0.00
1.2	1.593	38.74	0.7884	— 18.94	1.503	— 8.91	7.712	— 0.57	52.03	— 0.02
1.4	1.447	48.52	0.7035	— 14.80	1.012	— 14.06	4.212	— 1.35	24.54	— 0.06
1.6	1.335	58.62	0.6453	— 8.84	0.7627	— 19.03	2.504	— 2.77	12.85	— 0.16
1.8	1.246	68.96	0.6019	— 1.61	0.6309	— 22.49	1.596	— 5.08	7.290	— 0.37
2.0	1.173	79.49	0.5676	+ 6.52	0.5573	— 23.69	1.086	— 8.44	4.405	— 0.79
2.2	1.112	90.15	0.5392	15.31	0.5130	— 22.56	0.7898	— 12.71	2.801	— 1.55
2.4	1.060	100.93	0.5152	24.57	0.4836	— 19.45	0.6158	— 17.32	1.861	— 2.80
2.6	1.014	111.81	0.4944	34.20	0.4624	— 14.75	0.5136	— 21.41	1.287	— 4.73
2.8	0.9743	122.75	0.4760	44.11	0.4457	— 8.84	0.4535	— 24.15	0.9265	— 7.46
3.0	0.9389	133.76	0.4597	54.24	0.4319	— 1.99	0.4175	— 25.09	0.6965	— 11.01
3.2	0.9071	144.82	0.4450	64.55	0.4198	+ 5.59	0.3952	— 24.19	0.5496	— 15.13
3.4	0.8785	155.92	0.4317	75.01	0.4090	13.73	0.3804	— 21.64	0.4566	— 19.29
3.6	0.8524	167.06	0.4195	85.58	0.3992	2.33	0.3698	— 17.71	0.3987	— 22.81
3.8	0.8286	178.23	0.4084	96.25	0.3901	31.29	0.3617	— 12.66	0.3631	— 25.11
4.0	0.8067	189.42	0.3980	107.01	0.3816	40.55	0.3549	— 6.72	0.3412	— 25.90
4.2	0.7865	200.64	0.3885	117.83	0.3737	50.06	0.3489	— 0.04	0.3275	— 25.14
4.4	0.7678	211.88	0.3796	128.72	0.3662	59.77	0.3434	+ 7.22	0.3187	— 22.95
4.6	0.7503	223.14	0.3713	139.65	0.3592	69.66	0.3383	14.97	0.3126	— 19.54
4.8	0.7341	234.42	0.3635	150.64	0.3525	79.70	0.3334	23.13	0.3081	— 15.10
5.0	0.7188	245.71	0.3562	161.66	0.3462	89.87	0.3287	31.62	0.3044	— 9.81
ka	E_5	γ_5	E_6	γ_6	E_7	γ_7	E_8	γ_8	E_9	γ_9
2.0	22.07	— 0.04°	130.8	0.00°	903.5	0.00°	7144	0.00	—	0.00
2.2	12.82	— 0.10	68.99	0.00	432.1	0.00	3099	0.00	—	0.00
2.4	7.834	— 0.22	38.65	— 0.01	221.4	0.00	1452	0.00	—	0.00
2.6	4.999	— 0.45	22.78	— 0.03	120.2	0.00	725.6	0.00	4941	0.00
2.8	3.309	— 0.86	14.03	— 0.06	68.58	0.00	383.4	0.00	2418	0.00
3.0	2.261	— 1.53	8.967	— 0.13	40.86	— 0.01	212.6	0.00	1248	0.00
3.2	1.590	— 2.59	5.922	— 0.25	25.27	— 0.02	122.9	0.00	674.6	0.00
3.4	1.149	— 4.18	4.025	— 0.47	16.16	— 0.04	73.80	0.00	380.0	0.00
3.6	0.8534	— 6.41	2.805	— 0.83	10.65	— 0.07	45.80	0.00	222.1	0.00
3.8	0.6539	— 9.35	2.000	— 1.41	7.200	— 0.14	29.29	— 0.01	134.1	0.00
4.0	0.5190	— 12.92	1.456	— 2.30	4.985	— 0.26	19.24	— 0.02	83.43	0.00
4.2	0.4287	— 16.83	1.082	— 3.60	3.526	— 0.46	12.95	— 0.04	53.32	0.00
4.4	0.3693	— 20.62	0.8211	— 5.42	2.542	— 0.77	8.907	— 0.08	34.92	— 0.01
4.6	0.3312	— 23.74	0.6374	— 7.85	1.865	— 1.26	6.252	— 0.14	23.40	— 0.01
4.8	0.3071	— 25.76	0.5081	— 10.88	1.391	— 2.00	4.471	— 0.25	16.00	— 0.02
5.0	0.2921	— 26.44	0.4177	— 14.40	1.054	— 3.06	3.251	— 0.42	11.15	— 0.04

TABLE VI

Legendre functions for spherical coordinates [see Eqs. (7.2.4)]

ϑ	$P_{-1} = P_0$	$P_1(\cos \vartheta)$	$P_2(\cos \vartheta)$	$P_3(\cos \vartheta)$	$P_4(\cos \vartheta)$
0°	1.0000	1.0000	1.0000	1.0000	1.0000
5	1.0000	0.9962	0.9886	0.9773	0.9623
10	1.0000	0.9848	0.9548	0.9106	0.8352
15	1.0000	0.9659	0.8995	0.8042	0.6847
20	1.0000	0.9397	0.8245	0.6649	0.4750
25	1.0000	0.9063	0.7321	0.5016	0.2465
30	1.0000	0.8660	0.6250	0.3248	+0.0234
35	1.0000	0.8192	0.5065	+0.1454	-0.1714
40	1.0000	0.7660	0.3802	-0.0252	-0.3190
45	1.0000	0.7071	0.2500	-0.1768	-0.4063
50	1.0000	0.6428	+0.1198	-0.3002	-0.4275
55	1.0000	0.5736	-0.0065	-0.3886	-0.3852
60	1.0000	0.5000	-0.1250	-0.4375	-0.2891
65	1.0000	0.4226	-0.2321	-0.4452	-0.1552
70	1.0000	0.3420	-0.3245	-0.4130	-0.0038
75	1.0000	0.2588	-0.3995	-0.3449	+0.1434
80	1.0000	0.1736	-0.4548	-0.2474	0.2659
85	1.0000	0.0872	-0.4886	-0.1291	0.3468
90	1.0000	0.0000	-0.5000	0.0000	0.3750
ϑ	$P_5(\cos \vartheta)$	$P_6(\cos \vartheta)$	$P_7(\cos \vartheta)$	$P_8(\cos \vartheta)$	$P_9(\cos \vartheta)$
0°	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.9437	0.9216	0.8962	0.8675	0.8358
10	0.7840	0.7045	0.6164	0.5218	+0.4228
15	0.5471	0.3983	+0.2455	+0.0962	-0.0428
20	0.2715	+0.0719	-0.1072	-0.2518	-0.3517
25	+0.0009	-0.2040	-0.3441	-0.4062	-0.3896
30	-0.2233	-0.3740	-0.4102	-0.3388	-0.1896
35	-0.3691	-0.4114	-0.3096	-0.1154	+0.0965
40	-0.4197	-0.3236	-0.1006	+0.1386	0.2900
45	-0.3757	-0.1484	+0.1271	0.2983	0.2855
50	-0.2545	+0.0564	0.2854	0.2947	+0.1041
55	-0.0868	0.2297	0.3191	+0.1422	-0.1296
60	+0.0898	0.3232	0.2231	-0.0763	-0.2679
65	0.2381	0.3138	+0.0422	-0.2411	-0.2300
70	0.3281	0.2089	-0.1485	-0.2780	-0.0476
75	0.3427	+0.0431	-0.2731	-0.1702	+0.1595
80	0.2810	-0.1321	-0.2835	+0.0233	0.2596
85	0.1577	-0.2638	-0.1778	0.2017	0.1913
90	0.0000	-0.3125	0.0000	0.2734	0.0000

TABLE VII

Bessel functions for spherical coordinates

$$j_n(x) = \sqrt{\pi/2x} J_{n+1/2}(x), \quad n_n(x) = \sqrt{\pi/2x} N_{n+1/2}(x)$$

[see Eqs. (7.2.11)]

x	$j_0(x)$	$n_0(x)$	$j_1(x)$	$n_1(x)$	$j_2(x)$	$n_2(x)$
0.0	1.0000	$-\infty$	0.0000	$-\infty$	0.0000	$-\infty$
0.1	0.9983	-9.9500	0.0333	-100.50	0.0007	-3005.0
0.2	0.9933	-4.9003	0.0664	-25.495	0.0027	-377.52
0.4	0.9735	-2.3027	0.1312	-6.7302	0.0105	-48.174
0.6	0.9411	-1.3756	0.1929	-3.2337	0.0234	-14.793
0.8	0.8967	-0.8709	0.2500	-1.9853	0.0408	-6.5740
1.0	0.8415	-0.5403	0.3012	-1.3818	0.0620	-3.6050
1.2	0.7767	-0.3020	0.3453	-1.0283	0.0865	-2.2689
1.4	0.7039	-0.1214	0.3814	-0.7906	0.1133	-1.5728
1.6	0.6247	+0.0183	0.4087	-0.6133	0.1416	-1.1682
1.8	0.5410	0.1262	0.4268	-0.4709	0.1703	-0.9111
2.0	0.4546	0.2081	0.4354	-0.3506	0.1985	-0.7340
2.2	0.3675	0.2675	0.4346	-0.2459	0.2251	-0.6028
2.4	0.2814	0.3072	0.4245	-0.1534	0.2494	-0.4990
2.6	0.1983	0.3296	0.4058	-0.0715	0.2700	-0.4121
2.8	0.1196	0.3365	0.3792	+0.0005	0.2867	-0.3359
3.0	+0.0470	0.3300	0.3457	0.0630	0.2986	-0.2670
3.2	-0.0182	0.3120	0.3063	0.1157	0.3084	-0.2035
3.4	-0.0752	0.2844	0.2623	0.1588	0.3066	-0.1442
3.6	-0.1229	0.2491	0.2150	0.1921	0.3021	-0.0890
3.8	-0.1610	0.2082	0.1658	0.2158	0.2919	-0.0378
4.0	-0.1892	0.1634	0.1161	0.2300	0.2763	+0.0091
4.2	-0.2075	0.1167	0.0673	0.2353	0.2556	0.0514
4.4	-0.2163	0.0699	+0.0207	0.2321	0.2304	0.0884
4.6	-0.2160	+0.0244	-0.0226	0.2213	0.2013	0.1200
4.8	-0.2075	-0.0182	-0.0615	0.2037	0.1691	0.1456
5.0	-0.1918	-0.0567	-0.0951	0.1804	0.1347	0.1650
5.2	-0.1699	-0.0901	-0.1228	0.1526	0.0991	0.1871
5.4	-0.1431	-0.1175	-0.1440	0.1213	0.0631	0.1850
5.6	-0.1127	-0.1385	-0.1586	0.0880	+0.0278	0.1856
5.8	-0.0801	-0.1527	-0.1665	0.0538	-0.0060	0.1805
6.0	-0.0466	-0.1600	-0.1678	+0.0199	-0.0373	0.1700
6.2	-0.0134	-0.1607	-0.1629	-0.0124	-0.0654	0.1547
6.4	+0.0182	-0.1552	-0.1523	-0.0425	-0.0896	0.1353
6.6	0.0472	-0.1440	-0.1368	-0.0690	-0.1094	0.1126
6.8	0.0727	-0.1278	-0.1172	-0.0915	-0.1243	0.0875
7.0	0.0939	-0.1077	-0.0943	-0.1029	-0.1343	0.0609
7.2	0.1102	-0.0845	-0.0692	-0.1220	-0.1391	0.0337
7.4	0.1215	-0.0593	-0.0429	-0.1294	-0.1388	+0.0068
7.6	0.1274	-0.0331	-0.0163	-0.1317	-0.1338	-0.0189
7.8	0.1280	-0.0069	+0.0095	-0.1289	-0.1244	-0.0427
8.0	0.1237	+0.0182	0.0336	-0.1214	-0.1111	-0.0637

TABLE VIII

Phase angles and amplitudes for radiation and scattering from a sphere

$$ka = 2\pi a/\lambda = \omega a/c$$

[see Eq. (7.2.15)]

ka	B_0	δ_0	B_1	δ_1	B_2	δ_2	B_3	δ_3	B_4	δ_4
0.0	∞	0.00°	∞	0.00°	∞	0.00°	∞	0.00°	∞	0.00°
0.1	100.5	0.02	2000	-0.01	—	0.00	—	0.00	—	0.00
0.2	25.50	0.15	250.1	-0.08	5637	0.00	—	0.00	—	0.00
0.4	6.731	1.12	31.35	-0.58	354.6	-0.01	5906	0.00	—	0.00
0.6	3.239	3.41	9.408	-1.82	70.73	-0.06	785.5	0.00	—	0.00
0.8	2.001	7.18	4.101	-3.80	22.67	-0.25	188.9	-0.01	2058	0.00
1.0	1.414	12.30	2.236	-6.14	9.434	-0.70	62.97	-0.02	547.8	0.00
1.2	1.085	18.56	1.426	-8.11	4.646	-1.59	25.82	-0.08	186.9	0.00
1.4	0.8778	25.75	1.021	-8.97	2.583	-3.07	12.22	-0.22	75.74	-0.01
1.6	0.7370	33.68	0.7931	-8.25	1.584	-5.19	6.426	-0.51	34.84	-0.02
1.8	0.6355	42.19	0.6529	-5.87	1.057	-7.77	3.667	-1.05	17.66	-0.06
2.0	0.5590	51.16	0.5590	-1.97	0.7629	-10.40	2.236	-1.97	9.669	-0.15
2.2	0.4993	60.49	0.4918	+3.21	0.5901	-12.49	1.444	-3.38	5.635	-0.32
2.4	0.4514	70.13	0.4411	9.44	0.4837	-13.51	0.9823	-5.34	3.459	-0.64
2.6	0.4121	80.01	0.4011	16.50	0.4148	-13.14	0.7036	-7.80	2.220	-1.18
2.8	0.3792	90.08	0.3686	24.23	0.3676	-11.31	0.5308	-10.54	1.481	-2.02
3.0	0.3514	100.32	0.3415	32.49	0.3333	-8.11	0.4214	-13.16	1.024	-3.27
3.2	0.3274	110.70	0.3184	41.18	0.3071	-3.73	0.3508	-15.17	0.7334	-5.00
3.4	0.3066	121.20	0.2985	50.23	0.2862	+1.65	0.3042	-16.17	0.5443	-7.23
3.6	0.2883	131.79	0.2811	59.57	0.2688	7.86	0.2723	-15.94	0.4195	-9.83
3.8	0.2721	142.47	0.2657	69.15	0.2540	14.75	0.2496	-14.41	0.3364	-12.58
4.0	0.2577	153.22	0.2519	78.92	0.2411	22.20	0.2326	-11.67	0.2807	-15.10
4.2	0.2448	164.03	0.2396	88.88	0.2296	30.12	0.2193	-7.84	0.2432	-17.00
4.4	0.2331	174.91	0.2285	98.97	0.2194	38.44	0.2084	-3.08	0.2174	-17.95
4.6	0.2225	185.83	0.2184	109.20	0.2101	47.08	0.1992	+2.47	0.1994	-17.79
4.8	0.2128	196.79	0.2091	119.55	0.2016	56.00	0.1912	8.70	0.1863	-16.47
5.0	0.2040	207.79	0.2006	129.98	0.1939	65.16	0.1840	15.48	0.1764	-14.04
ka	B_5	δ_5	B_6	δ_6	B_7	δ_7	B_8	δ_8	B_9	δ_9
2.0	51.31	-0.01°	323.7	0.00°	2370	0.00°	—	0.00°	—	0.00°
2.2	27.14	-0.02	155.2	0.00	1030	0.00	7790	0.00	—	0.00
2.4	15.25	-0.04	79.69	0.00	483.5	0.00	3343	0.00	—	0.00
2.6	9.021	-0.10	43.38	-0.01	242.2	0.00	1541	0.00	—	0.00
2.8	5.573	-0.20	24.83	-0.01	128.2	0.00	755.6	0.00	5002	0.00
3.0	3.576	-0.37	14.83	-0.03	71.28	0.00	390.7	0.00	2407	0.00
3.2	2.371	-0.68	9.206	-0.06	41.34	0.00	211.7	0.00	1219	0.00
3.4	1.620	-1.16	5.907	-0.11	24.88	-0.01	119.5	0.00	645.8	0.00
3.6	1.137	-1.91	3.904	-0.21	15.49	-0.02	70.01	0.00	356.1	0.00
3.8	0.8183	-2.99	2.649	-0.38	9.933	-0.03	42.39	0.00	203.6	0.00
4.0	0.6043	-4.48	1.841	-0.66	6.545	-0.06	26.44	0.00	120.2	0.00
4.2	0.4583	-6.43	1.308	-1.09	4.418	-0.12	16.94	-0.01	73.09	0.00
4.4	0.3443	-8.79	0.9486	-1.73	3.050	-0.22	11.13	-0.02	45.66	0.00
4.6	0.2881	-11.45	0.7015	-2.65	2.148	-0.37	7.479	-0.04	29.24	0.00
4.8	0.2399	-14.15	0.5290	-3.92	1.542	-0.61	5.130	-0.07	19.16	0.00
5.0	0.2065	-16.56	0.4072	-5.58	1.126	-0.98	3.587	-0.12	12.82	-0.01

TABLE IX

Impedance functions for piston in infinite plane wall

$$\theta_0 - i\chi_0 = 1 - (2/w)J_1(w) - iM(w) = \tanh [\pi(\alpha_p - i\beta_p)]$$

$$w = 4\pi a/\lambda$$

[see Eqs. (7.4.31)]

w	θ_0	χ_0	α_p	β_p
0.0	0.0000	0.0000	0.0000	0.0000
0.5	0.0309	0.2087	0.0094	0.0655
1.0	0.1199	0.3969	0.0330	0.1216
1.5	0.2561	0.5471	0.0628	0.1663
2.0	0.4233	0.6468	0.0939	0.2020
2.5	0.6023	0.6905	0.1247	0.2316
3.0	0.7740	0.6801	0.1552	0.2572
3.5	0.9215	0.6238	0.1858	0.2800
4.0	1.0330	0.5349	0.2175	0.3008
4.5	1.1027	0.4293	0.2517	0.3194
5.0	1.1310	0.3231	0.2899	0.3353
5.5	1.1242	0.2300	0.3344	0.3460
6.0	1.0922	0.1594	0.3868	0.3456
6.5	1.0473	0.1159	0.4450	0.3207
7.0	1.0013	0.0989	0.4788	0.2600
7.5	0.9639	0.1036	0.4594	0.2050
8.0	0.9413	0.1220	0.4241	0.1887
8.5	0.9357	0.1456	0.3980	0.1958
9.0	0.9454	0.1663	0.3839	0.2132
9.5	0.9661	0.1782	0.3799	0.2344
10.0	0.9913	0.1784	0.3845	0.2565
10.5	1.0150	0.1668	0.3964	0.2774
11.0	1.0321	0.1464	0.4153	0.2958
11.5	1.0397	0.1216	0.4410	0.3097
12.0	1.0372	0.0973	0.4734	0.3158
12.5	1.0265	0.0779	0.5101	0.3083
13.0	1.0108	0.0662	0.5421	0.2810
13.5	0.9944	0.0631	0.5490	0.2409
14.0	0.9809	0.0676	0.5316	0.2117
14.5	0.9733	0.0770	0.5073	0.2032
15.0	0.9727	0.0881	0.4877	0.2092
15.5	0.9784	0.0973	0.4758	0.2231
16.0	0.9887	0.1021	0.4718	0.2406
16.5	1.0007	0.1013	0.4750	0.2591
17.0	1.0115	0.0948	0.4852	0.2767
17.5	1.0187	0.0843	0.5017	0.2914
18.0	1.0209	0.0719	0.5247	0.3007
18.5	1.0180	0.0602	0.5522	0.3010
19.0	1.0111	0.0515	0.5798	0.2879
19.5	1.0021	0.0470	0.5968	0.2610
20.0	0.9933	0.0473	0.5940	0.2314

TABLE X

Zeros and associated values of Bessel functions

$$J_m(j_{mn}) = 0; j_{mn} = \pi\beta_{mn}; J'_m(j'_{mn}) = 0; j'_{mn} = \pi\alpha_{mn}$$

n	j_{0n}	$J'_0(j_{0n})$	j_{1n}	$J'_1(j_{1n})$	j_{2n}	$J'_2(j_{2n})$	j_{3n}	$J'_3(j_{3n})$
0			0	0.5000	0	0	0	0
1	2.4048	-0.5191	3.8317	-0.4028	5.1356	-0.3397	6.3802	-0.2983
2	5.5201	+0.3403	7.0156	+0.3001	8.4172	+0.2714	9.7610	+0.2494
3	8.6537	-0.2715	10.1735	-0.2497	11.6198	-0.2324	13.0152	-0.2183
4	11.7915	+0.2325	13.3237	+0.2184	14.7960	+0.2065	16.2235	+0.1964

$$j_{mn} \rightarrow \pi(n + \frac{1}{2}m - \frac{1}{4}) - \frac{4m^2 - 1}{8\pi(n + \frac{1}{2}m - \frac{1}{4})} \quad n \gg m$$

n	j'_{0n}	$J_0(j'_{0n})$	$J'_1(j'_{1n})$	$J_1(j'_{1n})$	j'_{2n}	$J_2(j'_{2n})$	j'_{3n}	$J_3(j'_{3n})$
0	0	1.0000	1.8412	+0.5819	3.0542	+0.4865	4.2012	+0.4344
1	3.8317	-0.4028	5.3314	-0.3461	6.7061	-0.3135	8.0152	-0.2912
2	7.0156	+0.3001	8.5363	+0.2733	9.9695	+0.2547	11.3459	+0.2407
3	10.1735	-0.2497	11.7060	-0.2333	13.1704	-0.2209	14.5859	-0.2110
4	13.3237	+0.2184	14.8636	+0.2070	16.3475	+0.1979	17.7888	+0.1904

$$j'_{mn} \rightarrow \pi(n + \frac{1}{2}m - \frac{3}{4}) - \frac{4m^2 + 3}{8\pi(n + \frac{1}{2}m - \frac{3}{4})} \quad n \gg m$$

TABLE XI

Angle-distribution functions for transmission through a membrane
values of $F_n^m(x)$

[see Eq. (10.2.14)]

x	$F_1^0(x)$	$F_2^0(x)$	$F_3^0(x)$	$F_1^1(x)$	$F_2^1(x)$	$F_3^1(x)$	$F_1^2(x)$	$F_2^2(x)$	$F_3^2(x)$
0.0	1.6020	-1.0648	0.8514	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5	1.5714	-1.0076	0.8017	+0.3194	-0.2313	+0.1912	+0.0354	-0.0269	+0.0227
1.0	1.4821	-0.8424	0.6603	0.6120	-0.4267	0.3498	0.1369	-0.1020	0.0857
1.5	1.3421	-0.5884	0.4493	0.8539	-0.5554	0.4490	0.2909	-0.2099	0.1747
2.0	1.1633	-0.2744	+0.2014	1.0273	-0.5963	0.4723	0.4768	-0.3274	0.2692
2.5	0.9600	+0.0648	-0.0449	1.1217	-0.5409	0.4165	0.6702	-0.4283	0.3463
3.0	0.7489	0.3930	-0.2517	1.1354	-0.3941	0.2924	0.8460	-0.4875	0.3857
3.5	0.5446	0.6769	-0.3869	1.0749	-0.1737	+0.1227	0.9819	-0.4855	0.3735
4.0	0.3601	0.8904	-0.4300	0.9533	+0.0929	-0.0615	1.0614	-0.4118	0.3059
4.5	0.2053	1.0175	-0.3741	0.7896	0.3729	-0.2262	1.0756	-0.2671	0.1898
5.0	0.0856	1.0532	-0.2270	0.6041	0.6324	-0.3400	1.0254	-0.0630	+0.0423
5.5	+0.0026	1.0025	-0.0098	0.4173	0.8415	-0.3798	0.9154	+0.1793	-0.1120
6.0	-0.0462	0.8841	+0.2470	0.2649	0.9786	-0.3340	0.7630	0.4323	-0.2452
6.5	-0.0661	0.7165	0.5081	0.1062	1.0321	-0.2047	0.5856	0.6668	-0.3313
7.0	-0.0643	0.5255	0.7390	+0.0026	1.0015	-0.0070	0.4028	0.8558	-0.3503
7.5	-0.0489	0.3352	0.9112	-0.0619	0.8993	+0.2332	0.2331	0.9784	-0.2923
8.0	-0.0273	0.1661	1.0053	-0.0905	0.7421	0.0840	+0.0908	1.0233	-0.1591
8.5	-0.0058	+0.0326	1.0142	-0.0903	0.5544	0.7122	-0.0147	0.9887	+0.0356
9.0	-0.0111	-0.0580	0.9421	-0.0704	0.3609	0.8884	-0.0802	0.8852	0.2681
9.5	0.0213	-0.1053	0.8048	-0.0406	0.1837	0.9917	-0.1079	0.7286	0.5089
10.0	0.0242	-0.1148	0.6244	-0.0097	0.0400	1.0121	-0.1046	0.5419	0.7270

TABLE XII

Impedance functions for transmission through a membrane

values of $\theta_n^m(x)$ and of $\chi_n^m(x)$

[see Eq. (10.2.14)]

x	θ_{11}^m	χ_{11}^m	θ_{22}^m	χ_{22}^m	θ_{33}^m	χ_{33}^m	θ_{12}^m	χ_{12}^m	θ_{13}^m	χ_{13}^m	θ_{23}^m	χ_{23}^m
$m = 0$												
0	2.566x	5.541	1.134x	4.516	0.724x	3.704	-1.706x	-1.803	1.363x	1.481	-0.906x	-1.170
1	2.310	4.699	0.847	4.021	0.527	3.582	-1.399	-1.008	1.105	0.838	-0.668	-0.723
2	3.421	2.805	0.645	3.555	0.389	3.456	-1.409	+0.464	1.082	+0.283	-0.524	-0.231
3	3.204	1.099	0.526	4.301	0.285	3.700	-0.360	1.108	+0.323	-0.697	-0.387	-0.444
4	2.398	0.233	1.822	5.138	0.595	3.892	+0.477	0.664	-0.148	-0.384	-1.024	-0.440
5	1.722	0.018	3.867	4.451	0.578	3.986	0.501	+0.014	-0.098	-0.113	-1.385	+0.564
6	1.340	0.017	4.758	2.428	0.461	4.646	+0.166	-0.196	-0.026	-0.137	-0.490	1.499
7	1.116	0.014	4.147	0.711	1.625	5.910	-0.013	-0.098	-0.127	-0.138	+0.749	1.155
8	0.978	0.004	3.053	0.088	4.088	5.796	-0.014	-0.018	-0.173	-0.033	1.001	+0.101
9	0.845	0.001	2.401	0.060	5.770	3.647	-0.001	-0.013	-0.082	+0.091	0.428	-0.420
$m = 1$												
0	0.280x ³	4.072	0.151x ³	3.779	0.103x ³	3.582	-0.205x ³	-0.710	0.170x ³	0.615	-0.125x ³	-0.545
1	0.246	4.494	0.118	3.932	0.078	3.712	-0.170	-0.925	0.140	0.791	-0.097	-0.658
2	1.559	4.991	0.588	3.976	0.374	3.731	-0.953	-0.885	0.767	0.748	-0.471	-0.593
3	3.308	4.255	0.712	3.700	0.430	3.516	-1.505	+0.083	1.157	+0.000	-0.553	-0.222
4	4.144	2.468	0.434	4.050	0.261	3.638	-0.893	1.173	+0.687	-0.742	-0.338	-0.214
5	3.717	0.848	1.047	5.202	0.413	3.948	+0.239	1.214	-0.036	-0.700	-0.654	-0.480
6	2.810	0.137	3.087	5.562	0.630	4.007	0.759	+0.386	-0.218	-0.267	-1.352	+0.046
7	2.131	0.026	4.989	4.054	0.442	4.276	0.512	-0.242	-0.064	-0.149	-1.107	1.204
8	1.753	0.039	5.197	1.752	0.841	5.613	+0.042	-0.261	-0.108	-0.235	+0.244	1.649
9	1.539	0.020	4.146	0.358	2.932	6.555	-0.032	-0.079	-0.271	-0.126	1.194	0.723
$m = 2$												
0	0.010x ⁵	3.918	0.006x ⁵	3.633	0.006x ⁵	3.451	-0.009x ⁵	-0.465	0.007x ⁵	0.417	-0.005x ⁵	-0.378
1	0.009	4.108	0.005	3.696	0.004	3.585	-0.007	-0.533	0.006	0.472	-0.004	-0.416
2	0.257	4.752	0.116	3.939	0.090	3.702	-0.179	-0.771	0.148	0.664	-0.105	-0.538
3	1.316	5.405	0.492	4.000	0.315	3.794	-0.803	-0.845	0.647	0.702	-0.393	-0.536
4	3.144	5.204	0.720	3.902	0.429	3.687	-1.481	-0.147	1.137	+0.160	-0.554	-0.250
5	4.607	3.664	0.491	4.047	0.289	3.663	-1.254	+1.046	0.935	-0.650	-0.374	-0.117
6	4.683	1.677	0.667	5.153	0.315	4.069	-0.061	1.536	+0.171	-0.901	-0.457	-0.393
7	3.755	0.398	2.348	6.137	0.589	4.284	+0.917	0.852	-0.273	-0.486	-1.148	-0.263
8	2.840	0.051	4.740	5.374	0.539	4.358	0.861	-0.100	-0.158	-0.175	-1.426	+0.848
9	2.321	0.058	5.905	3.020	0.497	5.375	0.279	-0.403	-0.079	-0.253	-0.369	1.807

Entries for $x = 0$ give limiting values for $x \rightarrow 0$.

TABLE XIII

Transmission factor for transmission through a membrane

values of $H_m(uvw) = H_m(vuw) = H_m(wvu)$

[see Eq. (10.2.22)]

uvw	$H_m(uvw)$	uvw	$H_m(uvw)$	uvw	$H_m(uvw)$	uvw	$H_m(uvw)$
$m = 0$							
000	0.062500	044	0.004531	228	-0.000534	448	-0.000162
002	0.044104	046	0.001040	244	0.002575	466	-0.001249
004	0.011379	048	-0.000184	246	0.000088	468	-0.000267
006	-0.003373	066	0.001570	248	-0.000150	488	-0.000297
008	-0.000886	068	0.000440	266	0.000427	666	-0.001224
022	0.031701	088	0.000582	268	0.000124	668	-0.000195
024	0.009236	222	0.022937	288	0.000147	688	0.000103
026	-0.001680	224	0.006954	444	-0.000625	888	0.000582
028	-0.000715	226	-0.001020	446	-0.001290		
$m = 1$							
000	0.001302 <u>uvw</u>	044	0.002516 <u>u</u>	228	-0.000606	448	-0.000356
002	0.002015 <u>uv</u>	046	0.000728 <u>u</u>	244	0.003986	466	0.000076
004	0.001680 <u>uv</u>	048	-0.000228 <u>u</u>	246	0.001087	468	-0.000267
006	0.000199 <u>uv</u>	066	0.000630 <u>u</u>	248	-0.000345	488	-0.000023
008	-0.000284 <u>vu</u>	068	0.000211 <u>u</u>	266	0.000845	666	-0.000857
022	0.003133 <u>u</u>	088	0.000317 <u>u</u>	268	0.000282	668	-0.000683
024	0.001965 <u>u</u>	222	0.004415	288	0.000403	688	-0.000545
026	-0.000710 <u>u</u>	224	0.004249	444	0.003693	888	-0.000348
028	0.000116 <u>u</u>	226	0.000733	446	0.000948		
$m = 2$							
000	0.0000068 <u>u</u> ² <u>v</u> ² <u>w</u> ²	044	0.000324 <u>u</u> ²	228	-0.000030	448	0.000086
002	0.0000221 <u>u</u> ² <u>v</u> ²	046	0.000661 <u>u</u> ²	244	0.001069	466	0.001232
004	0.0000458 <u>u</u> ² <u>v</u> ²	048	-0.000023 <u>u</u> ²	246	0.000694	468	0.000253
006	0.0000259 <u>u</u> ² <u>v</u> ²	066	0.000174 <u>u</u> ²	248	0.000000	488	0.000165
008	-0.0000042 <u>u</u> ² <u>v</u> ²	068	0.000264 <u>u</u> ²	266	0.000573	666	0.000794
022	0.000072 <u>u</u> ²	088	0.000043 <u>u</u> ²	268	0.000153	668	-0.000016
024	0.001154 <u>u</u> ²	222	0.000237	288	0.000128	688	-0.000133
026	-0.000210 <u>u</u> ²	224	0.000497	444	0.002334	888	-0.000422
028	-0.000089 <u>u</u> ²	226	0.000295	446	0.001586		

Entry for 024, for example, gives limiting value for u small. Second-order interpolation of $(1 + u^2 + v^2 + w^2)H_0$, of $(1 + u^3 + v^3 + w^3)H_1/uvw$, or of $(1 + u^4 + v^4 + w^4)H_2/u^2v^2w^2$ will produce fairly accurate intermediate values.

TABLE XIV

A. Density, sound velocity, viscosity, and thermal properties of a few gases and liquids, at 0°C and 1 atm, in cgs units

MATERIAL	ρ_0	c_0	μ	γ	C_p	K
Air	0.00129	33,100	171×10^{-6}	1.40	0.240	0.055×10^{-3}
Argon	0.00178	31,900	210×10^{-6}	1.67	0.125	0.039×10^{-3}
CO ₂	0.00198	25,800	138×10^{-6}	1.30	0.199	0.034×10^{-3}
Helium	0.00018	97,000	186×10^{-6}	1.66	1.250	0.344×10^{-3}
H ₂	0.00009	127,000	84×10^{-6}	1.41	3.389	0.416×10^{-3}
Neon	0.00090	43,500	311×10^{-6}	1.64		0.110×10^{-3}
O ₂	0.00143	31,700	189×10^{-6}	1.40	0.218	0.057×10^{-3}
Pure water	(4°C) 1.000	143,000	179×10^{-4}		1.009	1.20×10^{-3}
Sea water	1.023	144,700				

At pressure of P atm, temperature $T^\circ\text{K}$, for the gases, $\rho = \rho_0 P(273^\circ/T)$ and $c = c_0 \sqrt{T/273^\circ}$ for moderate ranges of T and P . For sea water (32 g salt per 1,000 g water) $c = c_0 + 317(T - 273^\circ)$

B. Density, Young's modulus Q , and values of $\gamma^2 h/\omega = [3\rho(1 - s^2)/Q]^{1/2}$, where γ is the propagational parameter for a plate of Eq. (5.3.2)

MATERIAL	ρ	Q	$\gamma^2 h/\omega$
Aluminum	2.70	7×10^{11}	3.21×10^{-6}
Brass	8.44	9×10^{11}	5.00×10^{-6}
Copper	8.89	12×10^{11}	4.45×10^{-6}
Iron	7.2	9×10^{11}	4.62×10^{-6}
Nickel	8.7	20×10^{11}	3.41×10^{-6}
Silver	10.5	8×10^{11}	5.92×10^{-6}
Steel	7.8	20×10^{11}	3.22×10^{-6}

C. Comparison of mks and cgs units

MKS UNIT	CGS UNIT
1 meter	= 100 centimeters
1 kilogram	= 1,000 grams
1 newton	= 10^5 dynes
1 joule	= 10^7 ergs
1 kg/m ³	= 10^{-3} g/cm ³
1 newton/m ²	= 10 dynes/cm ²
1 kg/(m ²)(sec)	= 1/10 g/(cm ²)(sec) (ρc)
1 amp/m ²	= 3×10^5 statamp/cm ²
1 volt	= 1/300 statvolt
1 weber/m ²	= 10^4 B in emu
1 newton-sec/m ²	= 10 poises (viscosity)
1 joule/(sec)(m)(deg)	$\cong 1/420$ cal/(sec)(cm)(deg) (thermal conductivity)
1 joule/(mole)(deg)	$\cong 1/4200M$ cal/g (deg) (specific heat) (M = molecular weight)
1 atm	$\cong 10^5$ newtons/m ² = 10^6 dynes/cm ²

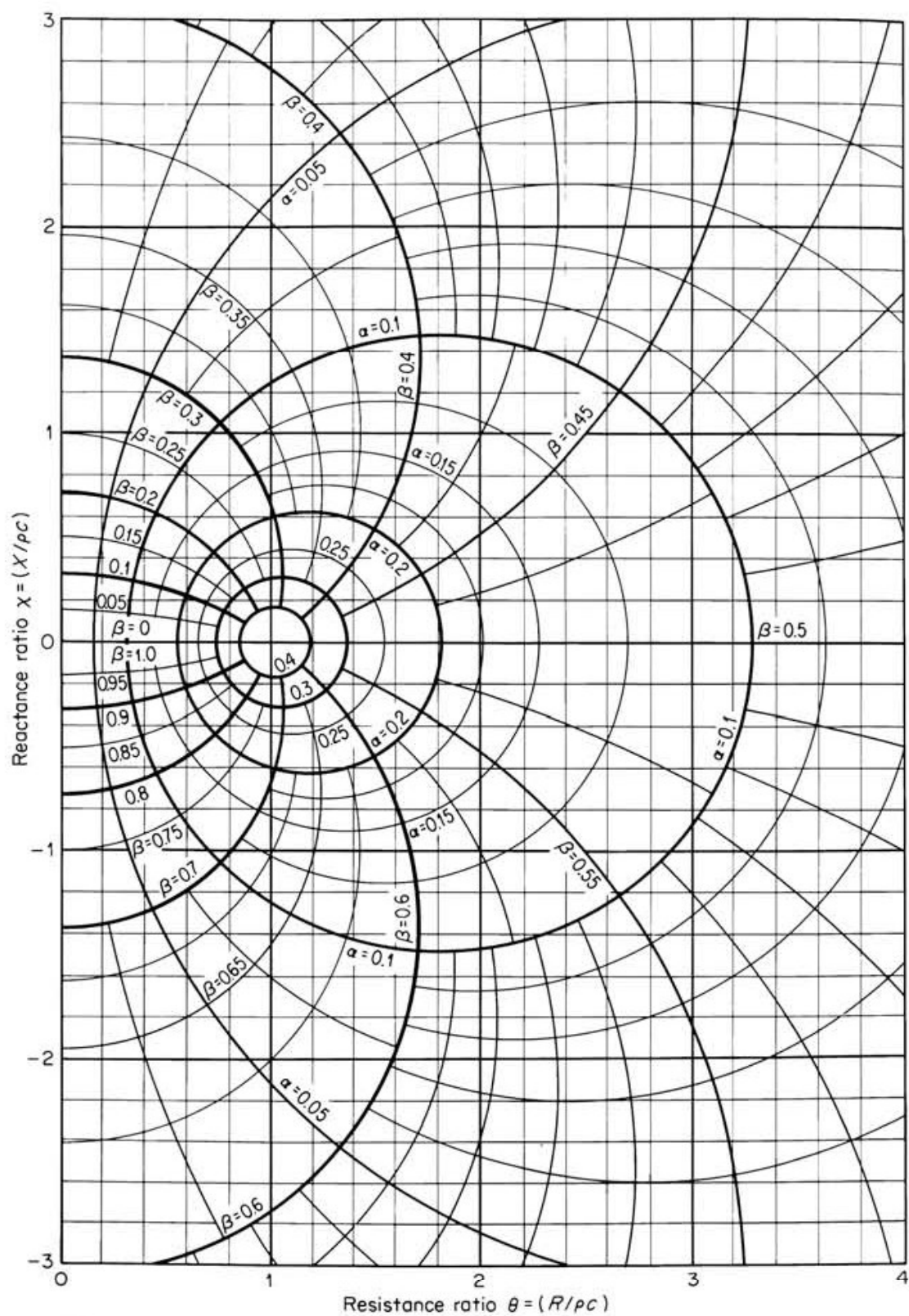


PLATE I

Conformal transformation from specific acoustic impedance $\zeta = \theta - i\chi = z/\rho c = \tanh[\pi(\alpha - i\beta)]$ of duct end, to wave-reflection parameter $(\Phi/\pi) + \frac{1}{2}i = \alpha - i\beta$. See Eqs. (9.1.4) and (9.3.1).

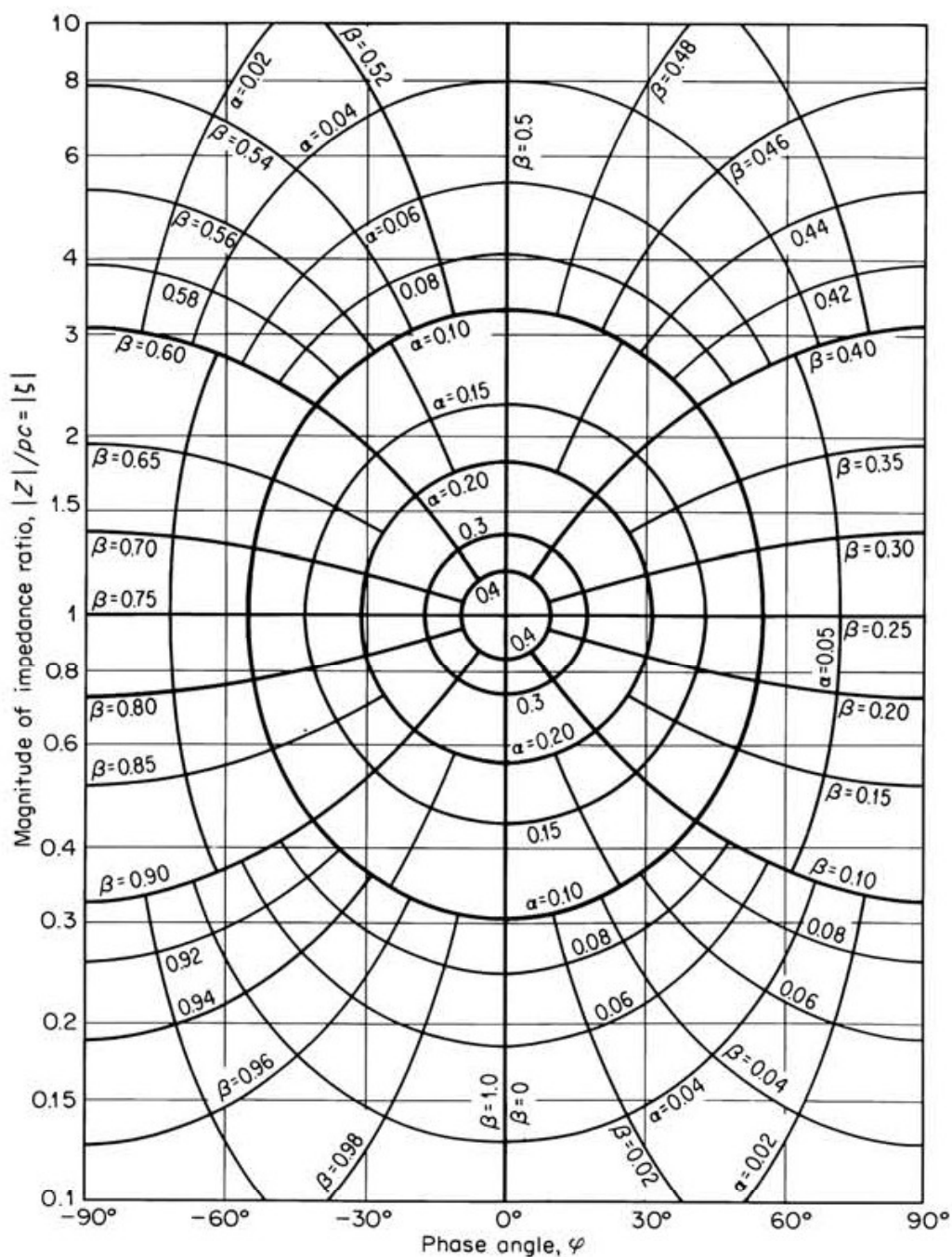


PLATE II

Conformal transformation from impedance magnitude and phase parameter $\ln \zeta = \ln |\zeta| - i\varphi$ to wave-reflection parameter $(\Phi/\pi) + \frac{1}{2}i = \alpha - i\beta$. See Eqs. (9.1.4) and (9.3.1).

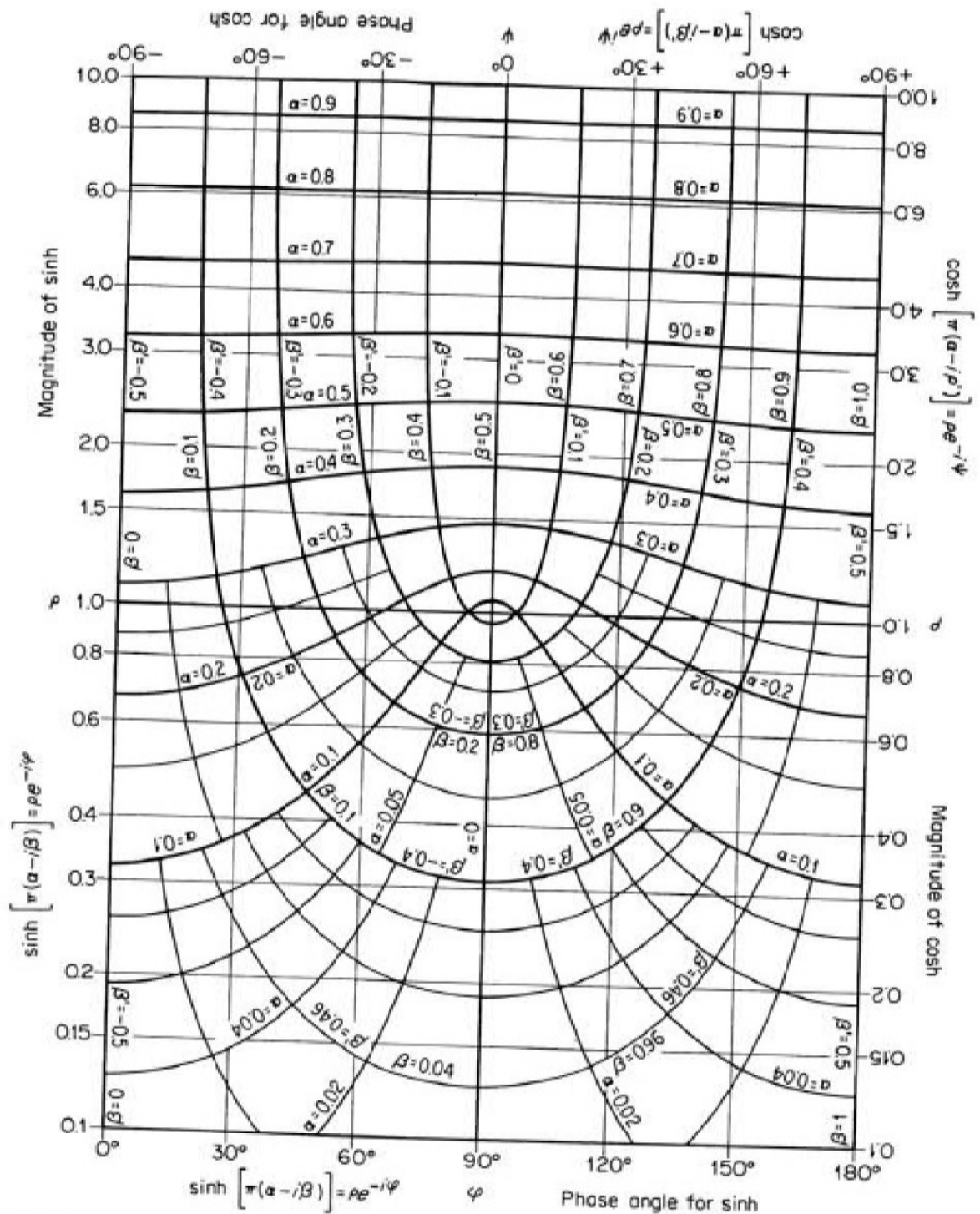


PLATE III

Conformal transformation from $\ln \left[\frac{\sinh}{\cosh} \pi(\alpha - i\beta) \right] = \ln \rho - i\varphi$ to argument $\alpha - i\beta$. See Eq. (9.1.5).

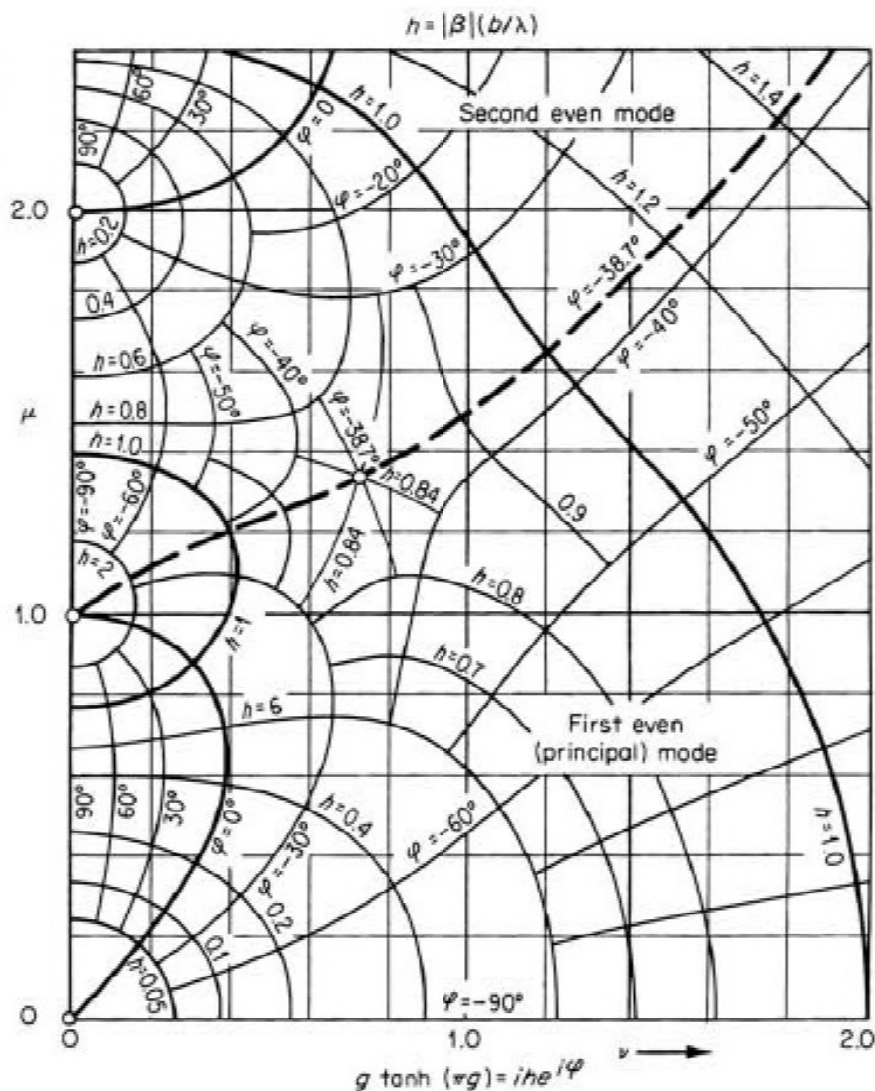
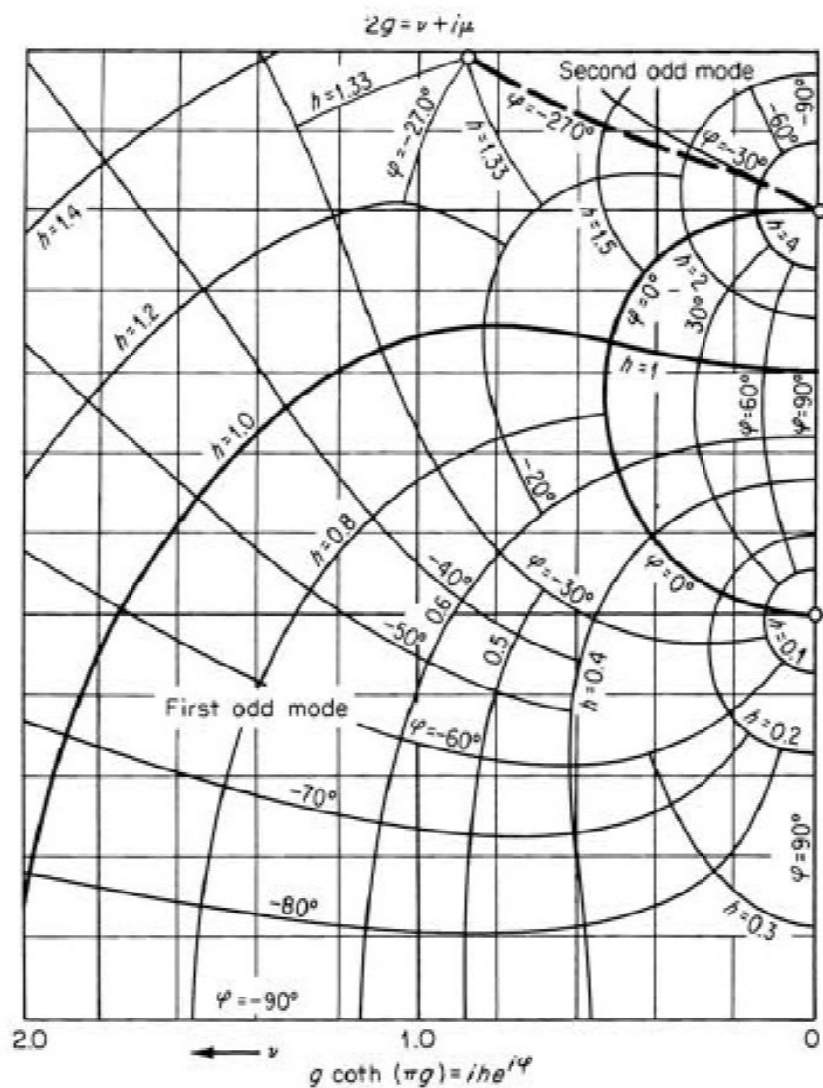


PLATE IV

Conformal transformation between admittance magnitude-and-phase parameter $\ln(\beta b/\lambda) = \ln h + i\varphi$ and wavenumber parameter $iq_m = 2g = \nu + i\mu$, for a rectangular duct of distance b between two walls, both having specific acoustic admittance $\beta = \rho c/z = |\beta| e^{i\varphi}$. If only one wall has admittance β , the other being rigid, use the right-hand chart only, and set $h = |2\beta b/\lambda|$; $q_m = \mu - i\nu$ then has half the values given on the chart. See Eqs. (9.2.19) and (9.2.20).

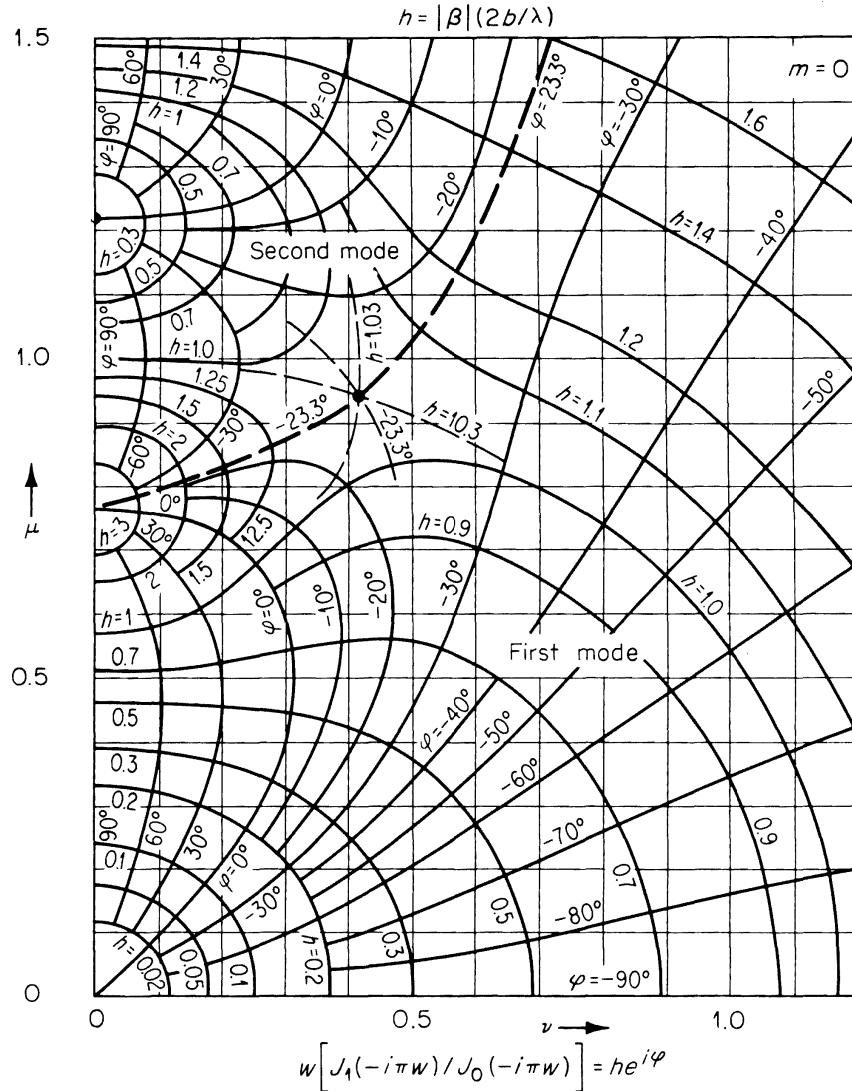
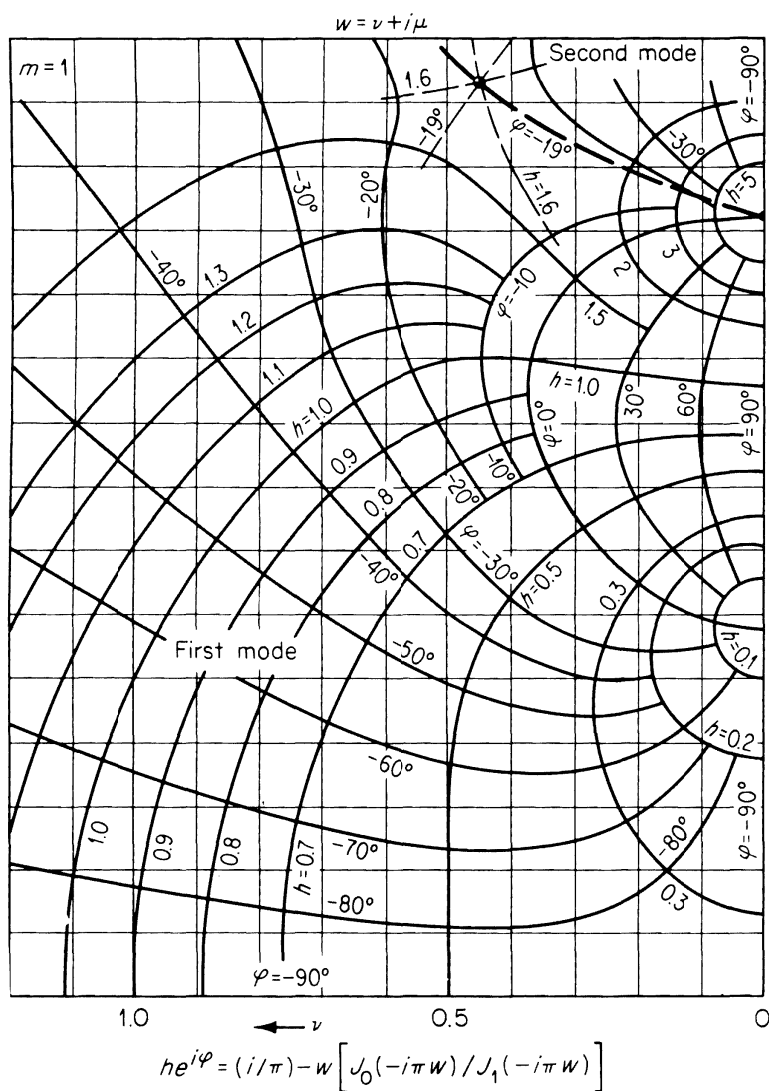


PLATE V

Conformal transformation between admittance magnitude-and-phase parameter $\ln(\beta b/\lambda) = \ln h + i\varphi$ and wavenumber parameter $i q_m = w = v + i\mu$, for a circular duct of radius b , the wall of the duct having specific acoustic admittance $\beta = \rho c/z = \beta e^{i\varphi}$. For modes symmetric about the duct axis ($m = 0$), use the right-hand chart; for modes with angle factor $\frac{\cos \phi}{\sin \phi}$ ($m = 1$), use the left-hand chart. See Eq. (9.2.25).

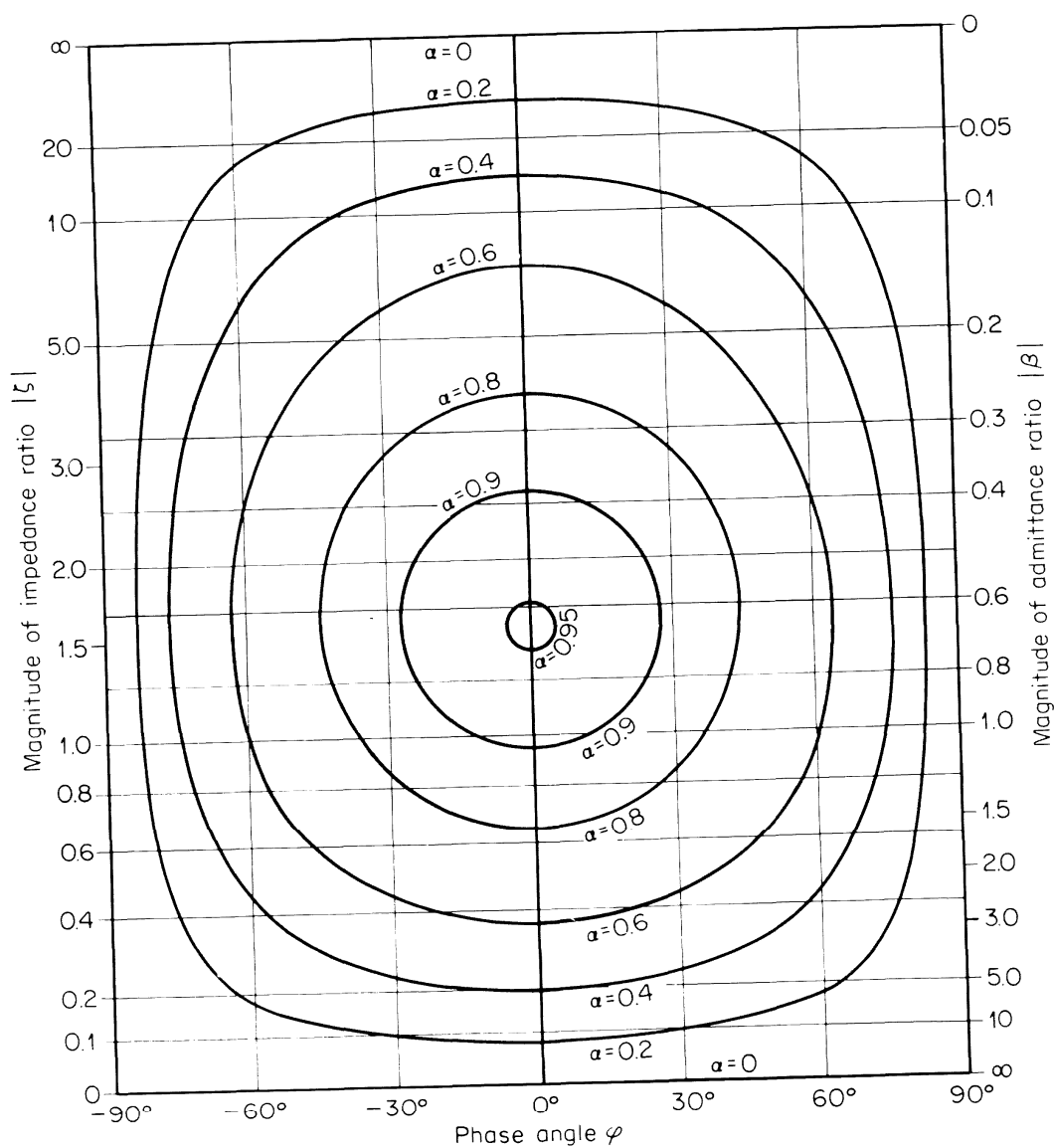


PLATE VI

Contour plot for mean absorption coefficient α for various values of magnitude and phase angle of specific acoustic impedance $z/\rho c = |\zeta| e^{-i\varphi} = 1/\beta$ of wall. Valid only when geometrical acoustics is applicable. See Eq. (9.5.8).